From Coulomb to Maxwell

Electro-dynamics is concerned with one particular attribute of matter – charge.

Experimentally, it was observed that certain bodies exert long range forces on each other that are certainly not gravitational – they are not proportional to the mass and they can be repulsive as well as attractive. The source of this new force was defined to be the "charge" of the object.

Electrostatics

Coulomb's Law - for charge $q_1$ at $\vec{r}_1$ and charge $q_2$ at $\vec{r}_2$, if separation $|\vec{r}_2 - \vec{r}_1|$ is much greater than the size of either charge, then

$$\vec{F}_{21} = \frac{k_1 q_1 q_2}{r_{12}^2} \hat{r}_{12} = \frac{q_2}{|\vec{r}_{12}|}$$

force on 2 due to 1

Central force - points from 1 to 2 - inverse square law

$$\vec{F}_{21} = \frac{q_1 q_2}{r_{12}^2} \hat{r}_{12}$$

0, $\vec{r}_2$, $\vec{g}_2$
$k_1$ is a universal constant of nature that determines the strength of the force when $q$ is expressed in terms of some arbitrary reference charge.

Since we only know about charge by measuring the Coulomb force, we are in principle free to choose $k_1$ to be anything we like – our choice then determines the units that charge is measured in.

In MKS system of units (same as SI system),
charge is measured in the historical unit, the "coulomb."
Then $k_1$ has the value $k_1 = \frac{1}{4\pi\varepsilon_0} = 10^{-7}\,\text{C}^2$,
where $c$ is speed of light in a vacuum. The units of $k_1$ are $\text{N}\cdot\text{m}^2/\text{Coul}^2$.

In CGS system of units (also called esu - electrostatic units),
one fixes $k_1 = 1$ and charge is measured in "statcoulombs."
$k_1$ is taken dimensionless, so $\text{statcoulomb} = (\text{N}\cdot\text{m}^2)^{1/2}$

Another reasonable modern choice would be to measure charge in integer multiples of the electron charge. This would yield a different value for $k_1$.

In this class we will be using CGS units.
But we keep $k_1$ general for now.
Superposition

For charges $q_i$ at positions $\vec{r}_i$, the force on charge $Q$ at position $\vec{r}$ is

$$\vec{F} = kQ \sum_i q_i \frac{\vec{r} - \vec{r}_i}{|\vec{r} - \vec{r}_i|^3}$$

forces add linearly

Conservation of charge - charge is neither created nor destroyed

$$\frac{d}{dt} \sum_i q_i = 0$$

where sum is over all charges in system

Continuum charge density

for charges $q_i$ at positions $\vec{r}_i$, define,

$$\rho(\vec{r}) = \sum_i q_i \delta(\vec{r} - \vec{r}_i)$$

$\delta(\vec{r} - \vec{r}_i)$ is Dirac $\delta$-function with properties:

$$\int_V d^3r \delta(\vec{r} - \vec{r}_i) = \begin{cases} 1 & \text{if } \vec{r} \in V \\ 0 & \text{otherwise} \end{cases}$$

$$\int_V d^3r f(\vec{r}) \delta(\vec{r} - \vec{r}_i) = \begin{cases} f(\vec{r}_i) & \text{if } \vec{r} \in V \\ 0 & \text{otherwise} \end{cases}$$

for any scalar function $f(\vec{r})$
\[ \int d^3r \rho(\vec{r}) = \sum \int d^3r \ n(\vec{r} - \vec{r}_i) \]

\[ = \text{total charge enclosed by volume } V \]

\[ \Rightarrow \rho \text{ has units of charge per volume} \]

\[ \Rightarrow n(\vec{r}) \text{ has units of } 1/\text{vol} \]

\[ \vec{F} = k_4 \vec{r} \rho(\vec{r}) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \]

We will often forget that \( \rho(\vec{r}) \) is in principle made up of a distribution of point charges, and take it to be a smooth continuous function.

\[ \text{charge conservation} - \frac{d}{dt} \int d^3r \rho(\vec{r}) = 0 \]

assuming \( V \) is so big that it contains all the charge, and no charge flows through the surface of \( V \)
Electric Field

$\vec{E}(\vec{r})$ is the force per unit charge that would be

felt by an infinitesimal test charge $q_0$ at position $\vec{r}$.

$$\vec{E}(\vec{r}) = \frac{1}{q_0} \vec{F} = k_0 \int \frac{\vec{q}(\vec{r}')} \cdot \frac{\vec{r} - \vec{r}'}{\left| \vec{r} - \vec{r}' \right|^3} \, \text{d}^3r' \quad (*)$$

In principle, the above is solution to all

electrostatic problems. In practice, we may

not always know $\vec{q}(\vec{r})$, but may need to

solve for it self-consistently with $\vec{E}$. It

will help to have another formulation of the

above in terms of differential equations.

We get these by taking the divergence and

curl of eq. $(*)$ above to get (see proof later)

$$\begin{cases}
\nabla \cdot \vec{E} = 4\pi \rho \quad (1) \quad \text{Gauss' Law} \\
\n\nabla \times \vec{E} = 0 \quad (2) \quad \text{true only for statics!}
\end{cases}$$

The proof of the above will follow on next page.

we also can recast $(1)$ and $(2)$ in integral form as follows:

by Gauss' theorem

$$\int \frac{\vec{q}}{\varepsilon} \cdot \vec{E} = \int \frac{d\sigma}{\varepsilon} \vec{A} \cdot \vec{E} = 4\pi k_0 \int \frac{d\sigma}{\varepsilon} \rho$$

by Stokes

$$\int \frac{d \sigma}{\varepsilon} \vec{A} \cdot \nabla \times \vec{E} = \int \frac{d\sigma}{\varepsilon} \vec{E} \cdot \nabla \cdot \vec{E} = 0$$

$$\int \frac{d \sigma}{\varepsilon} \vec{A} \cdot \nabla \times \vec{E}$$

by Stokes theorem

$$\int \frac{d \sigma}{\varepsilon} \vec{A} \cdot \nabla \times \vec{E} = \int \frac{d\sigma}{\varepsilon} \vec{E} = 0$$

by Gauss' theorem

$$\int \frac{d\sigma}{\varepsilon} \vec{A} = 0$$

$S$ enclosed in $V$

$\rho$ total charge

$S_0$

$S$ total charge

$\vec{A}$ enclosed in $S$