Electromagnetism

Clearly $\mathbf{E}$ and $\mathbf{B}$ must transform into each other under Lorentz transform.

In material frame K

Stationary line charge $\mathcal{L}$

\[ E \perp \mathcal{L} \]

Cylindrical outward electric field

No B-field

In frame $K'$ moving with $\mathbf{v} \parallel \mathbf{L}$, to write

\[ \mathbf{B} \]

Moving line charge gives current $\Rightarrow \mathbf{B}$ circulating around wire as well as outward radial $E$

Lorentz Force

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]

What is the velocity $\mathbf{v}$ here? Velocity with respect to what material frame? Clearly $\mathbf{E}$ and $\mathbf{B}$ must change from material frame to another if this force law can make sense.

Charge density

Consider charge $\Delta q$ contained in a vol $\Delta V$.

$\Delta q$ is a Lorentz invariant scalar.

Consider the reference frame in which the charge is instantaneously at rest. In this frame
\[ \Delta Q = \hat{\sigma} \Delta V \]
\[ \hat{\sigma} \text{ is charge density in the rest frame} \]
\[ \Delta V \text{ is volume in the rest frame} \]

\hat{\sigma} \text{ is Lorentz invariant by definition} \]

Now, transform to another frame moving with \( \hat{\gamma} \) with respect to the rest frame.

\[ \Delta Q \text{ remains the same} \]
\[ \Delta V = \frac{\Delta V}{\gamma} \]

volume contracts in direction \( \hat{\gamma} \) to \( \hat{\gamma} \)

\[ \hat{\sigma} = \frac{\Delta Q}{\Delta V} = \frac{\Delta Q}{\Delta V} \gamma = \hat{\sigma} \hat{\gamma} \]

Current density is \[ \hat{j} = \hat{\sigma} \hat{\gamma} = \gamma \hat{\sigma} \cdot \frac{\hat{\gamma}}{\gamma} = \hat{\sigma} \hat{\gamma} \]

Define 4-current \[ \hat{j}_{\mu} = (\hat{j}, \hat{\sigma} \cdot \hat{\gamma}) = \hat{\sigma}(\hat{\gamma}, i \hat{\gamma}) = \hat{\sigma} U_{\mu} \]

\( U_{\mu} \) is a 4-vector since \( U_{\mu} \) is 4-vector and \( \hat{\sigma} \) is Lorentz invariant scalar.

Charge conservation

\[ \nabla \cdot \hat{j} + \frac{\partial \hat{\sigma}}{\partial t} = \frac{\partial \hat{j}_{\mu}}{\partial x_{\mu}} = 0 \]
Equation for potentials in Lorentz gauge

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \vec{A} = \frac{-4\pi}{c} \vec{J}
\]

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = -4\pi \rho
\]

\[
\frac{\partial^2}{\partial x_\mu^2} \left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \phi = 0
\]

Lorentz gauge condition is

\[
\nabla \cdot \vec{A} + \frac{\partial \phi}{\partial t} = \frac{\partial A_\mu}{\partial x_\mu} = 0
\]

Electric and magnetic fields

\[
B_i = \frac{\partial A_k - \partial A_j}{\partial x_j} \quad \varepsilon_{ijk} \text{ cyclic permutation of } 1, 2, 3
\]

\[
E_i = -\frac{\partial \phi}{\partial x_i} - \frac{\partial A_i}{\partial t} = i \left( \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \right)
\]

Define field stress tensor

\[
F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} = -F_{\nu\mu}
\]

\[
\begin{pmatrix}
0 & B_3 & -B_2 & -iE_1 \\
-B_3 & 0 & B_1 & -iE_2 \\
B_2 & -B_1 & 0 & -iE_3
\end{pmatrix}
\]

"Curl" of a 4-ve is a 4x4 anti-symmetric 2nd rank tensor.
Inhomogeneous Maxwell's equations can be written in the form:

\[
\frac{\partial F_{\mu\nu}}{\partial x^\mu} = 4\pi \frac{\partial F_{\mu\nu}}{\partial x^\mu} = \begin{bmatrix}
\nabla \cdot E = 4\pi \rho \\
\nabla \times B = \mu_0 \frac{\partial E}{\partial t} = \frac{4\pi}{c} \frac{\partial B}{\partial t}
\end{bmatrix}
\]

\[
\frac{\partial}{\partial x^\nu} \left( \frac{\partial A_{\nu}}{\partial x^\mu} - \frac{\partial A_{\mu}}{\partial x^\nu} \right) = \frac{\partial}{\partial x^\mu} \left( \frac{\partial A_{\nu}}{\partial x^\nu} \right) - \frac{\partial^2 A_{\mu}}{\partial x^\nu \partial x^\nu}
\]

\[
\Rightarrow - \frac{\partial^2 A_{\mu}}{\partial x^\nu \partial x^\nu} = 4\pi \frac{\partial F_{\mu\nu}}{\partial x^\nu}
\]

This agrees with previous equation for \( A_{\mu} \).

Transformation law for 2nd rank tensor \( F_{\mu\nu} \):

\[
F'_{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\mu} \frac{\partial x'^\nu}{\partial x^\nu} F_{\mu\nu}
\]

Use \( A'_{\mu} = \alpha_{\mu\sigma} A_{\sigma} \)

\[
\Rightarrow \frac{\partial A'_{\mu}}{\partial x^\mu} = \alpha_{\mu\nu} \frac{\partial A_{\nu}}{\partial x^\nu}
\]

\[
F_{\mu\nu} = \alpha_{\mu\sigma} a_{\nu\lambda} F_{\sigma\lambda} \left( \text{if one knows } \vec{E} \text{ and } \vec{B} \right)
\]

\[
T_{\mu_1 \mu_2 ... \mu_n} = \alpha_{\mu_1 \nu_1} a_{\mu_2 \nu_2} ... a_{\mu_n \nu_n} T_{\nu_1 \nu_2 ... \nu_n}
\]
\[ \frac{\partial F_{\mu}}{\partial x^\nu} = a_{\mu\nu} \]

\[ a_{\nu\lambda} = a_{\lambda\nu} \text{ since inverse = transpose} \]

\[ a_{\nu\lambda} a_{\lambda\nu} = a_{\nu\nu} a_{\nu\nu} = \delta_{\nu\nu} \]

\[ \frac{\partial F_{\mu}}{\partial x^\nu} = a_{\mu\nu} \frac{\partial F_{\nu}}{\partial x^\lambda} \delta_{\lambda\nu} = a_{\mu\nu} \frac{\partial F_{\nu}}{\partial x^\lambda} \text{ transforms like a vector} \]

To write the homogeneous Maxwell Equations:

Construct 3rd rank co-variant tensor

\[ G_{\mu\nu\lambda} = \frac{\partial F_{\mu}}{\partial x^\lambda} + \frac{\partial F_{\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda}}{\partial x^\lambda} \]

transforms as \[ G'_{\mu\nu\lambda} = a_{\mu\rho} a_{\nu\lambda} a_{\lambda\sigma} G_{\rho\sigma} \]

in principle \[ G \] has \[ 4^3 = 64 \] components.

But can show that \[ G \] is antisymmetric in exchange of any two indices

\[ G_{\mu\nu\lambda} = \frac{\partial F_{\mu}}{\partial x^\lambda} + \frac{\partial F_{\nu}}{\partial x^\lambda} + \frac{\partial F_{\lambda}}{\partial x^\lambda} = - \frac{\partial F_{\mu}}{\partial x^\lambda} - \frac{\partial F_{\lambda}}{\partial x^\lambda} - \frac{\partial F_{\nu}}{\partial x^\lambda} \text{ as } F \text{ anti-symmetric} \]

\[ = - G_{\mu\nu\lambda} \]
Also $G_{\mu \nu \lambda} = 0$ if any two indices are equal

⇒ only 4 independent components

$G_{012}, G_{013}, G_{023}, G_{123}$

all other components either vanish or are ± one of the above.

The 4 homogeneous Maxwell Equations:

$\nabla \cdot \mathbf{E} = 0$, $\nabla \times \mathbf{E} + \frac{1}{c^2} \frac{\partial \mathbf{B}}{\partial t} = 0$

can be written as

$G_{\mu \nu \lambda} = 0$

to see, substitute in definition of $G$ the definition of $F$

$G_{\mu \nu \lambda} = \frac{\partial^2 A_\mu}{\partial x_\nu \partial x_\lambda} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda} + \frac{\partial^2 A_\mu}{\partial x_\lambda \partial x_\nu} - \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu} - \frac{\partial^2 A_\nu}{\partial x_\mu \partial x_\lambda} + \frac{\partial^2 A_\lambda}{\partial x_\mu \partial x_\nu}$

all terms cancel in pairs

$= 0$

$G_{123} = 0 \Rightarrow \nabla \cdot \mathbf{B} = 0$

$G_{012} = -i \left[ \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right]_3 = 0$ 3 component Faraday's law
Another way to write homogeneous Maxwell Equation's

Define \( \epsilon_{\mu \nu \lambda \sigma} = \begin{cases} +1 & \text{if } \mu \nu \lambda \sigma \text{ is even permutation} \\ -1 & \text{if } \mu \nu \lambda \sigma \text{ is odd permutation} \\ 0 & \text{otherwise} \end{cases} \)

4-d Levi-Civita symbol

Define \( \tilde{F}_{\mu \nu} = \frac{i}{2} \epsilon_{\mu \nu \lambda \sigma} F_{\lambda \sigma} \) \[ \text{pseudo-tensor has wrong sign under pauly } \]

\[ \begin{pmatrix} 0 & -E_3 & E_2 & -iB_1 \\ -E_3 & 0 & -E_1 & -iB_2 \\ -E_2 & E_1 & 0 & -iB_3 \\ iB_1 & iB_2 & iB_3 & 0 \end{pmatrix} \]

\( \partial \tilde{F}_{\mu \nu} = 0 \) gives homogeneous Maxwell equations

\[ \frac{1}{2} F_{\mu \nu} F_{\mu \nu} = B^2 - E^2 \] \[ \text{density invariant scalars} \]

\[ -\frac{1}{4} F_{\mu \nu} \tilde{F}_{\mu \nu} = B \cdot E \]
From $F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ we can set Lorentz transform for $E$ and $B$.

For a transformation from $K$ to $K'$ with $K'$ moving with $\nu$ along $x_1$ with respect to $K$, we have:

\[
E'_1 = E_1, \quad B'_1 = B_1, \\
E'_2 = \gamma (E_2 - \frac{\nu}{c} B_3), \quad B'_2 = \gamma (B_2 + \frac{\nu}{c} E_3), \\
E'_3 = \gamma (E_3 + \frac{\nu}{c} B_2), \quad B'_3 = \gamma (B_3 - \frac{\nu}{c} E_2).
\]

**Kinematics**

\[\dot{x} = \frac{d}{ds}\]

**4-momentum**

\[p_\mu = m \dot{x}_\mu = m u_\mu = (m \gamma \dot{\nu}, \gamma \dot{\nu} c, \gamma m c)\]

\[p_\mu^2 = m^2 u_\mu^2 = -m^2 c^2\]

**4-force**

\[K_\mu = (\gamma^2, i K_0) \quad \text{"Minkowski force"}\]

**Newton's 2nd law**

\[m \frac{d^2 x_\mu}{ds^2} = K_\mu\]

\[\Rightarrow m \frac{d u_\mu}{ds} = \frac{d p_\mu}{ds} = K_\mu\]

\[p_\mu^2 = -m^2 c^2 \Rightarrow \frac{d}{ds} (p_\mu^2) = p_\mu \frac{d p_\mu}{ds} = p_\mu K_\mu = 0\]

\[\Rightarrow m \gamma \dot{\nu} \cdot \vec{K} - m c \gamma K_0 = 0 \quad \text{or} \quad K_0 = \frac{\nu}{c} \cdot \vec{K}\]
Define the usual 3-force by

\[ \frac{d\vec{p}}{dt} = \vec{F} \]

\[ \frac{d\vec{p}}{ds} = \vec{K} \quad \text{and} \quad \frac{d\vec{p}}{ds} = \hat{e} \frac{d\vec{p}}{dt} = \hat{e} \vec{F} \quad \Rightarrow \quad \vec{K} = \hat{e} \vec{F} \]

\[ K_0 = \hat{e} \frac{\vec{u}}{c} \cdot \vec{F} \]

Consider 4th component of Newton's eqn

\[ m \frac{d\vec{u}_4}{ds} = m \frac{d(\hat{e} \vec{u}_4)}{ds} = \hat{e} \vec{K}_0 = \hat{e} \frac{\vec{u}}{c} \cdot \vec{F} \]

\[ d(m \vec{r}) = \frac{\gamma \vec{u}}{c^2} \cdot \vec{F} \frac{ds}{c^2} = \frac{dt}{c^2} \vec{v} - \frac{\vec{F}}{c^2} \frac{dt}{c^2} \]

Work-energy theorem: \( d(m \gamma c^2) = d\vec{p} \cdot \vec{F} = \text{work done} \)

\[ \Rightarrow d(m \gamma c^2) \] is change in \( \text{kinetic energy} \)

\[ E = m \gamma c^2 \] is \( \text{relativistic energy} \)

\[ \begin{aligned} p^0 &= (\vec{p}, \frac{E}{c}) \\ \vec{p} &= m \gamma \vec{v} \\ E &= m \gamma c^2 \end{aligned} \]

\[ E = \frac{mc^2}{\sqrt{1 - \frac{v^2}{c^2}}} \]

\[ \begin{aligned} \text{small} \frac{v}{c} \quad \text{rest mass} \quad \text{kinetic energy} \quad \text{energy} \end{aligned} \]

\[ \frac{dp^0}{ds} = K_0 \quad \text{Therefore} \]

relativistic analog of Newton's 3rd law as well as law of conservation of energy
Lorentz force

\[ \frac{d\nu}{ds} = K \nu \]

What is the \( K \) that represents the Lorentz force and how can we write it in a covariant way?

\( K \) should depend on the fields \( F_{\mu\nu} \) and the particle's trajectory \( x_\mu \)

as \( v \to 0 \), \( K = \frac{q}{\mu} E \)

\( K \) can't depend directly on \( x_\mu \) as should be independent of origin of coordinates. So can depend only on \( x_\mu, x_\nu, \) etc.

as \( v \to 0 \), \( K \) does not depend on the acceleration, so \( K \) does not depend on \( x_\mu \).

\( K \) only depends on \( F_{\mu\nu} \) and \( x_\mu \).

we need to form a \( \eta \)-vector out of \( F_{\mu\nu} \) and \( x_\mu \) that is linear in the fields \( F_{\mu\nu} \) and proportional to the charge \( q \).

The only possibility is

\[ q \cdot f (x_\mu^2) F_{\mu\nu} x_\nu \]
But $\xi_{\mu} = \xi^2$ is a constant. Choose $f(\xi_{\mu}) = \frac{1}{\xi}$

$$K_{\mu} = \frac{\xi}{c} F_{\mu \nu} \xi_{\nu}$$ is only possibility

The gives force

$$\vec{F} = \frac{1}{8} \vec{K}$$

$$F_i = \frac{1}{8} K_i = \frac{\xi}{\gamma c} (F_{ij} \xi_j + F_{i4} \xi_4)$$

$$= \frac{\xi}{\gamma c} \left( \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \right) \xi_j + \frac{\xi}{\gamma c} \left( -i E_i \right) \left( i c \gamma \right)$$

$$= \frac{\xi}{\gamma c} \left[ E_{ijk} B_k \gamma \xi_j \right] + \frac{\xi}{\gamma c} E_i \gamma \xi$$

$$= \frac{\xi}{\gamma c} E_i + \frac{\xi}{\gamma c} E_{ijk} \frac{\xi_j}{c} B_k$$

$$\vec{F} = \frac{\xi}{\gamma c} \vec{E} + \frac{\xi}{\gamma c} \frac{\xi_j}{c} \vec{B}$$

Lorentz force is the same form in all inertial frames.

No relativistic modification is needed.
Relativistic Larmor's formula

\[ P = \frac{2}{3} \frac{e [a(t_0)]^2}{c^3} \]

Consider motion frame in which charge is instantaneously at rest. Call this rest frame \( K \).

Power radiated in \( K \) is

\[ P = \frac{d \mathbf{E}(t)}{dt} \]

where \( \mathbf{E} \) is energy radiated. In \( K \), the momentum density

\[ \frac{\partial}{\partial t} = \frac{1}{\gamma mc} \mathbf{E} \times \mathbf{B} \sim \mathbf{E} \]

is in outward radial direction. Integrating over all directions, the radiated momentum vanishes

\[ \mathbf{P} = 0 \]

Energy-momentum is a 4-vector \((\mathbf{P}, E)\).

To get radiated energy in original frame \( K \), we can use Lorentz transform

\[ \frac{\mathbf{E}}{c} = \gamma \left( \frac{\mathbf{E}}{c} - \frac{\mathbf{v}}{c}, \mathbf{E} \right) \Rightarrow E = \gamma E \quad \text{as} \quad \mathbf{P} = 0 \]

\[ \frac{d \mathbf{x}}{dt} = \gamma d \mathbf{x} \quad \text{in time interval in} \ K \]

\( (d\mathbf{P} = 0 \quad \text{as charge stays at origin in} \ K) \)

\[ \gamma \frac{d \mathbf{E}}{dt} = \frac{d \mathbf{E}^E}{dt} = \frac{d \mathbf{E}^E}{dt} \Rightarrow P = \mathbf{P} \]

Radiated power \( \text{in Lorentz} \) invariant!
In $\mathbf{K}$ we can use non-relativistic Larmor's formula since $v = 0$. So
\[
P = \frac{2}{3} \frac{q A^2}{c^3} \quad \text{a } \text{acceleration in } \mathbf{K}
\]

To write an expression without explicitly making mention of frame $\mathbf{K}$, we need to find a Lorentz invariant scalar that reduces to $a^2$ as $v \to 0$.

The only choice is $\alpha^2$ the 4-acceleration $\alpha^2 = \frac{d\alpha}{ds}$

\[
\alpha^2 = \frac{d\nu}{dt} = \gamma \frac{d\nu}{dt} = \gamma \frac{d}{dt} \left( \gamma \frac{d\nu}{dt} \right)
\]

\[
\alpha^2 = \gamma^2 \frac{d\nu}{dt} + \gamma \frac{d\nu}{dt} \frac{d\gamma}{dt}
\]

\[
\alpha^4 = \gamma \frac{d\gamma}{dt}
\]

\[
\frac{d\gamma}{dt} = \frac{\nu - \frac{d\nu}{dt}}{\nu (1 - \frac{v^2}{c^2})^{3/2}} = \frac{\gamma^2}{c^2} \frac{d\nu}{dt} = \frac{1}{c^2} \gamma^3 \frac{d\nu}{dt}
\]

As $v \to 0$, $\gamma \to 1$, $\frac{d\gamma}{dt} \to 0$ so
\[
\alpha^2 \to \frac{d\nu}{dt} \quad \alpha^4 \to 0
\]

\[
\alpha^2 \to |A|^2 \quad \text{as desired}
\]

Relativistic Larmor's formula
\[
P = \frac{2}{3} \frac{q \nu^2}{c^3} \quad \text{or } \quad P = \frac{2}{3} \frac{q \nu}{c^3} \left( \frac{d\nu}{dt} \right)^2
\]
\[ \alpha'_\mu = \left( \gamma^2 \frac{d\gamma}{dt} + \gamma \nu \frac{d\nu}{dt} \right) + c \gamma \frac{dy}{dt} \]

\[ \frac{d\gamma}{dt} = \frac{1}{c^2} \gamma^3 \nu \cdot \tilde{a} \]

\[ \alpha'_\mu = \left( \gamma^2 \frac{d\gamma}{dt} + \gamma \nu \frac{d\nu}{dt} \right) + c \gamma \frac{dy}{dt} \]

\[ \frac{\alpha'^2}{c^2} = \gamma^4 a^2 + \gamma^8 \left( \nu \cdot \tilde{a} \right)^2 \nu^2 + 2 \gamma^6 \left( \nu \cdot \tilde{a} \right)^2 \nu^2 \]

\[ \frac{\alpha'^2}{c^2} = \gamma^4 \left[ \nu^2 + \gamma^4 \left( \nu \cdot \tilde{a} \right)^2 \left( \frac{\nu^2}{c^2} - 1 \right) + 2 \gamma \left( \nu^2 - \tilde{a}^2 \right) \right] \]

\[ \frac{\alpha'^2}{c^2} = \gamma^4 \left[ \nu^2 - \gamma^2 \left( \nu \cdot \tilde{a} \right)^2 + 2 \gamma^2 \left( \nu \cdot \tilde{a} \right)^2 \right] \]

\[ \frac{\alpha'^2}{c^2} = \gamma^4 \left[ \nu^2 + \gamma^2 \left( \nu \cdot \tilde{a} \right)^2 \right] \]

As \( \nu \to 0 \), \( \alpha'^2 \to a^2 \)

\[ \alpha'^2 = \nu^2 \text{ Lorentz invariant} \]

\( \tilde{a} = \text{acceleration in instantaneous rest} \)

For a charge accelerating, frame

in linear motion, \( \left( \nu \cdot \tilde{a} \right)^2 = \nu^2 a^2 \)

\[ \alpha'^2 = \gamma^4 a^2 \left( 1 + \gamma^2 \nu^2 \right) = \gamma^6 a^2 \]

\[ P = \frac{2}{3} \frac{a^2}{c^3} \gamma^6 \]

For a charge in circular motion, \( \left( \nu \cdot \tilde{a} \right) = 0 \)

\[ \alpha'^2 = \gamma^4 a^2 \]

\[ P = \frac{2}{3} \frac{a^2}{c^3} \gamma^4 \]