Multiple Expansion

region with $r \neq 0$

We want to find the potential $\phi$ for an arbitrary localized distribution of charge, $\sigma$, at distances far away $r \gg R$.

$$\phi(r) = \int \frac{\sigma(r')}{|r-r'|}$$

General Coulomb formula

we want an expansion of $\frac{1}{|r-r'|}$ in powers of $\left(\frac{r'}{r}\right)$ for $r \gg r'$

view this as the potential at $r$ due to a point point charge located at position $r'$.

We take $r'$ on the $z$ axis.

The problem has azimuthal symmetry $\Rightarrow \phi$ depends only on $r$ and $\phi$, so we can express it as an expansion in Legendre polynomials.

For $r > r'$

$$\phi(r, \theta) = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos \theta)$$

all $A_l = 0$ as need $\phi \to 0$

as $r \to \infty$

$$= \frac{1}{r} \sum_{l=0}^{\infty} \frac{B_l}{r^l} P_l(\cos \theta)$$
We know \( \phi(r, \theta=0) = \frac{1}{r-r'} \) (for \( r > r' \))

\[ \Rightarrow \phi(r, 0) = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} P_\ell(1) \]

\[ = \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} \quad \text{as} \quad P_\ell(1) = 1 \]

\[ = \frac{1}{r} \left( 1 - \frac{r'}{r} \right) \quad \text{exact result from Coulomb} \]

Now Taylor expansion \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots \)

\[ \Rightarrow \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} = \frac{1}{r} \left( 1 + \frac{r'}{r} + \left( \frac{r'}{r} \right)^2 + \left( \frac{r'}{r} \right)^3 + \cdots \right) \]

\[ \Rightarrow B_\ell = (\sigma')^\ell \quad \text{in solution} \]

So for \( r > r' \)

\[ \frac{1}{|r-r'|} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^\ell P_\ell(\cos \theta) \]

So for the charge distribution \( \rho \),

\[ \phi(r) = \int d^3r' \frac{\rho(r')}{|r-r'|} = \int d^3r' \frac{\rho(r')}{r} \sum_{\ell=0}^{\infty} \left( \frac{r'}{r} \right)^\ell P_\ell(\cos \theta) \]

\[ = \sum_{\ell=0}^{\infty} \frac{1}{r^{\ell+1}} \int d^3r' \rho(r') \left( \frac{r'}{r} \right)^\ell P_\ell(\cos \theta) \]

where \( \theta \) is the angle between the fixed observation point \( \hat{r} \) and the integration variable \( \hat{r}' \).
This is the multipole expansion, which expresses the potential far from a localized source as a power series in $(r'/r)$. It is exact provided one adds all the infinite $l$ terms. In practice, one generally approximates by summing only up to some finite $l$.

Note: in doing the integrals

$$\int d^3r' \frac{1}{r'}(r')^l p_l(\cos \theta)$$

$\theta$ is defined as the angle of $\vec{r}'$ with respect to observation point $\vec{r}$. We therefore in principle have to repeat the integration every time we change $\vec{r}$.

We will find a way around this by

(i) just looking explicitly at the few lowest order terms

(ii) a general method involving spherical harmonics $Y_{lm}(\theta, \phi)$
monopole: $l=0$ term

$$\phi^{(0)}(\vec{r}) = \frac{Q}{4\pi} \int d^3r' f(\vec{r}') \quad p_0(\cos\theta) = 1$$

$$= \frac{Q}{r} \quad \text{where} \quad g = \int d^3r' f(\vec{r}') \quad \text{is the total charge}$$

dipole: $l=1$ term

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \int d^3r' f(\vec{r}') r' \cdot p_1(\cos\theta)$$

$$= \frac{1}{r^2} \int d^3r' f(\vec{r}') r' \cos\theta$$

Now $\vec{r} \cdot \vec{r}' = rr' \cos\theta \Rightarrow \hat{r} \cdot \vec{r}' = r' \cos\theta$

$$\phi^{(1)}(\vec{r}) = \frac{1}{r^2} \hat{r} \cdot \int d^3r' f(\vec{r}') \vec{r}'$$

$$= \frac{\hat{p} \cdot \vec{r}}{r^2} \quad \text{where} \quad \hat{p} = \int d^3r' f(\vec{r}') \vec{r}'$$

is the dipole moment

For a set of point charges $q_i$ at $\vec{r}_i$,

$$\vec{p} = \sum q_i \vec{r}_i$$
quadrupole: \( l = 2 \) Term

\[ \phi^{(2)}(\hat{r}) = \frac{1}{r^2} \int d^3r' \rho(\hat{r}') r'^2 P_2(\cos \theta) \]

\[ = \frac{1}{r^3} \int d^3r' \rho(\hat{r}') r'^2 \frac{1}{2} \left( 3 \cos^2 \theta - 1 \right) \]

Use \( \cos \theta = \hat{r}', \hat{r} \)

\[ \phi^{(2)}(\hat{r}) = \frac{1}{r^3} \int d^3r' \rho(\hat{r}') \frac{1}{2} \left( 3 \hat{r}' \cdot \hat{r}^2 - (\hat{r}')^2 \right) \]

\[ = \frac{1}{r^3} \hat{r} \cdot \left[ \int d^3r' \rho(\hat{r}') \frac{1}{2} (3 \hat{r}' \hat{r'} - (\hat{r}')^2) \right] \hat{r} \]

where \( \hat{I} \) is the identity tensor such that for any two vectors \( \vec{u} \) and \( \vec{v} \), \( \vec{u} \cdot \hat{I} \cdot \vec{v} = \vec{u} \cdot \vec{v} \)

and \( \hat{r}' \hat{r'} \) is the tensor such that for any two vectors \( \vec{u} \) and \( \vec{v} \), \( \vec{u} \cdot [\hat{r}' \hat{r'}] \cdot \vec{v} = (\vec{u} \cdot \hat{r}') (\hat{r'}^T \vec{v}) \)

Define quadrupole tensor \( \mathbf{Q} = \int d^3r' \rho(\hat{r}') \left( 3 \hat{r}' \hat{r'} - (\hat{r}')^2 \right) \)

\[ \phi^{(2)}(\hat{r}) = \frac{1}{r^3} \frac{1}{2} \hat{r} \cdot \mathbf{Q} \cdot \hat{r} \]

So to lowest three terms

\[ \phi(\hat{r}) = \frac{\hat{r}}{r} + \frac{\hat{r} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \mathbf{Q} \cdot \hat{r}}{2r^3} + \ldots \]

defined in form of the moments \( \mathbf{q}, \mathbf{P}, \mathbf{Q} \) of the charge distribution.
Note, the moments \( \hat{q}, \hat{p}, \hat{\xi} \) do not depend on the observation point \( \vec{r} \) — we can calculate them once and then use them to get \( \phi(\vec{r}) \) at all \( \vec{r} \).

**Monopole:**  \( \hat{q} = \int d^3r \, p(\vec{r}) \)  scalar integral

**Dipole:**  \( \vec{p} = \int d^3r \, p(\vec{r}) \vec{r} \)  vector integral

If we pick a coordinate system, we have to do 3 integrations to get the three components of \( \vec{p} \):

\[
\vec{e}_i \cdot \vec{p} = p_i = \int d^3r \, p(\vec{r}) \vec{r}_i
\]

**Quadrupole:**  \( \vec{Q} = \int d^3r \, p(\vec{r}) (\vec{r} \vec{r} - \vec{r}^2 \delta_{ij}) \)  tensor integral

If we pick a coordinate system \( x, y, z \) then

\( \vec{Q} \) is a matrix with components \( \hat{e}_1 = x, \hat{e}_2 = y, \hat{e}_3 = z \)

\[
\vec{e}_i \cdot \vec{Q} \cdot \vec{e}_j = (Q_{ij}) = \int d^3r \, p(\vec{r}) \left[ 3 \vec{r}_i \vec{r}_j - r^2 \delta_{ij} \right]
\]

There are 9 elements of the \( 3 \times 3 \) matrix \( Q_{ij} \) but \( Q_{ij} = Q_{ji} \) is symmetric so there are only 6 independent elements to compute.
General method

\[ \phi(\vec{r}) = \sum_{l=0}^{\infty} \frac{1}{r^l+1} \int d^3r' \rho(\vec{r}') \rho(\vec{r})^l Y_l^m(\cos \theta) \]

\[ \theta \text{ is the angle between } \vec{r} \text{ and } \vec{r}' \]

If we think of \( \vec{r} \) and \( \vec{r}' \) as the spherical coordinates, then in effect, above is choosing \( \vec{r} \) to be on the \( z \) axis. We would like a representation in which \( \vec{r} \) is positioned arbitrarily with respect to the axes used in describing \( \rho \).

Use the addition theorem for spherical harmonics. See Jackson 3.6 for discussion and proof.

\[ P_l (\cos \theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_l^m(\theta', \phi') Y_l^m(\theta, \phi) \]

where \((\theta, \phi)\) are the angles of \( \vec{r} \), \((\theta', \phi')\) are the angles of \( \vec{r}' \), and \( \theta \) is the angle between \( \hat{z} \) and \( \hat{r} \), i.e., \( \cos \theta = \hat{z} \cdot \hat{r} \)

\[ \cos \theta = \frac{\hat{z} \cdot \hat{r}}{2} \]

\[ \cos \theta' = \frac{\hat{z} \cdot \hat{r}'}{2} \]

\[ \Rightarrow \]

\[ \phi(\vec{r}) = \frac{1}{r^l+1} \int d^3r' \rho(\vec{r}') \rho(\vec{r})^l Y_l^m(\theta', \phi') Y_l^m(\theta, \phi) \]

Define the moment

\[ \mathcal{M}_{lm} = \int d^3r' \rho(\vec{r}') \rho(\vec{r})^l Y_l^m(\theta', \phi') \]

independent of observation point.
Then
\[ \Phi(\hat{r}) = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{g_{lm} Y_{lm}(\theta, \phi)}{(2l+1)r^{l+1}}. \]

So the Jackson eqn (4.4), (4.5), (4.6) to relate \( \Phi \) to \( g_1, \hat{p}, \hat{r} \).

\[ \Phi(\hat{r}) = \frac{q}{r} + \frac{\hat{p} \cdot \hat{r}}{r^2} + \frac{\hat{r} \cdot \vec{r}}{2r^3}. \]

Electric field \( \vec{E} = -\vec{\nabla} \Phi = -\frac{q}{r^2} \hat{r} - \frac{1}{r} \frac{\partial \Phi}{\partial \theta} \hat{\theta} - \frac{1}{r \sin \theta} \frac{\partial (\Phi \sin \theta)}{\partial \phi} \hat{\phi}. \)

For the monopole term \( \vec{E} = \frac{q}{r^2} \hat{r}. \)

For the dipole term, choose \( \hat{p} \) along \( \hat{\theta} \)-axis so

\[ \Phi(\hat{r}) = \frac{p \cos \theta}{r^2}. \]

\[ \vec{E} = \frac{2p \cos \theta \hat{r}}{r^3} + \frac{p \sin \theta \hat{\theta}}{r^3}. \]

\[ \vec{E} = \frac{p}{r^3} (2 \cos \theta \hat{r} + \sin \theta \hat{\theta}). \]

Note \( p \cos \theta \hat{r} = (\vec{p} \cdot \hat{r}) \hat{r} \)

\( p \sin \theta \hat{\theta} = - (\vec{p} \cdot \hat{\theta}) \hat{\theta}. \)

Now \( \vec{p} = (\vec{p} \cdot \hat{r}) \hat{r} + (\vec{p} \cdot \hat{\theta}) \hat{\theta} \)

\[ \Rightarrow - (\vec{p} \cdot \hat{\theta}) \hat{\theta} = (\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}. \]

So
\[ \vec{E} = \frac{1}{r^3} [2(\vec{p} \cdot \hat{r}) \hat{r} + (p \cdot \hat{\theta}) \hat{\theta} - \vec{p}]. \]

Expresses \( \vec{E} \) in coord free form.