Reflection and Transmission of Waves at Interfaces

For simplicity, assume \( E_a \) is real and positive, \( E_b \) may be complex, \( \mu_a \) and \( \mu_b \) are real and constant.

**\( k_0 \)** is incident wave, \( \theta_0 \) = angle of incidence

**\( k_1 \)** is reflected wave, \( \theta_1 \) = angle of reflection

**\( k_2 \)** is the transmitted or "refracted" wave, \( \theta_2 \) = angle of refracted

Let each wave be given by

\[
\tilde{F}_n(r, t) = F_n e^{i(k_n \cdot r - \omega_n t)}
\]

Where \( \tilde{F}_n \) can be either \( \tilde{E}_n \) or \( \tilde{H}_n \) for the electric or magnetic component of the wave.

Boundary condition: Tangential component \( \tilde{E} \) must be continuous at \( z = 0 \). If \( \hat{\tau} \) is a vector in xy plane, then

\[
\hat{\tau} \cdot \tilde{E}_0 e^{-i\omega_0 t} + \hat{\tau} \cdot \tilde{E}_1 e^{-i\omega_1 t} = \hat{\tau} \cdot \tilde{E}_2 e^{-i\omega_2 t}
\]

must be true for all time. Can only happen if

\[
\omega_0 = \omega_1 = \omega_2 = \omega
\]

All frequencies are equal.
Now consider the same boundary condition for \( \vec{p} \), a position vector in the xy plane at \( z = 0 \). Since \( \alpha \)'s are all equal, we can cancel out the common \( \vec{e}_z \) factors to get

\[
\vec{a} \cdot \vec{E}_0 e^{i \vec{k}_0 \cdot \vec{p}} + \vec{a} \cdot \vec{E}_1 e^{i \vec{k}_1 \cdot \vec{p}} = \vec{a} \cdot \vec{E}_2 e^{i \vec{k}_2 \cdot \vec{p}}
\]

This must be true for all \( \vec{p} \). Can only happen if the projections of the \( \vec{k}_n \) in the xy plane are all equal:

\[
\begin{align*}
\vec{k}_0 &= \vec{k}_1 = \vec{k}_2 \\
\vec{k}_0 &= \vec{k}_1 = \vec{k}_2 \\
\end{align*}
\]

Only \( \vec{e}_z \) components of \( \vec{k} \) vectors can be different.

Choose coordinate system as in diagram so that all \( \vec{k} \) vectors lie in the xy plane (y out of page).

Since \( \alpha \) is real and positive, \( \vec{k}_0 \) and \( \vec{k}_1 \) are real vectors.

\[
\begin{align*}
k_0 &= k_1 \\
\end{align*}
\]

Since \( k_0 = \frac{\omega}{c^2} \sqrt{\mu \varepsilon_0} \) and \( k_1 = \frac{\omega}{c^2} \sqrt{\mu \varepsilon_0} \), then \( k_0 = k_1 \) so \( \sin \theta_0 = \sin \theta_1 \)

\[\theta_0 = \theta_1\]

Angle of incidence = angle of reflection.
If $e_b$ is also real and positive ($B$ is transparent), then $|k_2|$ is real.

$$k_{0x} = k_{2x} \Rightarrow |k_0| \sin \theta_0 = |k_2| \sin \theta_2$$

$$k_2^2 = \frac{\omega^2}{c^2} \sqrt{\mu_b \varepsilon_b}$$

$$\Rightarrow \sqrt{\mu_0 \varepsilon_0} \sin \theta_0 = \sqrt{\mu_b \varepsilon_b} \sin \theta_2$$

In terms of index of refraction $M = \frac{k_0}{\omega} = \frac{\omega \sqrt{\mu \varepsilon}}{c} = \frac{c}{\omega}$

$$M = \sqrt{\mu \varepsilon}$$

$$\Rightarrow m_a \sin \theta_0 = m_b \sin \theta_2$$

Snell's Law: 
true for all types of waves, not just EM waves.

If $m_a > m_b$, then $\theta_2 > \theta_0$.

In this case, when $\theta_0$ is too large, we will have $\frac{m_a \sin \theta_0}{m_b} > 1$ and thus will be no solution for $\theta_2$.

$\Rightarrow$ no transmitted wave

This is "total internal reflection" - wave does not exit medium A. The critical angle, above which one has total internal reflection, is given by

$$\frac{m_a}{m_b} \sin \theta_c = 1 \Rightarrow \theta_c = \arcsin \left( \frac{m_b}{m_a} \right)$$
Since \( n = \sqrt{\mu \varepsilon} \) and \( \varepsilon \) grows with density of the material, one usually has total internal reflection when one goes from a denser to a less dense medium.

Examples: diamonds sparkle due to total internal reflection. Diamonds have large \( n \) \( \Rightarrow \) small \( \theta_c \) \( \Rightarrow \) light bounces around inside many times before it can exit.

Can also see total internal reflection when swimming under water.

More general case. \( \sqrt{\mu \varepsilon} \) is complex so \( \bar{k}_2 \) is complex.

\[
\bar{k}_2 = \bar{k}'_2 + i \bar{k}''_2
\]

\( k'_2 = |\bar{k}'_2| \)

\( k''_2 = |\bar{k}''_2| \)

Real part: imaginary part.

Note \( \bar{k}'_2 \) and \( \bar{k}''_2 \) need not be in the same direction!

Condition \( k_0 x = \bar{k}_2 x \) \( \Rightarrow \) \( k_0 x = k'_2 x \) \( \Rightarrow \) equate real and imaginary parts.

\[
k_0 \sin \theta_0 = k'_2 \sin \theta'_2
\]

\[
\theta = k''_2 \sin \theta''_2
\]
\[ \Theta_2'' = 0 \]
\[ \frac{1}{k_2''} = k_2'' \] attenuation factor for the transmitted wave is \( e^{-k_2'' y} \)

planes of constant amplitude are parallel to the interface no matter what the angle of incidence \( \Theta_0 \)

\[ k_0 \sin \Theta_0 = k_2' \sin \Theta_2' \]

\[ k_0 = \frac{\omega}{c} \sqrt{\mu_0 \varepsilon_0} = \frac{\omega}{c} n_0 \]

planes of constant phase are \( \perp \) to \( k_2' \)

dispersion relation

\[ \frac{1}{k_2'} = \frac{1}{k_2} \cdot \frac{1}{k_2'} = (k_2')^2 - (k_2'')^2 + 2i \frac{k_2'}{k_2} \cdot k_2'' = \frac{\omega^2}{c^2} \mu_0 \varepsilon_0 \]

\[ \frac{k_2'}{k_2''} = \frac{k_2' k_2''}{k_2} \cos \Theta_2' \]

equate real and imaginary parts

\[ (k_2')^2 - (k_2'')^2 = \frac{\omega^2}{c^2} \mu_0 \varepsilon_0 \]

\[ \varepsilon_0 = \varepsilon_{01} + i \varepsilon_{02} \]

\[ 2k_2' k_2'' \cos \Theta_2' = \frac{\omega^2}{c^2} \mu_0 \varepsilon_{02} \]

solve

\[ (k_2')^2 - (k_2'')^2 + \frac{\omega^2}{c^2} \mu_0 \varepsilon_{01} \]

\[ (k_2')^2 = \left( \frac{\omega^2 \mu_0 \varepsilon_{02}}{2k_2' \cos \Theta_2'} \right)^2 + \frac{\omega^2}{c^2} \mu_0 \varepsilon_{01} \]
\[
(k'_2)^4 - \frac{\omega^2}{c^2} \mu_b E_{b1} (k'_2)^2 - \frac{\omega^4}{c^4} \frac{\mu_b^2 E_{b2}^2}{4 \cos^2 \theta'_2} = 0
\]

**Quadratic Formula**

\[
(k'_2)^2 = \frac{\omega^2 \mu_b E_{b1}}{c^2} + \sqrt{\frac{\omega^4 \mu_b^2 E_{b1}^2}{c^4} + \frac{\omega^4 \mu_b^2 E_{b2}^2}{c^4} \frac{1}{4 \cos^2 \theta'_2}}
\]

\[
k'_2 = \frac{\omega}{c} \sqrt{\mu_b} \left[ \frac{1}{2} E_{b1} + \frac{1}{2} \sqrt{E_{b1}^2 + \frac{E_{b2}^2}{\cos^2 \theta'_2}} \right]^{1/2}
\]

And

\[
(k''_2)^2 = (k'_2)^2 - \frac{\omega^2}{c^2} \mu_b E_{b1}
\]

\[
k''_2 = \frac{\omega}{c} \sqrt{\mu_b} \left[ \frac{1}{2} E_{b1} + \frac{1}{2} \sqrt{E_{b1}^2 + \frac{E_{b2}^2}{\cos^2 \theta'_2}} \right]^{1/2}
\]

Note, these reduce to what we had earlier for a plane wave, if we take \( \theta'_2 = 0 \)

Both \( k'_2 \) and \( k''_2 \) depend on angle of refraction \( \theta'_2 \)

Finally,

\[
k'_2 \sin \theta'_2 = \frac{\omega}{c} \mu_a \sin \theta_0
\]

\[
\Rightarrow \mu_a \sin \theta_0 = \sqrt{\mu_b E_{b1}} \left[ \frac{1}{2} + \frac{1}{2} \sqrt{1 + \frac{E_{b2}^2}{E_{b1}^2 \cos^2 \theta'_2}} \right]^{1/2} \sin \theta'_2
\]

Determine \( \theta'_2 \) in terms of given \( \theta_0 \)

*Cases*

1. For a nearly transparent material with \( E_{b2} \ll E_{b1} \)

   \[ \mu_b = \sqrt{\mu_b E_{b1}} \] index of refraction
\[ m_a \sin \theta_0 = m_b \sin \theta'_2 \left[ 1 + \frac{E_{b2}^2}{4 E_{bi}^2 \cos^2 \theta'_2} \right]^{1/2} \]

\[ \sim m_b \sin \theta'_2 \left[ 1 + \frac{E_{b2}^2}{8 E_{bi}^2 \cos^2 \theta'_2} \right] \]

Small correction to Snell's law

for \( \frac{E_{b2}}{E_{bi}} \ll 1 \), can solve iteratively

to lowest order:
\[ m_a \sin \theta_0 \approx m_b \sin \theta'_2 \]
\[ \Rightarrow \cos^2 \theta'_2 = 1 - \sin^2 \theta'_2 = 1 - \left( \frac{m_a \sin \theta_0}{m_b} \right)^2 \]

so to next order

\[ m_a \sin \theta_0 \approx m_b \sin \theta'_2 \left[ 1 + \frac{E_{b2}^2}{8 E_{bi}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_0 \right)} \right] \]

or \( \sin \theta'_2 \approx \frac{m_a \sin \theta_0}{m_b} \left[ 1 + \frac{E_{b2}^2}{8 E_{bi}^2 \left( 1 - \frac{m_a^2}{m_b^2} \sin^2 \theta_0 \right)} \right] \)

\[ \leq \frac{m_a}{m_b} \sin \theta_0 \]

result is that \( \theta'_2 \) is smaller than Snell's law would predict.
for a good conductor, or absorbing region of a dielectric \( \varepsilon_{b2} \gg \varepsilon_{b1} \)

to lowest order

\[
\sin \theta_0 = \sqrt{\frac{M_b \varepsilon_{b1}}{2 \left( \frac{\varepsilon_{b2}}{\varepsilon_{b1} \cos \theta_2'} \right)^{1/2} \sin \theta_2'}}
\]

\[
\sin \theta_0 = \sqrt{\frac{M_b \varepsilon_{b2}}{2}} \frac{\sin \theta_2'}{\sqrt{\cos \theta_2'}} \quad \text{very different from Snell's Law!}
\]

Snell's Law only holds if both media are transparent.