Electrostatic energy of interaction

\[ E = \frac{1}{8\pi} \int d^3r \ E^2 \]

Suppose the charge density \( \rho \) that produces \( E \)
can be broken into two pieces, \( \rho = \rho_1 + \rho_2 \)
with \( E = E_1 + E_2 \) where \( \nabla \cdot E_1 = 4\pi \rho_1 \) and \( \nabla \cdot E_2 = 4\pi \rho_2 \)
Then

\[ E = \frac{1}{8\pi} \int d^3r \left[ E_1^2 + E_2^2 + 2E_1 \cdot E_2 \right] \]

\[ \text{"self-energy"} \quad \text{"self-energy"} \quad \text{"interaction" energy} \]

\[ \rho_1 \quad \rho_2 \quad \rho_1 \text{ with } \rho_2 \]

\[ E_{\text{int}} = \frac{1}{4\pi} \int d^3r \ \vec{E}_1 \cdot \vec{E}_2 \]

\[ = \int d^3r \ \rho_1 \phi_2 = \int d^3r \ \rho_2 \phi_1 \]

where \( \vec{E}_1 = -\nabla \phi_1, \vec{E}_2 = -\nabla \phi_2 \), by similar manipulations

as earlier

integrals are over all space

Apply to the interaction energy of a dipole in
an external \( \vec{E} \) field

\[ E_{\text{int}} = \int d^3r \ \rho_1 \phi_2 \]

\( \phi \) potential of external \( \vec{E} \) field

charge distribution of dipole
Assuming $\Phi_2$ varies on length scale of $\rho_1$, then we can expand $\Phi_2(\mathbf{r}) = \Phi_2(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \Phi_2(\mathbf{r}_0)$ where $\mathbf{r}_0$ is the center of mass or any other convenient reference position within $\rho_1$.

\[
E_{\text{int}} = \int d^3r \, \rho(\mathbf{r}) \left[ \Phi_2(\mathbf{r}_0) + (\mathbf{r} - \mathbf{r}_0) \cdot \nabla \Phi_2(\mathbf{r}_0) \right]
\]

\[
= g \, \Phi_2(\mathbf{r}_0) + \left[ \int d^3r \, \rho(\mathbf{r}) (\mathbf{r} - \mathbf{r}_0) \right] \cdot \nabla \Phi_2(\mathbf{r}_0)
\]

\[
= g \, \Phi_2(\mathbf{r}_0) - \mathbf{p} \cdot \mathbf{E}
\]

where $g$ is total charge in $\rho_1$ and $\mathbf{p}$ is dipole moment with respect to $\mathbf{r}_0$. $\mathbf{E} = -\nabla \Phi_2$ is external $E$-field.

For a neutral charge distribution $g = 0$, and $\mathbf{p}$ is independent of the origin about which it is computed, so

\[
E_{\text{int}} = -\mathbf{p} \cdot \mathbf{E}
\]  \(\leftarrow\) does not include the energy needed to make the dipoles or to make $\mathbf{E}$.

$E_{\text{int}}$ is lowest when $\mathbf{p} \parallel \mathbf{E}$

$\Rightarrow$ in thermal ensemble, dipoles tend to align parallel to an applied $\mathbf{E}$.
Energy of magnetic dipole in external field

We had that the force on the dipole was

\[ \mathbf{F} = -\nabla (\mathbf{m} \cdot \mathbf{B}) \]

if we regard the force as coming from the gradient of a potential energy \( U \) then \( \mathbf{F} = -\nabla U \Rightarrow \)

\[ U = -\mathbf{m} \cdot \mathbf{B} \]

or equivalently, energy = work done to move dipole into position from \( \mathbf{r} \)

\[ W = -\int_{\mathbf{F} \cdot d\mathbf{r}} = -\int_{\nabla U} \mathbf{d\mathbf{r}} = -\mathbf{m} \cdot \mathbf{B} \]

This is the correct energy to use in cases where \( \mathbf{m} \) is due to intrinsic magnetic moments of atom or molecule - say from electron or nuclear spin. For a thermal ensemble magnetic moments tend to align \( \parallel \) to \( \mathbf{B} \).

The answer comes out quite differently if we are talking about a magnetic moment produced by a classical current loop. To see this, consider what we would get if we tried to do the calculation in a similar way to how we did for the energy of an electric dipole in an electric field...
Magnetic energy of interaction

\[ \mathcal{E} = \frac{1}{8\pi} \int d^3r \, B^2 \]

Suppose current \( \mathbf{j} \) that produces \( \mathbf{B} \) can be divided \( \mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2 \) with \( \mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2 \) where \( \mathbf{\nabla} \times \mathbf{B}_1 = \frac{4\pi}{c} \mathbf{j}_1 \) and \( \mathbf{\nabla} \times \mathbf{B}_2 = \frac{4\pi}{c} \mathbf{j}_2 \). Then

\[ \mathcal{E} = \frac{1}{8\pi} \int d^3r \left[ \mathbf{B}_1^2 + \mathbf{B}_2^2 + 2 \mathbf{B}_1 \cdot \mathbf{B}_2 \right] \]

Self energy, self energy, interaction energy

\[ \mathcal{E}_{\text{int}} = \frac{1}{4\pi} \int d^3r \, \mathbf{B}_1 \cdot \mathbf{B}_2 \]

\[ = \frac{1}{c} \int d^3r \, \mathbf{A}_1 \cdot \mathbf{A}_2 = \frac{1}{c} \int d^3r \, \mathbf{\dot{A}}_2 \cdot \mathbf{\dot{A}}_1 \]

where \( \mathbf{\dot{B}}_1 = \mathbf{\nabla} \times \mathbf{A}_1 \), \( \mathbf{\dot{B}}_2 = \mathbf{\nabla} \times \mathbf{A}_2 \), by similar manipulations as earlier.

Integrals are over all space.

Apply to the interaction energy of a magnetic dipole in an external \( \mathbf{B} \) field.

\[ \mathcal{E}_{\text{int}} = \frac{1}{c} \int d^3r \, \mathbf{\dot{A}}_1 \cdot \mathbf{\dot{A}}_2 \]

\( \mathbf{\dot{A}} \) vector potential of external \( \mathbf{B} \) field

Current distribution of dipole
Assuming \( \vec{A} \) varies slowly on length scale of \( \ell \), then expand \( A_i(r) = A_i(r_0) + (\vec{r} - \vec{r}_0) \cdot \nabla A_i(r_0) \)

\[
\mathcal{E}_{\text{int}} = \frac{1}{c} \int d^3r \frac{\vec{r}}{r^3} \cdot \vec{A}(\vec{r}_0)
+ \frac{1}{c} \int d^3r \sum \frac{\hat{\epsilon}_{ij}}{2} (r - r_0)_i \cdot \partial_j A_i(\vec{r}_0)
\]

**Superscript and subscript notation:**

From magnetostatic computation of magnetic dipole moment, we had \( \int d^3r \frac{\vec{r}}{r^3} = 0 \)

for magnetostatics

\( \Rightarrow \) 1st term above vanishes. So does the piece of 2nd term \( (\int d^3r \hat{\epsilon}_{ij}) \) \( r_0 \cdot \partial_j A_i(\vec{r}_0) \)

We are left with

\[
\mathcal{E}_{\text{int}} = \left[ \frac{1}{c} \int d^3r \hat{\epsilon}_{ij} \frac{r_j}{r^3} \right] \partial_j A_i(\vec{r}_0)
\]

From computation of magnetic dipole approxim we had

\[
\int d^3r \hat{\epsilon}_{ij} \frac{r_j}{r^3} = - \int d^3r \hat{\epsilon}_{ij} \frac{r_j}{r^3}
= \frac{1}{2} \int d^3r \left[ \hat{\epsilon}_{i}^{\kappa} \frac{r_j}{r^3} - \hat{\epsilon}_{i}^{\kappa} \frac{r_j}{r^3} \right]
= \frac{1}{2} \varepsilon_{kij} \int d^3r \left( \frac{\vec{r} \times \vec{r}}{r^2} \right)_k
\]

Recall:

\[
\vec{m} = \frac{1}{2c} \int d^3r \frac{\vec{r}}{r^3} \times \vec{r}
\]

\( \Rightarrow \) \( \hat{\epsilon}_{ij} \frac{r_j}{r^3} = - \varepsilon_{kij} m_k \) \( - \) mag dipole moment
\[ E_{\text{int}} = -m_k \varepsilon_{kij} \partial_j A_i = m_k \varepsilon_{kij} \partial_j A_i \]

\[ = \vec{m} \cdot (\vec{\nabla} \times \vec{A}) = \vec{m} \cdot \vec{B} = E_{\text{int}} \]

This is opposite in sign to what we found earlier!

Why the difference?

1. When we integrate the work done against the magnetostatic force to move \( \vec{m} \) into position from infinity, we found the energy:

\[ U = -m \cdot \vec{B} \]

2. When we compute the interaction energy from

\[ E_{\text{int}} = \frac{1}{2} \int d^3r \frac{\vec{A}_1 \cdot \vec{A}_2}{c^2} = \frac{1}{2} \int d^3r \int d^3r' \frac{\vec{F}_1(r) \cdot \vec{F}_2(r')}{(r-r')^2} \]

we found the energy \( E_{\text{int}} = +m \cdot \vec{B} \)

To see which is correct, let us consider computing the interaction energy \( \delta \) directly via method 1.
Consider two loops with currents $I_1$ and $I_2$.

What is the work done to move loop 2 in from infinity to its final position with respect to loop 1?

Magnetostatic force on loop 2 due to loop 1 is

$$ F = \frac{I_2}{C} \oint \mathbf{dl}_2 \times \mathbf{B}_1 $$
Lorentz force

$$ \mathbf{B}_1(r) = \frac{I_1}{C} \oint \mathbf{dl}_1 \times \frac{(\mathbf{r} - \mathbf{r}_1)}{\lvert \mathbf{r} - \mathbf{r}_1 \rvert^3} $$

Biot-Savart law

$$ F = \frac{I_1 I_2}{C^2} \oint \oint \mathbf{dl}_2 \times \left( \frac{\mathbf{dl}_1 \times (\mathbf{r}_2 - \mathbf{r}_1)}{\lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert^3} \right) $$

Use triple product rule

$$ \mathbf{dl}_2 \times \left[ \mathbf{dl}_1 \times (\mathbf{r}_2 - \mathbf{r}_1) \right] = \mathbf{dl}_1 \left[ \mathbf{dl}_2 \cdot (\mathbf{r}_2 - \mathbf{r}_1) \right] - (\mathbf{r}_2 - \mathbf{r}_1) \left( \mathbf{dl}_1 \cdot \mathbf{dl}_2 \right) $$

from the 1st term

$$ \oint \mathbf{dl}_2 \cdot \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{\lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert^3} = -\oint \mathbf{dl}_2 \cdot \mathbf{V}_2 \left( \frac{1}{\lvert \mathbf{r}_2 - \mathbf{r}_1 \rvert} \right) = 0 $$

as integral of gradient around closed loop always vanishes!
\[ \mathbf{F} = \frac{-I_1 I_2}{c^2} \oint \oint d\mathbf{\ell}_1 \cdot d\mathbf{\ell}_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \]

Write \( \mathbf{r}_2 = \mathbf{R} + \delta \mathbf{r}_2 \) where \( \mathbf{R} \) is center of loop \( \mathcal{C} \).

Use \( \frac{\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|^3} = \frac{\mathbf{\nabla}_R}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \)

\[ \mathbf{F} = \frac{I_1 I_2}{c^2} \oint \oint d\mathbf{\ell}_1 \cdot d\mathbf{\ell}_2 \mathbf{\nabla}_R \left( \frac{1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \right) \]

To move loop \( \mathcal{C} \) we need to apply a force equal and opposite to the above magnetostatic force.

The work we do in moving loop \( \mathcal{C} \) from infinity to its final position at \( \mathbf{R}_0 \) is

\[ W_{\text{mech}} = -\int_{\infty}^{\mathbf{R}_0} \mathbf{F} \cdot d\mathbf{R} = -\frac{I_1 I_2}{c^2} \oint \oint d\mathbf{\ell}_1 \cdot d\mathbf{\ell}_2 \int_{\mathbf{R}_0} d\mathbf{R} \mathbf{\nabla}_R \left( \frac{1}{|\mathbf{R} + \delta \mathbf{r}_2 - \mathbf{r}_1|} \right) \]

\[ = -\frac{I_1 I_2}{c^2} \oint \oint d\mathbf{\ell}_1 \cdot d\mathbf{\ell}_2 \frac{d^3 \mathbf{r}_2}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \]

where \( \mathbf{r}_2 = \mathbf{R}_0 + \delta \mathbf{r}_2 \)

Note the minus sign!

\[ = -\frac{I_1 I_2}{c^2} \int d^3 \mathbf{r} \int d^3 \mathbf{r}_2 \frac{\mathbf{f}_1(\mathbf{r}) \cdot \mathbf{f}_2(\mathbf{r}_2)}{|\mathbf{r}_2 - \mathbf{r}_1|} \]

This is just the negative of the interaction energy!!

\[ = -M_{12} I_1 I_2 \]

\( M_{12} \) mutual inductance

Why the minus sign!
The minus sign we have here is the same minus sign we get when we found \( U = -\mathbf{m} \cdot \mathbf{B} \) by integrating the force on the magnetic dipole.

Why don't we get \( + \frac{1}{c^2} \int d^3r_1 \int d^3r_2 \frac{\mathbf{J}_1(r_1) \cdot \mathbf{J}_2(r_2)}{|r_2 - r_1|} \) with the plus sign we expect from \( E = \frac{1}{8\pi} \int d^3r \mathbf{B}^2 \)?

**Answer:** we have left something out.

**Faraday's law** — when we move loop 2, the magnetic flux through loop 2 changes. This \( \frac{d\Phi}{dt} \) creates an emf \( \oint \mathbf{E} \cdot d\mathbf{l} \) around the loop that would tend to change the current in the loop.

If we are to keep the current fixed at constant \( I_2 \), then there must be a battery in the loop that does work to counter this induced emf (electromotive force).

Similarly, the flux through loop 1 is changing and a battery does work to keep \( I_1 \) constant. We need to add the work done by the batteries to the mechanical work computed above.

**Faraday**

\[
\begin{align*}
\mathcal{E}_1 &= -\frac{d\Phi_1}{c \, dt} \\
\mathcal{E}_2 &= -\frac{d\Phi_2}{c \, dt}
\end{align*}
\]

\( \Phi_1 = \) flux through loop 1

\( \Phi_2 = \) flux through loop 2
To keep the current constant, the batteries need to provide an emf that counters these Faraday induced emfs. The work done by the battery per unit time is therefore

\[ \frac{dW_{\text{battery}}}{dt} = -\varepsilon_1 I_1 - \varepsilon_2 I_2 \]

(check units: \( \varepsilon I \) is [length] \([\varepsilon] \cdot [B/s] \) = [length] \cdot [force/s] = energy/s)

\[ \frac{dW_{\text{battery}}}{dt} = \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \]

\[ W_{\text{battery}} = \int_0^T \left( \frac{d\Phi_1}{cdt} I_1 + \frac{d\Phi_2}{cdt} I_2 \right) dt \]

where \( t = 0 \) loop 2 is at infinity
\( t = T \) loop 2 is at final position
\( I_1, I_2 \) kept constant as loop moves

\[ W_{\text{battery}} = \frac{1}{c} \Phi_1 I_1 + \frac{1}{c} \Phi_2 I_2 \]

where \( \Phi_1 \) and \( \Phi_2 \) are fluxes in final position, and are assumed that fluxes = 0 at infinity

\[ \Phi_1 = CM_{12} I_2 \]
\[ \Phi_2 = CM_{21} I_1 = CM_{12} I_1 \]

as \( M_{12} = M_{21} \)

\[ \Rightarrow W_{\text{battery}} = 2M_{12} I_1 I_2 \]
add this to the mechanical work

\[ W_{\text{total}} = W_{\text{mech}} + W_{\text{battery}} = -M_{12} I_1 I_2 + 2M_{12} I_1 I_2 \]

\[ = M_{12} I_1 I_2 \]

\[ = + \frac{1}{c^2} \int \frac{d^3 r_1 \cdot d^3 r_2}{\mid r_1 - r_2 \mid} \quad \frac{\vec{f}_1(r_1) \cdot \vec{f}_2(r_2)}{\mid r_1 - r_2 \mid} \]

we get back the correct interaction energy!

**Conclusion:** The magnetostatic interaction energy

\[ \frac{1}{c^2} \int d^3 r_1 \cdot d^3 r_2 \quad \frac{\vec{f}_1(r_1) \cdot \vec{f}_2(r_2)}{\mid r_1 - r_2 \mid} \]

includes the work done to maintain the currents stationary as the current distributions move.

When we computed the interaction energy of a current loop dipole \( \vec{m} \) and find

\[ E_{\text{int}} = + \vec{m} \cdot \vec{B} \]

this includes the energy needed to maintain the constant current producing the constant \( \vec{m} \).

When we integrated the force on the dipole to find the potential energy

\[ U = -\vec{m} \cdot \vec{B} \]

this did not include the energy needed to maintain the constant current that creates \( \vec{m} \).

This is the correct energy expression to use when \( \vec{m} \) comes from intrinsic magnetic moments due to particles intrinsic spin, which cannot be viewed as arising from a current loop!
Electromagnetic Waves in a Vacuum

No sources \( \mathbf{f} = 0, \mathbf{q} = 0 \)

1) \( \nabla \cdot \mathbf{E} = 0 \)
2) \( \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{c \partial t} \)
3) \( \nabla \cdot \mathbf{B} = 0 \)
4) \( \nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \)

\( \nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{2}{c^2} \left( \frac{\partial \mathbf{E}}{\partial t} \right) \)

\( \Rightarrow \quad -\nabla^2 \mathbf{E} = -\frac{2}{c^2} \left( \nabla \times \mathbf{B} \right) = -\frac{2}{c^2} \left( \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \right) \)

\( \nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \)

Similarly
\( \nabla^2 \mathbf{B} = \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \)

\( \nabla^2 \mathbf{E} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \)

\( \nabla^2 \mathbf{E} - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \)

Note: In MKS units, above wave equation looks like

It was noticed that the speed of electromagnetic waves,
\( \sqrt{\varepsilon_0 \mu_0} = 3 \times 10^8 \text{ m/s} \), was the same as the speed of light! This observation was a key element in showing that light was in fact electromagnetic waves.
Harmonic

Plane waves

\[ \vec{E}(\vec{r}, t) = \text{Re} \left[ \vec{E}_k e^{i(k \cdot \vec{r} - \omega t)} \right] \]
\[ \vec{B}(\vec{r}, t) = \text{Re} \left[ \vec{B}_k e^{i(k \cdot \vec{r} - \omega t)} \right] \]

\( \vec{k} \) is wave vector
\( \omega \) is angular frequency
\( v = \frac{\omega}{2\pi} \) is frequency
\( \nu = \frac{1}{T} \) is period
\( \lambda = \frac{2\pi}{|k|} \) is wavelength

\( \frac{|\vec{E}_k|}{|\vec{B}_k|} \) is amplitude

\[ \vec{E}(\vec{r} + \lambda \hat{k}, t) = \vec{E}(\vec{r}, t) \] periodic in space with period \( \lambda \)

\[ \vec{E}(\vec{r}, t + T) = \vec{E}(\vec{r}, t) \] periodic in time with period \( T \)

"plane wave" \( \Rightarrow \vec{E}(\vec{r}, t) \) is constant on space or planes with constant \( \vec{m} \parallel \hat{k} \)

Properties of EM plane waves

\[ \nabla \cdot \vec{E} = 0 \] \( \Rightarrow \text{Re} \left[ \vec{E}_k \cdot \nabla \vec{E} \right] \]
\[ = \text{Re} \left[ i \vec{E}_k \cdot \hat{k} e^{i(k \cdot \vec{r} - \omega t)} \right] \]
\[ = 0 \]
\[ \Rightarrow \vec{E}_k \cdot \hat{k} = 0 \]

Amplitude is orthogonal to \( \vec{k} \)

\[ \nabla \cdot \vec{B} = 0 \] \( \Rightarrow \vec{B}_k \cdot \hat{k} = 0 \)

Amplitude orthogonal to \( \vec{k} \)
\[ \nabla \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \]

\[ \Rightarrow \text{Re} \left[ \nabla \times \vec{B}_h e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ \frac{1}{c} \vec{E}_h \frac{\partial}{\partial t} e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \text{Re} \left[ -\vec{B}_k \times \vec{E} e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ -i \omega \vec{E}_k e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \text{Re} \left[ i \vec{k} \times \vec{B}_h e^{i(k \cdot r - wt)} \right] = \text{Re} \left[ -i \omega \vec{E}_k e^{i(k \cdot r - wt)} \right] \]

\[ \Rightarrow \vec{k} \times \vec{B}_h = -\frac{\omega}{c} \vec{E}_k \]

\[ \vec{k} \times \vec{k} \times \vec{B}_h = -k^2 \vec{B}_h = -\frac{\omega}{c} \vec{k} \times \vec{E}_k \]

\[ \vec{B}_h = \frac{\omega}{ck^2} \vec{k} \times \vec{E}_k \]

Finally,

\[ \nabla^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \]

\[ \Rightarrow \text{Re} \left[ \vec{E}_k \nabla^2 e^{i(k \cdot r - wt)} - \frac{k^2}{c^2} \frac{\partial^2}{\partial t^2} e^{i(k \cdot r - wt)} \right] = 0 \]

\[ \Rightarrow k^2 \left[ \vec{E}_k (-k^2) e^{i(k \cdot r - wt)} + \frac{\omega^2}{c^2} \vec{E}_k e^{i(k \cdot r - wt)} \right] = 0 \]

\[ \Rightarrow k^2 = \frac{\omega^2}{c^2} \]

\[ \omega = \pm kc \]

\[ \text{Dispersion relation} \]

Consistent with above,

\[ \vec{B}_h = \hat{k} \times \vec{E}_h \]

\[ \hat{k} = \frac{\vec{E}_h}{|\vec{E}_h|} \]

\[ |\vec{B}_h| = |\vec{E}_h| \]