Separation of Variables

If the system has a rectangular boundary, we can look for solutions to $\nabla^2 \phi = 0$ of the form

$$\Phi(x, y, z) = X(x) Y(y) Z(z)$$

product of three functions

each of one variable only

$$\nabla^2 \phi = 0 \implies \frac{1}{\phi} \nabla^2 \phi = 0$$

$$= \frac{1}{X(x)} \frac{d^2 X}{d x^2} + \frac{1}{Y(y)} \frac{d^2 Y}{d y^2} + \frac{1}{Z(z)} \frac{d^2 Z}{d z^2} = 0$$

The only way this can happen for all values of $x, y, z$ is if each of the three terms is a constant, call them $a^2, b^2, c^2$

$$\frac{1}{X} \frac{d^2 X}{d x^2} = a^2 \implies X(x) = A_1 e^{-ax} + A_2 e^{ax}$$

$$\frac{1}{Y} \frac{d^2 Y}{d y^2} = b^2 \implies Y(y) = B_1 e^{-by} + B_2 e^{by}$$

$$\frac{1}{Z} \frac{d^2 Z}{d z^2} = c^2 \implies Z(z) = C_1 e^{-cz} + C_2 e^{cz}$$

with $a^2 + b^2 + c^2 = 0$

$\implies$ at least one of the $a^2, b^2, c^2$ is $< 0$

$\implies$ at least one of the $a, b, c$ is imaginary.
Above is one particular solution. But there are many solutions, each with different \(a_i, b_i, c_i\), but all obeying the constraint \(a_i^2 + b_i^2 + c_i^2 = 0\). The general solution is a superposition of these:

\[
\phi(x, y, z) = \sum_i \left( A_{i} e^{-a_i x} + A_{i} e^{a_i x} \right) \left( B_{i} e^{-b_i y} + B_{i} e^{b_i y} \right) \left( C_{i} e^{-c_i z} + C_{i} e^{c_i z} \right)
\]

**Example**

Consider a channel shaped as below – infinite along \(z\):

\[
\begin{align*}
\phi(0, y) &= 0 \\
\phi(a, y) &= 0 \\
\phi(x, y) &= 0 \text{ as } y \to \infty \\
\phi(x, 0) &= f(x) \text{ specified function}
\end{align*}
\]

Solution is independent of \(z\) \(\Rightarrow\)

\[
\phi(x, y) = \sum_i \left( A_{i} e^{-a_i x} + A_{i} e^{a_i x} \right) \left( B_{i} e^{-b_i y} + B_{i} e^{b_i y} \right)
\]

\(a_i^2 + b_i^2 = 0\)

we will see that the correct thing to choose is a magnitude

let \(a_i = i\alpha_i\)

\(b_i = \alpha_i\)

\[
\phi(x, y) = \sum_i \left( A_i \cos \alpha_i x + B_i \sin \alpha_i x \right) \left( C_i e^{-\alpha_i y} + D_i e^{\alpha_i y} \right)
\]

where

\[
\begin{align*}
A_i &= A_{i} + A_{i} \\
B_i &= i(A_{i} - A_{i}) \\
C_i &= B_{i} \\
D_i &= B_{i}
\end{align*}
\]
Now \( \phi(x,y) \to 0 \) as \( y \to \infty \) for all \( x \) \( \Rightarrow D_c = 0 \)

\[ \phi(x,y) = \sum_i \left[ A_i' \cos \alpha_i x + B_i' \sin \alpha_i x \right] e^{-\alpha_i y} \]

where \( A_i' = A_i C_i \), \( B_i' = B_i C_i \)

\[ \phi(0,y) = 0 \Rightarrow \sum_i A_i' e^{-\alpha_i y} = 0 \text{ all } y \Rightarrow \boxed{A_i' = 0} \]

\[ \phi(x,0) = 0 \Rightarrow \sum_i B_i' \sin \alpha_i x \cdot e^{-\alpha_i 0} = 0 \text{ all } y \]

\[ \Rightarrow \sin \alpha_i x = 0 \text{ or } \alpha_i x = n\pi \]

\[ \Rightarrow \boxed{\alpha_i = \frac{n\pi}{a}} \text{ integer } n \geq 1 \]

Finally,

\[ \phi(x,y) = \sum_{n=1}^{\infty} B_n' \sin \left( \frac{n\pi x}{a} \right) e^{-\frac{n\pi y}{a}} \]

\[ \phi(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} B_n' \sin \left( \frac{n\pi x}{a} \right) = f(x) \]

This is just the Fourier series for \( f(x) \)!

\[ B_n' = \frac{2}{a} \int_0^a f(x) \sin \left( \frac{n\pi x}{a} \right) \, dx \]

We have thus determined all unknown coefficients and found the solution.

See Jackson 2-8 if Fourier series needs review.
Recall orthonormality:
\[
\frac{2}{a} \int_0^a \sin \left( \frac{n \pi x}{a} \right) \sin \left( \frac{m \pi x}{a} \right) \, dx = \begin{cases} 
0 & m \neq n \\
1 & m = n
\end{cases}
\]

For \( f(x) = \Phi_0 \), a constant,

\[
B_n = \frac{\Phi_0}{a} \int_0^a \sin \left( \frac{n \pi x}{a} \right) \, dx = \frac{2\Phi_0}{\pi} \left[ -\frac{a}{n \pi} \cos \left( \frac{n \pi x}{a} \right) \right]_0^a
\]

\[
= \frac{2\Phi_0}{n \pi} (1 - \cos n \pi) = \begin{cases} 
0 & n \text{ even} \\
\frac{4\Phi_0}{n \pi} & n \text{ odd}
\end{cases}
\]
Polar Coordinates - still translationally invariant along \( z \) - so two dimensions

\[
\nabla^2 \phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \phi^2} = 0
\]

Assume \( \phi (r, \theta) = R(r) \Theta(\theta) \)

\[
\frac{r^2 \nabla^2 \phi}{\phi} = \frac{r}{R} \frac{d}{dr} \left( r \frac{dR}{dr} \right) + \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = 0
\]

Each term must be a constant

\[
\Rightarrow \quad \frac{r}{r^2} \frac{d}{dr} \left( r \frac{dR}{dr} \right) = \nu^2, \quad \frac{1}{\Theta} \frac{d^2 \Theta}{d\theta^2} = -\nu^2
\]

Solutions are

\[
R(r) = a r^\nu + b r^{-\nu} \\
\Theta(\theta) = A \cos (\nu \theta) + B \sin (\nu \theta)
\]

\[
R(r) = A_0 + B_0 \ln r
\]

\[
\Theta(\theta) = A_0 + B_0 \theta
\]

If \( \phi \) can take its entire range from \( 0 \) to \( 2\pi \)

(such as problem in which \( \phi \) is specified on the surface of a cylinder) then periodicity in \( \theta \rightarrow \theta + 2\pi \) requires \( B_0 = 0 \) and \( \nu = \text{integer} \ n \)

\[
\phi = A_0 + B_0 \ln r + \sum_{n=1}^{\infty} \left[ r^n (A_n \cos (n \theta) + B_n \sin (n \theta)) \right]
\]

\[
+ r^{-n} (C_n \cos (n \theta) + D_n \sin (n \theta))
\]
\[ \phi(r, \varphi) = a_0 + b_0 \ln r + \sum_{n=1}^{\infty} \left[ a_n r^n \sin(n \varphi + \alpha_n) + b_n r^{-n} \sin(n \varphi + \beta_n) \right] \]

If the region where there is no charge includes \( r = 0 \), then all \( b_n = 0 \) since \( \phi \) should not diverge at the origin.

If \( r = 0 \) is excluded from the region, then the \( b_n \) need not be zero. The case \( b_0 \neq 0 \) corresponds to a line charge \( \lambda \) along the \( x \)-axis.

Consider the case where \( \varphi \) has a restricted range, for example a wedge shaped opening of angle \( \beta \).

\[ 0 \leq \varphi \leq \beta \]

\[ \phi \text{ is constant in conductor} \]

\[ \Rightarrow \text{boundary conditions} \]

\[ \begin{align*}
\phi(r, 0) &= \phi_0 \\
\phi(r, \beta) &= \phi_0
\end{align*} \]

The general solution is the linear combination

\[ \phi(r, \varphi) = (A_0 + B_0 \ln r)(A_0 + B_0 \varphi) + \sum_{n \geq 1} \left( a_n r^n + b_n r^{-n} \right)(A_n \cos(n \varphi) + B_n \sin(n \varphi)) \]
1. The condition \( \phi(r, 0) = \phi_0 \) a constant independent of \( r \) then requires

\[
b_0 = 0, \quad A_0 = 0 \quad \forall \nu
\]

So,

\[
\phi(r, \varphi) = a_0 (A_0 + b_0 \varphi) + \sum_{\nu > 0} (a_\nu r^\nu + b_\nu r^{-\nu}) A_\nu \sin(\nu \varphi)
\]

2. Since \( \phi \) should be continuous as one approaches the conducting surface, and \( \phi = \phi_0 \) is a finite constant on the conducting surface, then \( \phi \) cannot diverge as one approaches the origin \( r = 0 \) along any fixed angle \( \varphi \). This requires

\[
b_\nu = 0 \quad \forall \nu
\]

So,

\[
\phi(r, \varphi) = a_0 (A_0 + b_0 \varphi) + \sum_{\nu > 0} a_\nu r^\nu \sin(\nu \varphi)
\]

3. The condition \( \phi(r, \beta) = \phi_0 \) a constant independent of \( r \) then requires

\[
\sin(\nu \beta) = 0 \Rightarrow \nu = \frac{n\pi}{\beta} \quad \forall \text{ integer } n > 1
\]

So,

\[
\phi(r, \varphi) = a_0 (A_0 + b_0 \varphi) + \sum_{n=1}^{\infty} a_n r^{n\pi/\beta} \sin(n\nu \varphi)
\]

4. As \( \phi \) must approach the constant \( \phi_0 \) as \( r \to 0 \) along any fixed angle \( \varphi \), we therefore must have

\[
b_0 = 0, \quad a_0 A_0 = \phi_0
\]
So finally we have

\[ \phi(r, \varphi) = \phi_0 + \sum_{n=1}^{\infty} a_n r^{\frac{n\pi}{\beta}} \sin\left(\frac{n\pi \varphi}{\beta}\right) \]

We still have all the unknown \(a_n\)s! These depend on how \(\phi(r, \varphi)\) behaves as \(r \to \infty\) (we can't make the choice here that \(\phi \to 0\) as \(r \to \infty\)) - this is additional information that must be specified to find the complete solution.

Nevertheless we can still get very interesting information near the origin at small \(r\). In this case, the leading term in the above series expansion for \(\phi\) is the \(n=1\) term, as it vanishes most slowly as \(r \to 0\).

\[ \phi(r, \varphi) \approx \phi_0 + a_1 r^{\frac{\pi \alpha}{\beta}} \sin\left(\frac{\pi \varphi}{\beta}\right) \]

The electric field is

\[ E_r(r, \varphi) = -\frac{\partial \phi}{\partial r} = -\frac{\pi \alpha}{\beta} r^{\frac{\pi \alpha}{\beta} - 1} \sin\left(\frac{\pi \varphi}{\beta}\right) \]

\[ E_\varphi(r, \varphi) = -\frac{1}{\beta} \frac{\partial \phi}{\partial \varphi} = -\frac{\pi \alpha}{\beta} r^{\frac{\pi \alpha}{\beta} - 1} \cos\left(\frac{\pi \varphi}{\beta}\right) \]

\( \Rightarrow E \sim r^{\frac{\pi \alpha}{\beta} - 1} \)

Induced surface charge given by \(4\pi \sigma = \vec{E} \cdot \hat{n}\)
for surface at $\varphi = 0$, $\hat{m} = \hat{\varphi}$

for surface at $\varphi = \beta$, $\hat{m} = -\hat{\varphi}$

$$\sigma(r, \varphi = 0) = \frac{E \varphi(r, 0)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

$$\sigma(r, \varphi = \beta) = -\frac{E \varphi(r, \beta)}{4\pi} = -\frac{a_1}{4\beta} r^{\frac{\pi}{\beta} - 1}$$

For $\frac{\pi}{\beta} > 1$, i.e. $\beta < \pi$, $\vec{E}$ and $\sigma$ vanish as

approach the origin.

For $\frac{\pi}{\beta} < 1$, i.e. $\beta > \pi$, $\vec{E}$ and $\sigma$ diverge as

approach the origin.

$\beta = \frac{\pi}{2}$

$E \propto r$

$\beta = \frac{3\pi}{2}$

$E \propto r^{-1/3}$

$\beta = 2\pi$

$E \propto r^{-1/2}$

$E$ diverges at an "external" corner

$E$ vanishes at an "internal" corner.

Remember, the above examples all had translational
symmetry along $z$, so the "corners" above are
really infinitely long straight "edges".