Image Charge Method

For simple geometries, one can try to obtain \( G_D \) or \( G_N \) by placing a set of "image charges" outside the volume of interest \( V \), on the "other side" of the system boundary surface \( S \). Because these image charges are outside \( V \), they contribute to the potential inside \( V \) because \( \nabla^2 \phi_{\text{image}} = 0 \), as necessary. Choose location of image charges so that total \( \phi \) has desired boundary condition.

1) Charge in front of infinite grounded plane.

\[
\begin{align*}
\nabla^2 \phi &= -4\pi q \delta(x) \delta(y) \delta(z-d) \\
\phi &= 0 \quad \text{for} \quad z = 0 \\
\n\text{If we find a solution to above, it is the unique solution.}
\end{align*}
\]

Solution - put fictitious image charge \(-q\) at \( z = -d \). Let \( \phi \) be the Coulomb potential from the real charge + the image charge.

\[
\phi(r) = \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{-q}{\sqrt{x^2 + y^2 + (z+d)^2}}
\]

\( \phi \) above satisfies \( \phi(x, y, 0) = 0 \) as required.

Also,
\[
\nabla^2 \phi = -4\pi q \delta(x-d) + 4\pi q \delta(-x+d) \\
= -4\pi q \delta(x-d) \quad \text{for region} \quad z > 0
\]
Can now find $\mathbf{E}$ for $z > 0$

$$\mathbf{E} = -\nabla \phi$$

In particular

$$E_z = -\frac{\partial \phi}{\partial z} = + \frac{q}{4\pi} \left[ (\frac{1}{2}) \left( \frac{2(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - (\frac{1}{2}) \frac{2(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right) \right]$$

$$E_z = \frac{q}{4\pi} \left[ \frac{(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}} \right]$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

$$-\frac{\partial \phi}{\partial n} = 4\pi \sigma$$

$$\Rightarrow \quad \sigma = -\frac{1}{4\pi} \frac{\partial \phi}{\partial z} = \frac{1}{4\pi} E_z (x,y,z=0)$$

$$\sigma(x,y) = \frac{q}{4\pi} \left[ \frac{-d}{(x^2+y^2+d^2)^{3/2}} - \frac{d}{(x^2+y^2+d^2)^{3/2}} \right]$$

$$\Rightarrow \quad \sigma = \frac{-q d}{2\pi (x^2+y^2+d^2)^{3/2}} = \frac{-q d}{2\pi (r_1^3+d^2)^{3/2}}$$

$$r_1 = \sqrt{x^2+y^2}$$
\[ q_{\text{induced}} = \int_{-\infty}^{\infty} dx dy \sigma(x,y) \]
\[ = 2\pi \int_{0}^{\infty} dr \, r \sigma(r) \]
\[ = 2\pi \int_{0}^{\infty} dr \, r \frac{(-qd)}{2\pi (r^2 + d^2)^{3/2}} \]
\[ = -qd \left[ \frac{-1}{(r^2 + d^2)^{1/2}} \right]_{0}^{\infty} \]
\[ = -qd \left[ 0 - \frac{-1}{d} \right] \]
\[ q_{\text{induced}} = -q \]

\[ \text{induced charge = image charge} \]

**Force on charge q in front of conducting plane**

Due to the induced \( \sigma \). The E field of this \( \sigma \)

\[ \vec{E} = \frac{-q^2 \hat{\jmath}}{(2d)^2} = \frac{-q^2 \hat{\jmath}}{4d^2} \]

\[ \text{attractive} \]

**Work done to move q into position from infinity**

\[ W = -\int_{-\infty}^{\infty} dx \int \vec{F} \cdot d\vec{r} = -\int_{0}^{\infty} dr \, F_{\jmath} \]

\[ \text{where}\ 
\vec{F} \text{ \text{is the electric force}} \]
\[ W = \int\limits_{\mathbb{V}} dV \left(-\frac{q^2}{4\varepsilon_0^2}\right) = -\frac{q^2}{4\varepsilon_0} \]

\[ W < 0 \Rightarrow \text{energy released} \]

**Note:** The above is not the electrostatic energy that would be present if the image charge were real. It is not the energy of an image charge \( q^* \) at \( \vec{r} = d\hat{\mathbf{z}} \):
\[ W = \int\limits_{\mathbb{V}} dV \left(-\frac{q^* q^2}{2\varepsilon_0} \right) = -\frac{q^2}{2\varepsilon_0}d \]

One way to see why is to note that as \( q \) is moved quasistatically in towards the conductor plane, the image charge also must be moving to stay equidistant on the opposite side.
2) point charge in front of a grounded (\( \phi = 0 \)) conducting sphere.

Charge \( q \) placed a distance \( s \) from center of grounded conducting sphere of radius \( R \).

Place image charge \( q' \) inside sphere so that the combined \( \phi \) from \( q \) and \( q' \) vanishes on surface of sphere.

By symmetry, \( q' \) should lie on the same radial line as \( q \) does, call the distance \( q \) from the origin \( "a" \).

Potential at position \( \vec{r} \) is

\[
\phi(\vec{r}) = \frac{q}{|\vec{r} - \frac{3a}{2}|} + \frac{q'}{|\vec{r} - a \hat{z}|}
\]

\[
= \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra \cos \theta)^{1/2}}
\]

Can we choose \( q' \) and \( a \) so that \( \phi(\vec{r}, \theta) = 0 \) for all \( \theta \)?
\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q'}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} \]

\[ \text{make denominators look alike} \]

\[ r^2 + a^2 - 2ar \cos \theta = \frac{a}{s} \left( \frac{s^2}{a} r^2 + sa - 2sr \cos \theta \right) \]

if choose \( sa = R^2 \), \( \alpha = R^2/s \), then \( sr^2 = s^2 \) and then the denominator of the 2nd term is:

\[ \left[ \frac{R^2}{s^2} (s^2 + r^2 - 2sr \cos \theta) \right]^{1/2} = \frac{R}{s} \left[ s^2 + r^2 - 2sr \cos \theta \right]^{1/2} \]

\[ \Rightarrow \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} + \frac{q'(s^2)}{(r^2 + s^2 - 2sr \cos \theta)^{1/2}} \]

So choose \( q'(s^2) = -\frac{q}{s} \Rightarrow q' = -\frac{q}{s} \)

to get \( \phi(r, \theta) = 0 \)

\[ \text{Solution is} \]

\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{qR/s}{\left( \frac{r^2 + s^2 - 2rs \cos \theta}{s^2} \right)^{1/2}} \]

\[ = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{\left( \frac{s^2 + r^2 - 2rs \cos \theta}{r^2} \right)^{1/2}} \]

Can get induced surface charge on sphere by

\[ 4\pi \sigma = \vec{E} \cdot \hat{n} = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} \]

see Jackson Eq. (2.5) for result
\[ \Theta(\theta) = -\frac{q}{4\pi R S} \frac{1}{\left(1 + \frac{R}{S}\right)^2 - 2\left(\frac{R}{S}\right)\cos \theta}^{3/2} \]

\(\Theta(\theta)\) is greatest at \(\theta = 0\), as one should expect.

One can integrate \(\Theta(\theta)\) to get total induced charge. One finds
\[ 2\pi \int \Theta \sin \theta R^2 \Theta = q' = -\frac{q}{R^2} \]

In general, total induced charge = sum of all image charges.

\[ \text{Force of attraction of charge to sphere} \]

\[ \mathbf{F} = \frac{q q'}{s^2} = -\frac{q^2}{(s-a)^2} = -\frac{q^2 R S}{(s^2 - R^2)^2} \]

Close to the surface of the sphere, \(s \approx R\), so write \(s = R + d\), where \(d \ll R\). Then
\[ \mathbf{F} = -\frac{q^2 R S}{(s-R)^2(s+R)^2} \approx -\frac{q^2 R}{d^2 (2R+d)^2} \]

get same result as for infinite flat grounded plane.

When \(q\) is so close to surface that \(d \ll R\), the charge does not "see" the curvature of the surface.
\[ F = \frac{g g' z^2}{(s-a)^2} = -\frac{z^2}{(s^2 r^2)} \approx -\frac{z^2 R^2}{s^3} \]

\[ F \sim \frac{1}{s^3} \text{ very different from flat plane} \]

\[ \text{also different from point charge} \]

Note: In preceding two problems, what we found was a \( \phi \) such that \( \nabla^2 \phi = -4\pi \delta(\mathbf{r} - \mathbf{x}_0) \) for a charge at \( \mathbf{x}_0 \), and \( \phi = 0 \) on the boundary. Such a \( \phi \) is nothing more than \( G_0 \), the corresponding Green function for Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we have a sphere with fixed net charge \( Q \).

We want to add new image charge to represent this case. If we put \( g' = -g/k \) at \( a = k/\lambda \) as before, the boundary condition of \( \phi = \text{const on surface } r = R \) is met, but the net charge on the sphere is \( Q' \) (the induced charge) not the desired \( Q \). We therefore need to add new image charge(s) of total charge \( Q - Q' \) (so total image charge is \( Q \)) in such a way that we keep \( \phi \) constant on the surface of the sphere. The way to do this is to put \( Q - Q' \) at the origin!
Solution 5

\[ \Phi(\rho, \theta) = \frac{Q + qR/s}{r} + \frac{q}{(r^2 + s^2 - 2rscos\theta)^{1/2}} - \frac{q}{r^2} \]

The force on the charge \( q \) is due to the \( \vec{E} \) field of the images.

\[ \vec{F} = \vec{F} = \frac{q}{s} \left( \frac{Q + qR/s}{s^2} + \frac{q}{(s-a)^2} \right) \]

\[ F = \frac{qQ}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s-R/s)^2} \]

\[ = \frac{qQ}{s^2} + \frac{q^2R}{s^3} \left[ \frac{1}{s^3} - \frac{1}{s^3(1-R^2/s^2)^2} \right] \]

\[ = \frac{qQ}{s^2} + \frac{q^2R}{s^3} \left[ 1 - \frac{1}{(1-R^2/s^2)^2} \right] \]

\[ F = \frac{qQ}{s^2} - \frac{q^2R^3}{s^3} \frac{2 - R^2}{(s^2 - R^2)^2} \]

For large \( s \gg R \) far from surface

\[ F \sim \frac{qQ}{s^2} - \frac{q^2R^3}{s^5} \]

leading term is quart

Coulomb force between \( q \)

and \( Q \) at origin

for \( Q > 0 \), \( F \) is always repulsive for large enough \( s \)
For $s = R+d$, $d \ll R$ close to surface

\[ F = \frac{qQ}{(R+d)^2} - \frac{q^2 R^3}{(R+d)} \cdot \frac{z - \frac{R^2}{(R+d)^2}}{(R^2 + d^2 + 2Rd - R^2)^2} \]

\[ \approx \frac{qQ}{R^2} - \frac{q^2 R^3}{R} \cdot \frac{(2-1)}{4R^2d^2} \]

\[ F \approx \frac{qQ}{R^2} - \frac{q^2}{4d^2} \approx -\frac{q^2}{4d^2} \text{ for } d \text{ small enough} \]

$F$ is always attractive for small enough $d$, and is equal to the force in front of a grounded plane, no matter what is the value of $Q$. This is because the image charge $q'$ lies so much closer to $q$ than does the $Q-q'$ at the origin, that it dominates the force.

The cross over from attractive to repulsive occurs at a distance $s$ that depends on $Q$. This distance is given by

\[ \frac{Q}{q} = \frac{R^3 s}{(s^2 - R^2)^2} \cdot \frac{(R^3)}{s^3} \cdot \frac{z - (\frac{R}{s})^2}{\left[1 - \left(\frac{R}{s}\right)^2\right]^2} \]

let $x = \frac{R}{s} \in (0,1)$

\[ \frac{Q}{q} = x^3 \frac{(z-x^2)}{(1-x^2)^2} \text{ gives 5th order polynomial in $x$} \]

we analytic solution can solve graphically
For $Q = 1$, crossover is at $R_3 = 0.62$

\[ S = 1.6 R \]

For $Q = 0.01$, crossover is at $R_3 = 0.36$

\[ S = 2.8 R \]