Spherical Coordinates

\[ \nabla^2 \phi = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \phi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \phi}{\partial \phi^2} = 0 \]

\[ \phi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

\[ r^2 \nabla^2 \Phi = \Theta \Phi \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R \Phi}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{R \Theta}{\sin^2 \theta} \frac{d^2 \Phi}{d\phi^2} = 0 \]

\[ \frac{r^2 \sin^2 \theta}{\Phi} \nabla^2 \Phi = \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) + \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

The solution depends only on \( \phi \) and \( \theta \)

\[ = -\text{const} \]

For \( \Phi \),

\[ = \text{const} \]

Take \( \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = -m^2 \)

\[ \Rightarrow \Theta = \sin^m \phi \]

\[ m \text{ integer for } 2\pi \text{ periodicity} \]

\[ \Rightarrow \frac{\sin^2 \theta}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{\sin \theta}{\Theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = m^2 \]

\[ \frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} \frac{1}{\Phi} \frac{d^2 \Phi}{d\phi^2} = 0 \]

The solution depends only on \( r \)

\[ = \text{const} \]

The solution depends only on \( \theta \)

\[ = -\text{const} \]
call the constant \(l(l+1)\)

For \( R \)

\[
\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = l(l+1) = 0
\]

Solutions are of the form \( R(r) = a_0 r^l + b_0 r^{-(l+1)} \)

Substitute \( a_0, b_0 \) to verify

\[
\frac{1}{r} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = \frac{d}{dr} \left( r^2 \left( la_0 r^{l-1} - (l+1)b_0 r^{-(l+2)} \right) \right)
\]

\[
= \frac{d}{dr} \left( la_0 r^{l+1} - (l+1)b_0 r^{-l} \right)
\]

\[
= l(l+1) a_0 r^l + b_0 r^{-(l+1)} = l(l+1) R
\]

For \( \theta \)

\[
\frac{1}{\sin \theta} \frac{d}{d\theta} (\sin \theta \frac{d\theta}{d\theta}) = \frac{m^2}{\sin^2 \theta} = -l(l+1)
\]

let \( x = \cos \theta \)

\[
dx = -\sin \theta \, d\theta
\]

\[
\frac{d\theta}{\sin \theta} = -\frac{dx}{x}
\]

0 < \( \theta \) < \pi

solutions for \(-1 \leq x \leq 1\) correspond to \( l \geq 0 \) integers

\[
\frac{d}{dx} \left[ \left( 1-x^2 \right) \frac{d\theta}{dx} \right] + \left[ \frac{l(l+1) - m^2}{1-x^2} \right] \theta = 0
\]

Called generalized Legendre Equation - solutions are called the associated Legendre functions.

Ordinary Legendre polynomials are solutions for \( m = 0 \)
For the special case \( m = 0 \), i.e., the solution has azimuthal symmetry and \( \Theta \) does not depend on the angle \( \phi \) (i.e., rotational symmetry about \( \hat{z} \) axis).

We want the solutions to

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d\Theta}{dx} \right] + \ell (\ell + 1) \Theta = 0
\]

The solutions are known as the Legendre polynomials, \( P_\ell(x) \).

They are given, for \( \ell \) integer, by

\[
P_\ell(x) = \frac{1}{2^{\ell} \ell!} \left( \frac{d}{dx} \right)^\ell (x^2 - 1)^\ell \quad \text{Rodriguez's formula}
\]

The lowest \( \ell \) polynomials are

\[
P_0(x) = 1 \\
P_1(x) = x \\
P_2(x) = \frac{1}{2} (3x^2 - 1) \\
P_3(x) = \frac{1}{2} (5x^3 - 3x)
\]

In general, \( P_\ell(x) \) is a polynomial of order \( \ell \) with

- only even powers of \( x \) if \( \ell \) is even, and
- only odd powers of \( x \) if \( \ell \) is odd.

\( P_\ell(x) \) is normalized so that \( P_\ell(1) = 1 \).
Note: Legendre polynomials are only for integer $l \geq 0$. What about solutions for non-integer $l$?

The $P_l(x)$ give one solution for each integer $l$.

But $P_l(x)$ are defined by a 2nd order differential equation - shouldn't there be a 2nd independent solution for each $l$?

It turns out that these "2nd" solutions, as well as solutions for non-integer $l$, all blow up at either $x = -1$ or $x = 1$, i.e. at $\theta = 0$ or $\theta = \pi$.

They therefore are physically unacceptable and we do not need to consider them. See Jackson 3.2.

The Legendre polynomials are orthogonal and form a complete set of basis functions on the interval $-1 \leq x \leq 1$.

$$\int_{-1}^{1} P_l(x) P_m(x) \, dx = \int_0^{\pi} \sin^2 \theta \, P_l(\cos \theta) P_m(\cos \theta) \, d\theta = \begin{cases} 0 & l \neq m \\ \frac{2}{2l+1} & l = m \end{cases}$$

$\Rightarrow$ we can expand any function $f(\theta)$, $0 \leq \theta \leq \pi$, as a linear combination of the $P_l(\cos \theta)$.

This is the reason they are useful for solving problems of Laplace's eqn with spherical boundary surfaces.
For \( m \neq 0 \), the solutions to (see Jackson 3.5)

\[
\frac{d}{dx} \left[ (1-x^2) \frac{d}{dx} \right] + \left[ (l+1) - \frac{m^2}{1-x^2} \right] \psi = 0
\]

are the associated Legendre functions \( P_{l}^{m}(x) \).

For \( P_{l}^{m}(x) \) to be finite in interval \(-1 \leq x \leq 1\), one again finds that \( l \) must be integer \( l \geq 0 \) and integer \( m \) must satisfy \( |m| \leq l \), i.e. \( m = -l, -l+1, \ldots, 0, \ldots, l-1, l \).

For each \( l \) and \( m \) there is only one such non-divergent solution.

It is typical to combine the solutions \( P_{l}^{m}(\cos \theta) \) to the \( \theta \)-part of the equation with the \( \Phi_{m}(\phi) = e^{im\phi} \) solutions to the \( \phi \)-part of the equation to define the spherical harmonics

\[
\Psi_{l}^{m}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_{l}^{m}(\cos \theta) e^{im\phi}
\]

The \( \Psi_{l}^{m} \) are orthogonal

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \sin \theta \, d\theta \, d\phi \, \Psi_{l}^{m*}^{*}(\theta, \phi) \Psi_{l'}^{m'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}
\]

and are a complete set of basis functions for expanding any function \( f(\theta, \phi) \) defined on the surface of a sphere.
Examples with azimuthal symmetry \( m = 0 \)

General solution to \( \nabla^2 \phi = 0 \) can be written in form

\[
\phi(r, \theta) = \sum_{l=0}^{\infty} \left[ A_l r^l + \frac{B_l}{r^{l+1}} \right] P_l \left( \cos \theta \right)
\]

determine the \( A_l \) and \( B_l \) from the boundary conditions of the particular problem.

1. Suppose one is given \( \phi(r, \theta) = \phi_0(\theta) \) on surface of sphere of radius \( R \).

To find solution of \( \nabla^2 \phi = 0 \) inside sphere, \( \phi \) should not diverge at origin \( \Rightarrow B_l = 0 \) for all \( l \).

\[
\phi(r, \theta) = \sum_{l=0}^{\infty} A_l r^l P_l \left( \cos \theta \right)
\]

\[
\Rightarrow \phi(r, \theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} A_l R^l P_l \left( \cos \theta \right)
\]

\[
\Rightarrow \int_0^{\pi} \int_0^{2\pi} \phi_0(\theta) P_m(\cos \theta) = \sum_{l=0}^{\infty} A_l R^l \int_0^{\pi} \sin \theta P_l(\cos \theta) P_m(\cos \theta)
\]

\[
= \sum_{l=0}^{\infty} A_l R^l \left( \frac{2}{2l+1} \right) \delta_{lm}
\]

\[
A_m = \frac{2m+1}{2R^m} \int_0^{\pi} \sin \theta \phi_0(\theta) P_m(\cos \theta)
\]

\[
A_m \sim R^m \frac{2}{2^{m+1}}
\]

\[
\Rightarrow \text{solution}
\]
To find solution of $\nabla^2 \phi = 0$ outside sphere

If $\phi \to 0$ as $r \to 0$, then $A_l = 0$ for all $l$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

$$\phi(R, \theta) = \phi_0(\theta) = \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)$$

This gives the solution:

$$B_m = \frac{2^{m+1}}{2} \left( \frac{m+1}{R^{m+1}} \right)^{\frac{\pi}{2}} \int_0^{\pi} \sin \theta \phi_0(\theta) P_m(\cos \theta) d\theta$$

$$B_m = A_m R^{2m+1}$$

2. Suppose one is given surface charge density $\sigma(\theta)$ fixed on surface of sphere of radius $R$. What is $\phi$ inside and outside?

From previous example:

$$\phi(r, \theta) = \begin{cases} 
\sum_{l=0}^{\infty} A_l r^l P_l(\cos \theta) & r < R \\
\sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta) & r > R 
\end{cases}$$

Boundary conditions at $r = R$ on surface

(i) $\phi$ continuous

$$-\sum_{l=0}^{\infty} \left[ A_l R^l - \frac{B_l}{R^{l+1}} \right] P_l(\cos \theta) = 0$$
If an expansion in Legendre polynomials vanishes for all \( \theta \), then each coefficient in the expansion must vanish.

\[ A_e R^l = \frac{B_e}{R^{l+1}} \Rightarrow B_e = A_e R^{2l+1} \]

(Cii) Jump in electric field at \( \sigma \)

\[ \frac{\partial \phi_\text{out}^\theta}{\partial r} \bigg|_{r=R} + \frac{\partial \phi_\text{in}^\theta}{\partial r} \bigg|_{r=R} = 4\pi \sigma \]

\[ \sum_{l=0}^{\infty} \left[ \frac{(l+1)B_e}{R^{l+2}} + lA_e R^{l-1} \int P_L(\cos \theta) \right] = 4\pi \sigma \]

\[ \sum_{l=0}^{\infty} \left[ \frac{(l+1)A_e R^{2l+1}}{R^{l+2}} + lA_e R^{l-1} \right] P_L(\cos \theta) \]

\[ \sum_{l=0}^{\infty} (2l+1) R^{l-1} A_e P_L(\cos \theta) = 4\pi \sigma \]

\[ (2m+1) R^{m-1} A_m \left( \frac{2}{2m+1} \right) = 4\pi \int_0^\pi \sin \theta \sigma(\theta) P_m(\cos \theta) \]

\[ A_m = \frac{4\pi}{2R^{m-1}} \int_0^\pi \sin \theta \sigma(\theta) P_m(\cos \theta) \]
Suppose $\sigma(\theta) = k \cos \theta$. What is $\phi$?

Note $\sigma(\theta) = k P_1(\cos \theta)$

Hence only $A_1 \neq 0$ by orthogonality of $P_0(\cos \theta)$

$A_1 = \frac{4\pi k}{2} \int_0^\pi \sin \theta \, P_1(\cos \theta) \, P_1(\cos \theta) \, d\theta$

$= \frac{4\pi k}{2} \left( \frac{2}{2+1} \right) = \frac{4\pi k}{3}$

$\Rightarrow \phi(r, \theta) = \begin{cases} \frac{4\pi k}{3} kr \cos \theta & r < R \\ \frac{4\pi k}{3} \frac{R^3}{r^2} \cos \theta & r > R \end{cases}$

We will see that potential outside the sphere is that of an ideal dipole with dipole moment $p = \frac{4\pi k R^3}{2}$

Inside the sphere, the potential $\phi = \frac{4\pi k}{3} z$

Where $z = r \cos \theta$. The electric field inside the sphere is therefore the constant $\vec{E} = -\nabla \phi = -\frac{4\pi k}{3} \hat{z}$
outside the sphere the field is

\[ \mathbf{E} = -\nabla \phi = -\frac{\partial \phi}{\partial r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} \]

\[ = \frac{8\pi}{3} k \frac{R^3}{r^3} \cos \theta \hat{r} + \frac{4\pi}{3} k \frac{R^3}{r^3} \sin \theta \hat{\theta} \]

\[ \mathbf{E} = \frac{4\pi R^3 k}{3} \frac{1}{r^3} \left[ 2 \cos \theta \hat{r} + \sin \theta \hat{\theta} \right] \]
Physical example with $\sigma(\theta) = k \cos \theta$.

Two spheres of radii $r$, with equal but opposite uniform charge densities $\rho$ and $-\rho$, displaced by small distance $d << R$.

Surface charge $\sigma$ builds up due to displacement. This is a uniformly "polarized" sphere.

\[ \sigma(\theta) = \rho \cos \theta \]

Surface charge: \[ \sigma(\theta) = \rho \cos \theta \]

Surface charge: \[ \sigma(\theta) = \rho S r \]

Surface charge: \[ \sigma(\theta) = \rho d \cos \theta \]

Total dipole moment is \((\rho d) \frac{4\pi}{3} R^3\).

Polarization = \frac{\text{dipole moment}}{\text{volume}} = \rho d.

\(\vec{E}\) field inside a uniformly polarized sphere is constant. \(\vec{E} = -\rho d \frac{4\pi}{3}\).
Conducting sphere in uniform electric field \( \hat{E} = E_0 \hat{z} \)

as \( r \to \infty \) for remote sphere, \( \hat{E} = E_0 \hat{z} \implies \phi = -E_0 z \)

boundary conditions:
\[
\begin{align*}
\phi(r, \theta) &= 0 \\
\phi(r \to \infty, \theta) &= -E_0 \cos \theta
\end{align*}
\]

solution outside sphere has the form
\[
\phi(r, \theta) = \sum_{l=0}^{\infty} \left( A_l r^l + \frac{B_l}{r^{l+1}} \right) P_l(\cos \theta)
\]

From boundary condition as \( r \to \infty \) we have:
\[
A_l = 0 \quad \text{all } l \neq 1
\]
\[
A_1 = -E_0 \quad \text{since } P_1(\cos \theta) = \cos \theta
\]

\[
\phi(r, \theta) = -E_0 \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)
\]

From \( \phi(r, \theta) = 0 \) we have:
\[
0 = -E_0 R \cos \theta + \sum_{l=0}^{\infty} \frac{B_l}{R^{l+1}} P_l(\cos \theta)
\]

\[
\implies B_l = 0 \quad \text{all } l \neq 1
\]

\[
B_1 = \frac{E_0 R}{R^2} \implies B_1 = +E_0 R^3
\]

\[
\frac{B_1}{R^2}
\]
\[
\phi(r, \theta) = -E_0 \left( r - \frac{R^3}{r^2} \right) \cos \theta
\]

1st term is just potential \(-E_0 \cos \theta\) of the uniform applied electric field.

2nd term is potential due to the induced surface charge on the surface—it is a dipole field.

Induced charge density is

\[
4\pi \sigma(\theta) = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} = E_0 \left( 1 + \frac{2R^3}{r^3} \right) \cos \theta
\]

\[
= 3E_0 \cos \theta
\]

\[
\sigma(\theta) = \frac{3}{4\pi} E_0 \cos \theta \quad \text{like uniformly polarized sphere} \quad k = \frac{3E_0}{4\pi}
\]

From (2) we know that the field inside the sphere due to this \(\sigma\) is just

\[
-\frac{1}{2} \pi k \hat{z} = -\frac{1}{2} \pi \frac{3E_0}{4\pi} \hat{z}
\]

\[
= -E_0 \hat{z}. \quad \text{This is just what is required so that the total field in the conducting sphere vanishes.}
\]

Can check that outside the sphere, \( \vec{E} = -\nabla \phi \)

is normal to surface of sphere at \( r = R \).
Behavior of fields near circular hole or slotted gap.

We now want to solve the $\nabla^2 \phi = 0$

with separation of variables,

but now $\phi$ is restricted to range

$0 \leq \phi \leq \beta$.

We still have azimuthal symmetry,

but now since we do not need solution to $\phi$ to be finite

for all $\theta \in [0, \pi]$, but only $0 \leq \theta \leq \beta$, we have more

solutions to the $\theta$ equation, i.e. $l$ does not have to

be integer, still need $l \neq 0$ to be finite at $\theta = 0$.

See Jackson sec. 3.4 for details.