Green's Functions - part I

\[- \nabla^2 \phi = 4\pi \rho \]

We already know that for a point charge \( q \) at position \( \vec{r} \)
\[ \rho(\vec{r}) = q \delta(\vec{r} - \vec{r}') \] the solution to the above is
\[ \phi(\vec{r}) = \frac{q}{4\pi |\vec{r} - \vec{r}'|} \]
\[ \implies - \nabla^2 \left( \frac{1}{4\pi |\vec{r} - \vec{r}'|} \right) = 4\pi \delta(\vec{r} - \vec{r}') \]

We call the special solution for a point source the Green's function for the differential operator
\[- \nabla^2 G(\vec{r}, \vec{r}') = - 4\pi \delta(\vec{r} - \vec{r}') \]

\( G(\vec{r}, \vec{r}') \) gives the potential at position \( \vec{r} \) due to a unit source at position \( \vec{r}' \).

Generally, one also has to specify a desired boundary condition for the Green function on the boundary of the system.

For the Coulomb solution for a point charge the simplest boundary condition is that the potential vanish infinitely far from the charge.

\[ G(\vec{r}, \vec{r}') \to 0 \quad \text{as} \quad |\vec{r} - \vec{r}'| \to \infty \]

boundary of the system is taken to infinity
If one knows the Green's function, then one can find the solution for any distribution of sources $\rho (\vec{r})$

$$\phi (\vec{r}) = \int d^3r' G(\vec{r}, \vec{r}') \rho (\vec{r}')$$

**Proof:**

$$- \nabla^2 \phi = \int d^3r' \left[ - \nabla^2 G(\vec{r}, \vec{r}') \right] \rho (\vec{r}')$$

$$= \int d^3r' \left[ 4\pi \delta (\vec{r} - \vec{r}') \right] \rho (\vec{r}')$$

$$= 4\pi \rho (\vec{r})$$

We will return to concept of Green's function when we discuss solution of Poisson's equation in a finite volume.

We will also see Green's functions again when we discuss solution of the inhomogeneous wave equation.
The Coulomb problem as a boundary value problem

Consider a conducting sphere of radius $R$ with net charge $Q$ (as $R \to 0$ we get a point charge).

What is $\phi(\vec{r})$? What is $\vec{E}(\vec{r})$?

Review: Properties of conductors in electrostatics

1) $\vec{E}=0$ inside conductor - if $\vec{E} \neq 0$ then a current $\vec{j} = \sigma \vec{E}$ flows and it is not static (\(\sigma\) is conductivity).

2) $j = 0$ inside conductor - if $\vec{E} = 0$ inside, then $\nabla \cdot \vec{E} = \nabla \cdot j = 0$.

3) Any net charge on the conductor must lie on the surface - follows from (2).

4) $\phi$ is constant throughout conductor - if $\vec{E} = 0$ then $\nabla \phi = -\vec{E}$.

5) Just outside the conductor, $\vec{E}$ is $\perp$ to surface.
   - If $\vec{E}$ has a component $\parallel$ to surface then it exerts a force on electrons at the surface leading to a surface current - so would not be static.

For conducting sphere, $j = 0$ for $r > R$ and $r < R$

all charge is on the surface $\Rightarrow \nabla^2 \phi = 0$ for $\{r \geq R \cap r < R\}$

Spherical symmetry $\Rightarrow$ expect spherically symmetric solution

$\Rightarrow \phi(\vec{r})$ depends only on $r = |\vec{r}|$
Solve Laplace's equation by writing \( \nabla^2 \phi \) in spherical coordinates. Only the radial terms do not vanish.

\[
\nabla^2 \phi = \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 0
\]

\[
\frac{r^2 d\phi}{dr} = -C_0 \quad \text{a constant}
\]

\[
\frac{d\phi}{dr} = -\frac{C_0}{r^2}
\]

\[
\phi(r) = \frac{C_0}{r} + C_1, \quad C_1 \text{ a constant}
\]

"outside" \( r > R \) \( \phi_{\text{out}}(r) = \frac{C_{0\text{out}}}{r} + C_{1\text{out}} \)

"inside" \( r < R \) \( \phi_{\text{in}}(r) = \frac{C_{0\text{in}}}{r} + C_{1\text{in}} \)

The solution "outside" does not necessarily go smoothly into the solution "inside" because of the charge layer at \( r = R \) that separates the two regions. We need to determine the constants \( C_{0\text{in}}, C_{0\text{out}}, C_{1\text{in}}, C_{1\text{out}} \) by applying boundary conditions corresponding to the physical situation.

1. For \( r > R \), assume \( \phi \to 0 \) as \( r \to \infty \) - boundary condition at infinity
   \[
   \Rightarrow C_{0\text{out}} = 0
   \]

\( \phi_{\text{out}}(r) = \frac{C_{0\text{out}}}{r} \) recover the expected Coulomb form.
2) For \( r < R \).

i) we could use the fact that the region \( r < R \) is a conductor with \( \phi = \text{constant} \) to conclude \( C_{\text{in}} = 0 \).

ii) or, if we were dealing with a charged shell instead of a conductor, we could argue as follows:

no charge at origin \( r=0 \) \( \Rightarrow \) expect \( \phi \) should be finite at origin \( \Rightarrow C_{\text{in}} = 0 \)

So \( \phi_{\text{in}}(r) = C_{\text{in}} \) a constant

3) Now we need boundary conditions at \( r = R \) where "inside" and "outside" meet.

\[ \text{Review: Electric field and potential at a surface charge density} \]

\[ \sigma(\vec{r}) \]

\[ \text{a general surface } S \text{ with surface charge density} \]

\[ \sigma(\vec{r}) \text{ for } \vec{r} \text{ on } S, \quad \sigma(\vec{r}) \text{d}a \text{ is total charge in area } \text{d}a \text{ on surface} \]

i) Take "Gaussian pillbox" surface about point \( \vec{r} \) on the surface \( S \)

\[ \text{top and bottom areas of pill box } \text{d}a \]

\[ \text{side view} \]

\[ \text{side of pillbox } \text{d}l \]

\[ \text{Gauss' Law in integral form} \quad \oint \text{d}a \hat{n} \cdot \vec{E} = 4\pi Q \text{ enclosed } S \]
expect $E$ is finite $\Rightarrow$ contribution from sides of pulldown vanish as $dl \to 0$.

\[
\int_S d\mathbf{a} \cdot \mathbf{E} = \int_{top} d\mathbf{a} \cdot \mathbf{E} + \int_{bottom} d\mathbf{a} \cdot \mathbf{E}
\]

\[
= (\mathbf{\hat{m}}_{top} \cdot \mathbf{E}_{top} + \mathbf{\hat{m}}_{bottom} \cdot \mathbf{E}_{bottom}) da \quad \text{since } da \text{ is small}
\]

$\mathbf{E}_{top}$ is electric field at $P$ just above the surface $S$.
$\mathbf{E}_{bottom}$ is electric field at $P$ just below the surface $S$.

$\mathbf{\hat{m}}_{top} \equiv \mathbf{\hat{m}}$ is outward normal on top.
$\mathbf{\hat{m}}_{bottom} = -\mathbf{\hat{m}}$ is outward normal on bottom.

\[
\Rightarrow (\mathbf{E}_{top} - \mathbf{E}_{bottom}) \cdot \mathbf{\hat{m}} = 4\pi \sigma \; \text{enclosed} = 4\pi \sigma(\mathbf{r}) \; da
\]

\[
(\mathbf{E}_{top} - \mathbf{E}_{bottom}) \cdot \mathbf{\hat{m}} = 4\pi \sigma(\mathbf{r}) \quad \text{discontinuity in normal component of } \mathbf{E}
\]

ii) Take "American loop" $C$ at surface about point $P$.

\[
\nabla \times \mathbf{E} = 0 \Rightarrow \oint_C d\mathbf{r} \cdot \mathbf{E} \quad \text{since } \mathbf{E} \text{ is finite at surface,}
\]

\[
\text{if take sides } dl' \to 0 \text{ then their contribution to integral vanishes}
\]

\[
\Rightarrow \oint_C d\mathbf{l} \cdot \mathbf{E} = (\mathbf{E}_{top} - \mathbf{E}_{bottom}) \cdot d\mathbf{l} = 0
\]

where $d\mathbf{l}$ is any infinitesimal tangent to the surface at $P$. 
\[ \text{tangential component of } \vec{E} \text{ is continuous} \]

Combine above to write \[ \vec{E} \text{ top} - \vec{E} \text{ bottom} = 4\pi \sigma (\hat{N}) \hat{N} \]

\( \text{iii) } \vec{E} = -\nabla \phi \Rightarrow \phi (r_2) - \phi (r_1) = -\int_{r_1}^{r_2} \nabla \cdot \vec{E} \]

Take \( r_2 \) just above \( r \) on surface \[ \int_{r} \vec{d}l \geq 0 \]

Since \( \vec{E} \) is finite \[ \int_{r} \vec{d}l \cdot \vec{E} \to 0 \]

\[ \Rightarrow \phi \text{ top } = \phi \text{ bottom} \]

Potential \( \phi \) is continuous at surface charge layer

Can rewrite (i) as

\[ \left( -\nabla \phi \text{ top } + \nabla \phi \text{ bottom } \right) \cdot \hat{N} = 4\pi \sigma \]

\[ \frac{-\partial \phi \text{ top}}{\partial m} + \frac{\partial \phi \text{ bottom}}{\partial m} = 4\pi \sigma \]

1 directional derivative of \( \phi \) in direction \( \hat{N} \)

Discontinuity in normal derivative of \( \phi \) at surface

Applying to conducting sphere

\[ \phi \text{ continuous} \Rightarrow \phi \text{ in} (R) = \phi \text{ out} (R) \]

\[ C_1 = \frac{C_0}{R} \]

Only one unknown left
normal derivative of \( \phi \) if discontinuous

\[
- \frac{\partial \phi_{\text{top}}}{\partial m} + \frac{\partial \phi_{\text{bottom}}}{\partial m} = 4 \pi \sigma
\]

here \( \vec{r} \) is the radial direction

\[
\left[ - \frac{d\phi^{\text{out}}}{dr} + \frac{d\phi^{\text{in}}}{dr} \right] = 4 \pi \sigma
\]

but \( \frac{d\phi^{\text{in}}}{dr} = 0 \) as \( \phi^{\text{in}} = \text{constant} \)

\[
- \frac{d\phi^{\text{out}}}{dr} \bigg|_{r=R} = 4 \pi \sigma
\]

charge \( q \) is uniformly distributed on surface at \( R \)

\[
- \frac{d}{dr} \left( \frac{C^{\text{out}}}{r} \right)_{r=R} = \frac{C^{\text{out}}}{R^2} = 4 \pi \sigma = 4 \pi \left( \frac{q}{4 \pi R^2} \right) = \frac{q}{R^2}
\]

\[
\Rightarrow C^{\text{out}} = q, \quad C^{\text{in}} = \frac{C^{\text{out}}}{R} = \frac{q}{R}
\]

\[
\phi(r) = \begin{cases} \frac{q}{r} & r \leq R \text{ inside} \\ \frac{q}{R} & r > R \text{ outside} \end{cases}
\]

\[
\Rightarrow \bar{E} = -\vec{\nabla} \phi = - \frac{d\phi}{dr} = \begin{cases} 0 & r \leq R \text{ inside} \\ \frac{q}{r^2} & r > R \text{ outside} \end{cases}
\]

we get familiar Coulomb solution!
Summary: We can view the preceding solution for $\phi_{out}$ as solving Laplace's equation $\nabla^2 \phi = 0$
subject to a specified boundary condition on the normal derivative of $\phi$ at the boundary $r=R$
of the "outside" region of the system.

Alternate problem:

Another physical situation would be to connect a condenser sphere to a battery that charges the
sphere to a fixed voltage $\phi_0$ (stat volts!) with respect to ground $\phi=0$ at $r=\infty$.

As before, outside the sphere $\phi = \frac{C_0}{r}$.

Now the boundary condition is to specify the value of $\phi$ on the boundary of the outside
region, i.e.

$$\phi(R) = \phi_0$$

$$\Rightarrow \frac{C_0}{R} = \phi_0 \Rightarrow C_0 = \phi_0 R$$

$$\phi(r) = \phi_0 \frac{R}{r}$$

(from preceding solution, we know that charging the sphere to voltage $\phi_0$ (stat volts) induces a net
charge $q = \phi_0 R$ on it)
These two versions of the conducting sphere
problem are examples of a more general
boundary value problem.

Solve $\nabla^2 \phi = 0$ in a given region of space
subject to one of the following two types of
boundary conditions on the boundary surfaces of
the region:

i) Neumann boundary condition

$$\frac{\partial \phi}{\partial n}$$ - normal derivative of $\phi$ is specified
on the boundary surface

ii) Dirichlet boundary condition

$$\phi$$ - value of $\phi$ is specified on the
boundary surfaces

If the boundary surfaces consist of disjoint
pieces, it is possible to specify either (i) or
(ii) on each piece separately to get a
mixed boundary value problem.
Some more problems

Infinite conducting wire of radius $R$ with line charge density $\lambda = \text{charge per unit length}

\[ \sigma = \frac{\lambda}{2\pi R} \]

Expect cylindrical symmetry $\Rightarrow \phi$ depends only on cylindrical coord $r$.

\[ \nabla^2 \phi = 0 \quad \text{for } \quad r > R, \quad r < R \]

Use $\nabla^2$ in cylindrical coords—only radial term nonvanishing

\[ \nabla^2 \phi = \frac{1}{r} \frac{1}{\partial r} (r \partial \phi/\partial r) = 0 \]

\[ r \frac{d\phi}{dr} = C_0 \quad \text{constant} \]

\[ \frac{d\phi}{dr} = \frac{C_0}{r} \]

\[ \phi(r) = C_0 \ln r + C_1 \quad \text{const} \]

Note: one cannot now choose $\phi \to 0$ as $r \to \infty$!

One needs to fix zero of $\phi$ at some other radius, a convenient choice is $r = R$, but any other choice could also be made.
\[
\phi_{\text{out}} = C_0 \ln r + C_1 \text{ out}
\]
\[
\phi_{\text{in}} = C_0 \ln r + C_1 \text{ in}
\]

\[
\phi_{\text{in}} = \text{const in conductor } \Rightarrow C_{1,\text{in}} = 0
\]
-or- \[
\phi_{\text{in}} \text{ should not diverge as } r \to 0 \Rightarrow C_{1,\text{in}} = 0
\]

So \[
\phi_{\text{in}} = C_{1,\text{in}} \text{ constant}
\]

Boundary condition at \( r = R \)

\[
\left[ -\frac{d\phi_{\text{out}}}{dr} + \frac{d\phi_{\text{in}}}{dr} \right]_{r=R} = 4\pi \sigma
\]

\[
\Rightarrow -\frac{C_{0,\text{out}}}{R} = 4\pi \sigma = 4\pi \left( \frac{\lambda}{2\pi R} \right) = \frac{2\lambda}{R}
\]

\[
C_{0,\text{out}} = -2\lambda
\]

\[
\phi_{\text{out}}(r) = -2\lambda \ln r + C_{1,\text{out}}
\]

Continuity of \( \phi \)

\[
\phi_{\text{in}}(R) = \phi_{\text{out}}(R) \Rightarrow C_{1,\text{in}} = -2\lambda \ln R + C_{1,\text{out}}
\]

Remaining const \( C_{1,\text{out}} \) is not too important as it is just a common additive constant to both \( \phi_{\text{in}} \) and \( \phi_{\text{out}} \) \( \Rightarrow \) does not change \( \varepsilon = -\nabla \phi \).

If use the condition \( \phi(R) = 0 \) then we can solve for \( C_{1,\text{out}} \).
\[ 0 = -2\lambda \ln R + c_{1 \text{out}} \Rightarrow c_{1 \text{out}} = 2\lambda \ln R \]

\[ \Rightarrow \phi(r) = \begin{cases} 
-2\lambda \ln (r/R) & r > R \\
0 & r < R 
\end{cases} \]

\[ E(r) = \begin{cases} 
\frac{2\lambda}{r} & r > R \\
0 & r < R 
\end{cases} \]

\text{infinite conducting half space}

\text{uniform surface charge density}

expect \( \phi \) depends only on \( x \)

\[ \nabla^2 \phi = \frac{d^2 \phi}{dx^2} = 0 \]

\[ \Rightarrow \begin{cases} 
\phi^>(x) = c_0^x x + c_1^> & x > 0 \\
\phi^<(x) = c_0^x x + c_1^< & x < 0 
\end{cases} \]

for \( x < 0 \), \( \phi = \text{const \ at \ conductor} \Rightarrow c_1^< = 0 \)

at \( x = 0 \), \( \phi \) continuous \( \Rightarrow \phi^>(0) = \phi^<(0) \)

\[ c_1^> = c_1^< \]

\[ \frac{d\phi}{dx} \text{ discontinuous} \Rightarrow \]

\[ -\frac{d\phi^>}{dx} \bigg|_{x=0} = 4\pi \sigma^< \]

\[ c_0^> = -4\pi \sigma^< \]

\[ \Rightarrow \phi(x) = \begin{cases} 
-4\pi \sigma x + c_1^> & x > 0 \\
c_1^> & x < 0 
\end{cases} \]

const \( c_1^> \) does not change value of \( E \)
as for the wire, we cannot choose \( \phi \to 0 \) as \( x \to 0 \).

Then set \( \phi \to 0 \) not

\[
- \vec{\nabla} \phi = \vec{E} = \begin{cases} 4\pi \sigma \hat{x} & x > 0 \\ 0 & x < 0 \end{cases}
\]

\text{infinite charged plane}

similar to previous problem, but now no conductor
at \( x < 0 \), just free space on both sides of the
charged plane at \( x = 0 \).

\[
\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} = 0 \Rightarrow \phi^x = C^x_0 x + C^x_1 \quad x > 0
\]

\[
\phi^x = C^x_0 x + C^x_1 \quad x < 0
\]

continuity of \( \phi \) at \( x = 0 \)

\[
\Rightarrow \phi^x(0) = \phi^x(0) \Rightarrow C^x_1 = C^x_1
\]

discontinuity of \( \frac{d\phi}{dx} \) at \( x = 0 \)

\[
- \frac{d\phi^x}{dx} + \frac{d\phi^x}{dx} = 4\pi \sigma
\]

\[
-C^x_0 + C^x_0 = 4\pi \sigma
\]

Define \( \zeta_0 = \frac{C^x_0 + C^x_0}{2} \)

Then we can write
\[ \phi = \begin{cases} -2\pi\sigma x + \bar{c}_0 x + c^\prime_1 & x > 0 \\ 2\pi\sigma x + \bar{c}_0 x + c^\prime_1 & x < 0 \end{cases} \]

\[ \frac{d\phi}{dx} = \begin{cases} (2\pi\sigma - \bar{c}_0) \hat{x} & x > 0 \\ (-2\pi\sigma - \bar{c}_0) \hat{x} & x < 0 \end{cases} \]

Constant \( c^\prime_1 \) does not affect \( \vec{E} \) - additive constant to \( \phi \)

\( \bar{c}_0 \) represents constant uniform electric field \( -\bar{c}_0 \hat{x} \), that exists independently of the charged surface

If we assumed that all \( \vec{E} \) fields are just those arising from the plane, then we can set \( \bar{c}_0 = 0 \).

Equivalently, if the plane is the only source of \( \vec{E} \), then we expect \( \phi \) depends only on \( |x| \) by symmetry.

\[ \Rightarrow \quad c^\prime_0 = -c^\prime_1 \quad \text{and again} \quad \bar{c}_0 = 0. \]

In this case:

\[ \phi(x) = \begin{cases} -2\pi\sigma x & x > 0 \\ 2\pi\sigma x & x < 0 \end{cases} \]

\[ \vec{E}(x) = \begin{cases} 2\pi\sigma \hat{x} & x > 0 \\ -2\pi\sigma \hat{x} & x < 0 \end{cases} \]

\( \vec{E} \) is constant but oppositely directed on either side of the charged plane.