Clausius-Mossotti equation

Elastic susceptibility & atomic polarizability

If a field \( \mathbf{E}_{ec} \) is applied to an atom, it gets polarized

\[
\mathbf{p} = \alpha \mathbf{E}_{ec}
\]

\( \alpha \) is what one calculates from a microscopic theory.

If \( \mathbf{E}_{ec} = \mathbf{E} \) the average field in the material, then electric susceptibility given by

\[
\mathbf{P} = m \mathbf{p} = m \alpha \mathbf{E}_{ec} = m \alpha \mathbf{E} = \chi_e \mathbf{E}
\]

\( \Rightarrow \chi_e = m \alpha \) where \( m \) = density of atoms

But a more careful consideration shows \( \mathbf{E}_{ec} \neq \mathbf{E} \).

The average field \( \mathbf{E} \) includes the electric field created by the polarized atom itself, \( \mathbf{E}_{ec} \), the local field the atom sees, should exclude its own self field.

\[
\mathbf{E} = \mathbf{E}_{ec} + \mathbf{E}_{atom}
\]

\( \Rightarrow \) average field average field excluding the atom
A polaronic material

cut-out sphere with volume $V_m$
the volume per atom

$\vec{E}_{loc}$ is field excluding the field of the polaron sphere of
volume $V_m$.

$\vec{E}_{atom}$ is field of the polaron sphere
$\vec{E}_{atom} = -\frac{4\pi}{3}\vec{P} = -\frac{4\pi}{3}m\vec{p}$

$\vec{E}_{loc} = \vec{E} - \vec{E}_{atom} = \vec{E} + \frac{4\pi}{3}\vec{P} = \vec{E} + \frac{4\pi}{3}m\vec{p}$

$\vec{P} = \alpha\vec{E}_{loc} = \alpha(\vec{E} + \frac{4\pi}{3}m\vec{p}) = \alpha\vec{E} + \frac{4\pi}{3}\alpha m\vec{p}$

$\vec{P} = m\vec{p} = \frac{2m}{1 - \frac{4\pi}{3}m\alpha} \frac{1}{\vec{E}} = \kappa_e \frac{1}{\vec{E}}$

$\kappa_e = \frac{m\alpha}{1 - \frac{4\pi}{3}m\alpha}$
on solve for $\alpha$ in terms of $\varepsilon$

$$\chi_e = \frac{m \alpha}{1 - \frac{4\pi}{3} \varepsilon m^2} \Rightarrow \chi_e - \frac{4\pi}{3} \varepsilon m^2 \chi_e = \alpha m$$

$$\Rightarrow \alpha = \frac{\chi_e}{m \left(1 + \frac{4\pi}{3} \varepsilon(\chi_e)\right)}$$

$$E = 1 + \chi_e \varepsilon \Rightarrow \alpha = \frac{E - 1}{4\pi m \left(1 + \frac{E - 1}{\varepsilon + 2}\right)}$$

relates atomic polarizability to measured dielectric constant

\[\alpha = \frac{3}{4\pi m} \left(\frac{E - 1}{E + 2}\right)\]

\[\text{Clapeyron-Mosotti}\]

or Lorentz-Lorenz equation

**Single model for $\alpha$**

**uniform**

atomic radius $a$

$$\rho = \frac{q}{\frac{4\pi}{3} a^3}$$

field inside $E_{\text{r}} = \frac{4\pi \rho r \hat{r}}{3}$

fringe

In external field $E_0$, net forces balance $\Rightarrow q E_0 = q \frac{4\pi \rho d}{3}$

$$\chi_e = \frac{ma^3}{1 - \frac{4\pi}{3} \varepsilon m^2 a^3}$$

$$p = \frac{q d}{4\pi} \Rightarrow q E_0 = \frac{3 \left(\frac{4\pi}{3} a^3\right) q d}{4\pi}$$

$$\Rightarrow \alpha = a^3 E_0$$

if $f = \frac{4\pi a^3}{2}$ fraction of vol that is occupied by atoms

$$\chi_e = \frac{1}{4\pi} \frac{3f}{1 - f}$$
Linear dielectrics

Bound charge is proportional to free charge

\[ \rho_b = - \nabla \cdot \mathbf{P} = - \nabla \cdot (\varepsilon \mathbf{E}) = - \nabla \cdot \left( \frac{\varepsilon_0 \mathbf{E}}{\varepsilon} \right) \]

If \( \varepsilon \) (and hence \( \mathbf{E} \)) is spatially constant, then

\[ \rho_b = - \frac{\varepsilon_0 \mathbf{E} \cdot \nabla \mathbf{E}}{\varepsilon} = - \frac{\varepsilon_0}{\varepsilon} \quad 4\pi \rho \]

\[ \rho_b = \frac{-4\pi \xi}{1 + 4\pi \xi} \int \rho \quad \text{when free charge} \; \rho = 0, \quad \text{then} \; \rho_b = 0 \]

\[ \rho_{\text{total}} = \rho + \rho_b = \rho \left[ 1 - \frac{4\pi \xi}{1 + 4\pi \xi} \right] = \frac{\rho}{1 + 4\pi \xi} = \frac{\rho}{\varepsilon} = \rho_{\text{total}} \]

Bound charge "screens" the free charge so the total charge is reduced compared to the free charge.
For linear dielectrics

**Statics**

\[ \nabla \cdot \vec{D} = 4\pi \rho \]
\[ \nabla \times \vec{E} = 0 \]

\[ \vec{D} = \epsilon \vec{E} \implies \nabla \cdot (\epsilon \vec{E}) = 4\pi \rho \]

If \( \epsilon \) is constant in space then \( \vec{E} \) \( \epsilon \nabla \cdot \vec{E} = 4\pi \rho \)
\[ \nabla \cdot \vec{E} = 4\pi \rho \]
\[ \nabla \times \vec{E} = 0 \]

Alternatively, could write \( \vec{E} = \vec{D}/\epsilon \)

\[ \nabla \times (\vec{D}/\epsilon) = 0 \]

\[ \nabla \times \vec{D} = 0 \text{ when } \epsilon \text{ constant in space} \]

\[ \vec{D} = 4\pi \rho \]

Complication arises at interface between dielectrics (or between dielectric and vacuum). At interface,

\( \epsilon \) \( \epsilon \) not constant \( \nabla \times \vec{D} \neq 0 \).

What we can do is to solve for \( \vec{E} \) or \( \vec{D} \) inside each dielectric separately and then use the boundary conditions

\[ \vec{n} \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = 4\pi \rho \]
\[ \hat{n} \cdot (\vec{E}_{\text{above}} - \vec{E}_{\text{below}}) = 0 \]

To match solutions across the interfaces.

A similar story holds for linear magnetic materials.
**Simple example**: parallel plate capacitor filled with a dielectric

\[ \sigma \quad \text{free charge} \]

\[ \begin{array}{c}
\sigma \\
\downarrow \\
\rightarrow \\
\delta \\
\end{array} \]

What is \( E \) between plates?

We know \( \vec{E} = \vec{D} = 0 \) outside plates.

Between plates \( \nabla \cdot \vec{D} = 0 \) as \( \rho = 0 \)

\[ \vec{D} = D(x) \hat{x} \Rightarrow \partial D/\partial x = 0 \Rightarrow D \text{ is constant} \]

**Boundary conditions**:

- **Left side plate** \[\begin{array}{c}
\langle \hat{m} = \hat{x} \\
\vec{D} = 0 \\
\end{array} \]

\[ \hat{x} \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = D = 4\pi \sigma \]

- **Right side plate** \[\begin{array}{c}
\langle \hat{m} = \hat{x} \\
\vec{D} = 0 \\
\end{array} \]

\[ \hat{x} \cdot (\vec{D}_{\text{above}} - \vec{D}_{\text{below}}) = -D = 4\pi (-\sigma) \]

\[ D = 4\pi \sigma \text{ as before} \]

\[ \Rightarrow \quad \vec{D} = 4\pi \sigma \hat{x} \]

\[ \boxed{\vec{E} = \vec{D}/\varepsilon = 4\pi \sigma \hat{x}} \]

Electric field reduced by factor \( 1/\varepsilon \) as compared to capacitor with vacuum between plates.

See Jackson section 4.4 for more interesting examples:
- Dielectric sphere in uniform applied \( \vec{E} \)

See Jackson section 5.11 for an interesting magnetic b.c. problem:
- Spherical permeable shell in uniform applied \( \vec{D} \)
point charge within a dielectric sphere

Charge \( q \) at center of dielectric sphere of radius \( R \), dielectric constant \( \varepsilon \)

\[
\nabla \cdot \vec{D} = 4\pi q = \oint_S d\hat{a} \cdot \vec{D} = 4\pi q \text{ enclosed}
\]

From symmetry \( \vec{D}(r) = D(r) \hat{r} \)

\[
\oint_S d\hat{a} \cdot \vec{D} = 4\pi \int_0^R r^2 D(r) = 4\pi q
\]

Sphere of radius \( r \)

\[
\vec{D} = \frac{q}{r^2} \hat{r} \quad \text{all } r
\]

\[
\Rightarrow \vec{E}(\vec{r}) = \begin{cases} \frac{\varepsilon}{\varepsilon + R^2} \hat{r} & r < R \\ \frac{q}{r^2} \hat{r} & r \geq R \end{cases}
\]

can check that tangential component of \( \vec{E} \) is continuous and normal component of \( \vec{D} \) is continuous as there is no free \( \sigma \) at surface of dielectric.

Normal component of \( \vec{E} \) jumps by

\[
\vec{n} \cdot (\vec{E} \text{ above } - \vec{E} \text{ below}) = \frac{q}{R^2} - \frac{q}{\varepsilon R^2} = \frac{q}{R^2} \left( 1 - \frac{1}{\varepsilon} \right) = \frac{q}{R^2} \left( \frac{\varepsilon - 1}{\varepsilon} \right)
\]

\[
= \frac{\varepsilon}{R^2} \left( \frac{4\pi \kappa_0}{1 + 4\pi \kappa_0} \right) = 4\pi \sigma_{\text{total}} = 4\pi \sigma_b
\]

\[
\Rightarrow \sigma_b = \frac{q}{4\pi R^2} \left( \frac{4\pi \kappa_0}{1 + 4\pi \kappa_0} \right) = \frac{\varepsilon \kappa_0}{R^2 \varepsilon}
\]

We can check this directly.
\[ \vec{\Phi} = \kappa \epsilon \vec{E} = \kappa \epsilon \frac{q}{r^2} \hat{r} \]

\[ \vec{p} = -\nabla \cdot \vec{P} = -\kappa \epsilon \frac{q}{\epsilon} 4\pi \delta(\vec{r}) \]

\[ \text{bound charge at origin} \quad q_b = -\frac{\kappa \epsilon}{\epsilon} 4\pi q \]

total charge at origin \quad \hat{q} + q_b = \hat{q} \left(1 - \frac{4\pi \kappa \epsilon}{\epsilon}\right)

\[ \epsilon = 1 + 4\pi \kappa \epsilon \quad = \hat{q} \left(\frac{\epsilon - 4\pi \kappa \epsilon}{\epsilon}\right) = \frac{\hat{q}}{\epsilon} \quad \text{screened charge} \]

at surface,

\[ \sigma_b = \hat{m} \cdot \vec{P} = \kappa \epsilon \frac{q}{\epsilon} \frac{q}{\epsilon^2} \]

agrees with what we get from jump in \( \hat{m} \cdot \vec{E} \).

Note: inside the dielectric the \( \vec{E} \) field is that of the screened point charge \( \hat{q} \).

outside the dielectric \( \vec{E} \) is just that of the free charge \( q \). There is no evidence in \( E_{\text{out}} \) that the dielectric even exists!
Now consider same problem but \( q \) is off center

\[ \vec{D} \cdot \vec{D} = 4\pi \rho \quad \text{where} \quad \rho = \rho \delta(r^2 - s^2) \]

\[ \vec{D} = \varepsilon \vec{E} \implies \vec{D} \cdot \vec{E} = 4\pi \rho / \varepsilon \]

\[ \vec{E} = -\vec{\nabla} \phi \implies \nabla^2 \phi = -4\pi \rho / \varepsilon = -4\pi q / \varepsilon (r^2 - s^2) \]

Solution for \( \phi \) will be of the form

\[ \phi(r) = \frac{q}{\varepsilon(r^2 - s^2)} + f(r) \]

where 1st term is due to the point charge \( q / \varepsilon \) and 2nd term satisfies \( \nabla^2 f = 0 \) and will be chosen to get the correct behavior at the boundary of the dielectric.

Since there is a azimuthal symmetry about \( \zeta \),
we can write

\[ f(r) = \sum_{l=0}^{\infty} a_{l} r^{l} \cos(l\zeta) \]

there are no \( x \) or \( y \) terms since \( F \) should not diverge at the origin.
\( \phi^\text{in}(r) = \frac{q}{r} + \sum_{l=0}^{\infty} a_l r^l P_l(\cos \theta) \)

From our discussion of electric multipole expansion, we know we can write for \( r > s \),

\[
\frac{1}{\sqrt{r^2 - s^2}} = \frac{1}{r} \sum_{l=0}^{\infty} \frac{(r/s)^l}{l!} P_l(\cos \theta)
\]

So for \( r > s \), (not true for \( r < s \! : \) ?)

\[
\phi^\text{in}(r) = \sum_{l=0}^{\infty} \left( \frac{q}{r^l} \left( \frac{r}{s} \right)^l + a_l r^l \right) P_l(\cos \theta)
\]

Outside the sphere there is no charge, so \( \vec{\nabla} \cdot \vec{E} = 0 \) or \( \nabla^2 \phi = 0 \)

\( \Rightarrow \phi^\text{out}(r) = \sum_{l=0}^{\infty} \frac{b_l}{r^{l+1}} P_l(\cos \theta) \)

there are no dipole terms since \( \phi^\text{out} \to 0 \) as \( r \to \infty \)

To determine the unknown \( a_l \) and \( b_l \) we use the boundary conditions at surface of dielectric at \( r = R \)
(1) Tangential component is continuous

\[ \mathbf{E} = \frac{3 \phi}{2r} \hat{r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \hat{\theta} = E_r \hat{r} + E_\theta \hat{\theta} \]

\[ \Rightarrow E_\theta \text{ is continuous at } r = R \]

Condition that \( E_\theta \) is continuous is the same condition that \( \phi \) is continuous (check this out for yourself if you are not sure)

\[ \Rightarrow \phi_{\text{in}}(R, \theta) = \phi_{\text{out}}(R, \theta) \]

\[ \frac{\phi_{\text{in}}(R)}{\epsilon R} + a R \ell = \frac{b \ell}{R^{2\ell+1}} \]

\[ \Rightarrow b \ell = \frac{\phi_{\text{in}}(R)}{\epsilon} s^\ell + a R \ell^{2\ell+1} \]

Normal component is continuous (since free surface charge \( \sigma = 0 \))

\[ \mathbf{D} = \epsilon \mathbf{E} \]

\[ \Rightarrow E_{r,\text{in}} = E_{r,\text{out}} \]

\[-\epsilon \frac{\partial \phi_{\text{in}}}{\partial r} = -\frac{\partial \phi_{\text{out}}}{\partial r} \]

\[ \Rightarrow (\ell+1) \frac{\phi_{\text{in}}(R)}{R^2} - l \epsilon a \ell R^{\ell-1} \frac{R^2}{R^{2\ell+2}} = \frac{(\ell+1) b \ell}{R^{2\ell+2}} \]
\[ q s^2 - \frac{d}{e+1} e a_{er} R^{2e+1} = b_e \]

Substitute \( b_e \) from previous boundary condition

\[ q s^2 - \frac{d}{e+1} e a_{er} R^{2e+1} = \frac{q}{e} s^2 + a_{er} R^{2e+1} \]

\[ q s^2 \left[ 1 - \frac{1}{e} \right] = a_{er} R^{2e+1} \left[ 1 + \frac{d}{e+1} e \right] \]

\[ a_{er} = \frac{q s^2}{R^{2e+1}} \frac{\left[ 1 - \frac{1}{e} \right]}{\left[ 1 + \frac{d}{e+1} e \right]} \]

\[ b_e = \frac{q s^2}{e} + a_{er} R^{2e+1} \]

\[ = \frac{q s^2}{e} + q s^2 \left[ 1 - \frac{1}{e} \right] \frac{1}{\left[ 1 + \frac{d}{e+1} e \right]} \]

\[ b_e = \frac{q s^2}{e} \left[ 1 + \frac{e - 1}{1 + \frac{d}{e+1} e} \right] \]

\[ = \frac{q s^2}{e} \left[ \frac{e (1 + \frac{d}{e+1})}{1 + \frac{d}{e+1} e} \right] \]

\[ b_e = \frac{q s^2}{e} \left[ 1 + \frac{\frac{d}{e+1}}{1 + \frac{d}{e+1} e} \right] \]
check the result:

as $s \to 0$, should recover previous answer

for $s = 0$, $a_0 = b_0 = 0$ for all $l \neq 0$

$$a_0 = \frac{q}{R} \left[ 1 - \frac{1}{2} \right]$$

$$b_0 = \frac{q}{R}$$

So $\phi^m (r^2) = \frac{q}{r} + \frac{q}{R} \left[ 1 - \frac{1}{2} \right]$.

$$E^m = - \nabla \phi^m = \frac{q}{r^2} \hat{r} \quad \text{as before}$$

$$\phi^\text{out} (r^2) = \frac{q}{r}$$

$$E^\text{out} = - \nabla \phi^\text{out} = \frac{q}{r^2} \hat{r} \quad \text{as before.}$$

Note: the constant that is the 2nd term in $\phi^m$

is just what is needed to make $\phi$ continuous at $r = R$.
Another check:

Let $\varepsilon \to \infty$ this models a conductor!

Again one finds $a_0 = b_0 = 0$ for all $l \neq 0$

$$a_0 = \frac{q}{R}$$

$$b_0 = \frac{q}{R}$$

$$\phi^m(\vec{r}) = \frac{q}{\varepsilon R} + \frac{q}{R} \to \frac{q}{R} \quad \text{as } \varepsilon \to \infty$$

$$\Rightarrow E^m(\vec{r}) = 0 \quad \text{as } \phi^m \text{ is a constant.}$$

$$\phi^{\text{out}}(\vec{r}) = \frac{q}{R} \quad \Rightarrow \vec{E}^{\text{out}} = \frac{q}{R^2} \hat{r}$$

Field outside is like point charge $q$ at the origin, independent of where $q$ is inside the sphere. This is the correct behavior of a conductor.

The mobile charges in the conductor completely screen the $q$ inside, and leave a uniform surface charge $\sigma_0 = \frac{q}{4\pi R^2}$ on the surface.
**Magnetostatics**

**Bar magnets** - \( \vec{J} = 0 \), \( \vec{M} \) fixed and given

\[
\nabla \cdot \vec{B} = 0
\]

\[
\nabla \times \vec{H} = \frac{4\pi}{c} \vec{J} = 0
\]

\[
\nabla \times \vec{H} = 0 \Rightarrow \vec{H} = -\nabla \Phi_M \text{ magnetic scalar potential}
\]

\[
\vec{B} = \vec{H} + 4\pi \vec{M}
\]

\[
\nabla \cdot \vec{B} = \nabla \cdot (\vec{H} + 4\pi \vec{M}) = 0
\]

\[
\nabla \cdot \vec{H} = -\nabla^2 \Phi_M = -4\pi \nabla \cdot \vec{M}
\]

\[
\nabla^2 \Phi_M = 4\pi \nabla \cdot \vec{M}
\]

so \( \Phi_M \equiv -\nabla \Phi \) looks like a magnetic "charge"

\( \Phi_M \) is source for \( \vec{H} \)

Also at surfaces of material \( \sigma_M = \vec{M} \cdot \hat{n} \) looks like surface charge

\[
\vec{H}(\vec{r}) = \int d^3r' \Phi_M(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} + \oint ds' \sigma_M(\vec{r}') \frac{(\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}
\]

Field lines for \( \vec{H} \) can start and end at sources and sinks given by \( \Phi_M \) and \( \sigma_M \)
\( \hat{M} = M \hat{z} \)

Bond currents:
\[
\vec{j}_b = c \hat{\alpha} \times \hat{M} = 0
\]
\[
\vec{E}_b = c \hat{M} \times \hat{M}
\]
\[
K_b = \left\{ \begin{array}{ll}
\text{CM} & \text{on side} \\
0 & \text{on top and bottom}
\end{array} \right.
\]

\( \vec{B} \) is like solenoid current.
Field lines of \( \vec{B} \) look like:

But \( \vec{H} \) is determined as follows:
\[
\vec{B}_M = -\nabla \times \vec{M} = 0
\]
\[
\vec{A}_M = \vec{A} \cdot \vec{M} = \left\{ \begin{array}{ll}
M & \text{on top} \\
-M & \text{on bottom}
\end{array} \right.
\]
Field lines of \( \vec{H} \) look like parallel plate capacitor.

Field lines of \( \vec{H} = \text{field lines of } \vec{B} \) outside magnet, but they are very different inside the magnet!