We want to show that the boundary value problem we described is well posed - i.e. there is a unique solution. We start by deriving Gauss' Theorems.

Consider \( \int \limits_{\mathcal{S}} d^3 \mathbf{r} \ \nabla \cdot \mathbf{A} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l} \) \quad \text{Gauss theorem}

Let \( \mathbf{A} = \phi \mathbf{\nabla} \psi \) \quad \phi, \psi \text{ any two scalar functions}

\[ \Rightarrow \quad \nabla \cdot \mathbf{A} = \phi \Delta^2 \psi + \nabla \phi \cdot \nabla \psi \]

\[ \phi \nabla \psi \cdot \hat{\mathbf{n}} = \phi \frac{\partial \psi}{\partial m} \]

\[ \Rightarrow \quad \int \limits_{\mathcal{V}} d^3 r \left( \phi \Delta^2 \psi + \nabla \phi \cdot \nabla \psi \right) = \frac{1}{2} \int \limits_{\mathcal{S}} d\mathbf{a} \phi \frac{\partial \psi}{\partial m} \] \quad \text{Green's 1st identity}

Let \( \phi \leftrightarrow \psi \)

\[ \int \limits_{\mathcal{V}} d^3 r \left( \psi \Delta^2 \phi + \nabla \phi \cdot \nabla \psi \right) = \frac{1}{2} \int \limits_{\mathcal{S}} d\mathbf{a} \psi \frac{\partial \phi}{\partial m} \]

Subtract

\[ \int \limits_{\mathcal{V}} d^3 r \left( \phi \Delta^2 \psi - \psi \Delta^2 \phi \right) = \frac{1}{2} \int \limits_{\mathcal{S}} d\mathbf{a} \left( \phi \frac{\partial \psi}{\partial m} - \psi \frac{\partial \phi}{\partial m} \right) \] \quad \text{Green's 2nd identity}

Apply Green's 2nd identity with \( \psi = \frac{1}{r - r'} \), \( r' \) in integration variable, \( \phi \) is the scalar potential, with \( \Delta^2 \phi = -4\pi \rho \). Use \( \Delta^2 \psi = \Delta^2 \phi = -4\pi \delta(r - r') \)

\[ \int \limits_{\mathcal{V}} d^3 r' \left[ \phi(r') \left(-4\pi \delta(r - r') \right) \right] \]

\[ = \frac{1}{2} \int \limits_{\mathcal{S}} d\mathbf{a'} \left[ \phi \frac{\partial}{\partial m'} \left( \frac{1}{|r - r'|} \right) - \frac{1}{|r - r'|} \frac{\partial \phi}{\partial m'} \right] \]
If \( \vec{r} \) lies within the volume \( V \), then

\[
(\star) \quad \phi(\vec{r}) = \frac{\int d^3r' \, \rho(\vec{r}')}{V} + \oint_{S} \, d\vec{a}' \left[ \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial m'} - \frac{\phi}{4\pi} \frac{\partial}{\partial m'} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \right]
\]

Note: if \( \vec{r} \) lies outside the volume \( V \), then

\[
0 = \frac{\int d^3r' \, \rho(\vec{r}')}{V} + \oint_{S} \, d\vec{a}' \left[ \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \phi}{\partial m'} - \frac{\phi}{4\pi} \frac{\partial}{\partial m'} \left( \frac{1}{|\vec{r} - \vec{r}'|} \right) \right]
\]

potential from a

potential from a

surface charge density

surface dipole layer of

\[ \sigma = \frac{1}{4\pi} \frac{\partial \phi}{\partial m'} \]

dipole strength density

\[ \Phi \]

From (\( \star \)), if \( S \to \infty \) and \( E \sim \frac{\partial \phi}{\partial m} \to 0 \) faster than \( \frac{1}{r} \),

then the surface integral vanishes and we recover

Coulomb's law

\[ \phi(\vec{r}) = \int d^3r' \, \Phi(\vec{r}') / |\vec{r} - \vec{r}'| \]

(\( \star \)) gives the generalization of Coulomb's law to a system

with a finite boundary

For a charge free volume \( V \), i.e. \( \varphi(r) = 0 \) in \( V \),

the potential everywhere is determined by the

potential and its normal derivative on the surface.

But one cannot in general specify both

\[ \phi \] and \[ \frac{\partial \phi}{\partial m} \] on the boundary surface, since the

resulting \( \phi \) from (\( \star \)) would not in general obey

Laplace's equation

\[ \nabla^2 \phi = 0. \]
Specifying both \( \phi \) and \( \phi' \) on surface is known as "Cauchy" boundary conditions — for Laplace's eqn, Cauchy b.c. over-specify the problem, and a solution cannot in general be found.

**Uniqueness**

If we have a system of charges in vol \( V \), and either the potential \( \phi \), or its normal derivative \( \phi'_n \), is specified on the surface of \( V \), then there is a unique solution to Poisson's equation inside \( V \). Specifying \( \phi \) is known as Dirichlet boundary conditions. Specifying \( \phi'_n \) is known as Neumann boundary conditions.

**Proof:** Suppose we had two solutions \( \phi_1 \) and \( \phi_2 \), both with \( -\nabla^2 \phi = \rho \) inside \( V \), and obeying specified b.c. on surface of \( V \).

Define \( U = \phi_2 - \phi_1 \), then \( \nabla^2 U = 0 \) inside \( V \)

and \( U = 0 \) on surface \( S \) — for Dirichlet b.c.,
or \( \frac{\partial U}{\partial n} = 0 \) on surface \( S \) — for Neumann b.c.

Use Green's 1st identity with \( \phi = \psi = U \)

\[
\oiint_{V} \left( \nabla^2 U + \nabla U \cdot \nabla U \right) \, d^3r = \oiint_{S} \left( \frac{\partial U}{\partial n} \right) \, d^2s
\]

as \( \nabla^2 U = 0 \), and \( U \) \( \frac{\partial U}{\partial n} = 0 \).
\[ \int dV |\nabla u|^2 = 0 \quad \Rightarrow \quad \nabla u = 0 \quad \Rightarrow \quad u = \text{const} \]

For Dirichlet b.c., \( u = 0 \) on surface \( S \), so const = 0 and \( \phi_1 = \phi_2 \). Solution is unique.

For Neumann b.c., \( \phi_1 \) and \( \phi_2 \) differ only by an arbitrary constant. Since \( \vec{E} = -\nabla \phi \), the electric fields \( E_1 = -\nabla \phi_1 \) and \( E_2 = -\nabla \phi_2 \) are the same.

Additionally, if boundary surface \( S \) consists of several disjoint pieces, then solution is unique if specify \( \phi \) on some pieces and \( \frac{\partial \phi}{\partial n} \) on other pieces.

Solution of Poisson's equation with both \( \phi \) and \( \frac{\partial \phi}{\partial n} \) specified on the same surface \( S \) (Cauchy b.c.) does not in general exist, since specifying either \( \phi \) or \( \frac{\partial \phi}{\partial n} \) alone is enough to give a unique solution.
Green's function - part II

Greens 2nd identity

\[ \int_V d^3 r' \left( \phi \nabla'^2 - 4 \frac{\partial^2 \phi}{\partial m'^2} \right) = \oint_S \left( \phi \frac{\partial \phi}{\partial m'} - 4 \frac{\partial \phi}{\partial m'} \right) \]

Apply above with \[ \phi(r') \] electrostatic potential with \[ \nabla'^2 \phi = -4 \pi \delta(r') \]
\[ \phi(r') = G(r, r') \] the Green function satisfying

\[ \nabla'^2 G(r, r') = -4 \pi \delta(r-r') \]

We saw one solution of above is \[ G(r, r') = \frac{1}{r-r'} \]

but a more general solution is

\[ G(r, r') = \frac{1}{r-r'} + F(r, r') \]

where \[ \nabla'^2 F(r, r') = 0 \] for \[ r' \] in volume \[ V \]

we will choose \[ F(r, r') \] to simplify solution of \[ \phi \]

\[ \Rightarrow \int_V d^3 r' \left( \phi(r') \nabla'^2 G(r, r') - G(r, r') \nabla'^2 \phi(r') \right) \]

\[ = \int_V d^3 r' \left( \phi(r') \left[ -4 \pi \delta(r-r') \right] - G(r, r') \left[ -4 \pi \rho(r') \right] \right) \]

\[ = -4 \pi \phi(r) + 4 \pi \int_V G(r, r') \rho(r') \]

\[ = \oint_S d a' \left( \phi \frac{\partial G}{\partial m'} - G \frac{\partial \phi}{\partial m'} \right) \]
\[ \phi(\vec{r}) = \frac{1}{V} \int d^3 r' \ G(\vec{r}, \vec{r}') \ f(\vec{r}') + \frac{1}{S} \int_{S_{B}} d\vec{a}' \ \left( \frac{\partial \phi(\vec{r}')}{\partial r^i_{B}} - \frac{\partial \phi(\vec{r})}{\partial r^j_{B}} \right) \delta_{iB} \delta_{jB} \]

Consider the *Dirichlet boundary problem*. If we can choose \( F(\vec{r}, \vec{r}') \) such that \( G(\vec{r}, \vec{r}') = 0 \) for \( \vec{r}' \) on the boundary surface \( S' \), then the above simplifies to

\[ \phi(\vec{r}) = \frac{1}{V} \int d^3 r' \ G(\vec{r}, \vec{r}') \ f(\vec{r}') - \frac{1}{S} \int_{S_{B}} d\vec{a}' \ \phi(\vec{r}') \ \partial_{iB} \ G_{B}(\vec{r}, \vec{r}') \]

Since \( f(\vec{r}) \) is specified in \( V \), and \( \phi(\vec{r}) \) is specified on \( S' \), above then gives desired solution for \( \phi(\vec{r}) \) inside volume \( V \).

Finally, \( G_{B} \) is therefore equivalent to finding an \( F(\vec{r}, \vec{r}') \) such that

\[ \nabla^2 F(\vec{r}, \vec{r}') = 0 \quad \text{for} \quad \vec{r}' \in V \] (solves Laplace eqn) and

\[ F(\vec{r}, \vec{r}') = \frac{-1}{|\vec{r} - \vec{r}'|} \quad \text{for} \quad \vec{r}' \text{ on boundary surface } S' \]

Always exists unique solution for \( F \).
Next consider Neumann boundary problem.

One might think to find \( F(\vec{r}, \vec{r}') \) such that \( \frac{\partial G}{\partial m'} (\vec{r}, \vec{r}') = 0 \) on boundary surface. But this is not possible.

Consider \( \int V(r, r') d^3r' = \int V' \cdot \nabla' G(\vec{r}, \vec{r}') d^3r' \nabla' \cdot \vec{G} = -4\pi \delta(\vec{r} - \vec{r}') \)

So we can't have \( \frac{\partial G}{\partial m'} = 0 \) for \( \vec{r}' \) on \( S' \)

Simplest choice is then \( \frac{\partial G_N}{\partial m'} (\vec{r}, \vec{r}') = -4\pi \quad \text{for} \quad \vec{r}' \quad \text{on} \quad S' \quad \text{area of surface} \)

Then

\[
\phi(\vec{r}) = \int d^3r' \ G_N(\vec{r}, \vec{r}') \phi(\vec{r}') + \oint \frac{da'}{4\pi} \cdot \left[ \phi(\vec{r}') + \frac{\partial \phi(\vec{r}')}{\partial m'} \cdot \nabla' \cdot \vec{G}(\vec{r}, \vec{r}') \frac{\partial G_N}{\partial m'} \right]
\]

Since \( \phi(\vec{r}) \) is specified in \( V \)

and \( \frac{\partial \phi}{\partial m} \) is specified on \( S' \)

This above gives solution \( \phi(\vec{r}) \) in \( V \) within additive constant \( \langle \phi \rangle_S \)

Since \( \vec{E} = -\nabla \phi \), the constant \( \langle \phi \rangle_S \) is of no consequence.
Finding \( G_\nu (\vec{r}, \vec{r}') \) is therefore equivalent to finding an \( F(\vec{r}, \vec{r}') \) such that

\[
\nabla^2 F(\vec{r}, \vec{r}') = 0 \quad \text{for } \vec{r}' \in V
\]

and \( \frac{\partial F(\vec{r}, \vec{r}')}{\partial n} = -\frac{\Phi}{\sigma} \) for \( \vec{r}' \) on surface \( S' \).

always exists a unique solution (within additive constant)

While \( G_D \) or \( G_N \) always exist in principle, they depend in detail on the shape of the surface \( S \) and are difficult to find except for single geometries.

In proceeding, we defined \( G \) by

\[
\nabla^2 G(\vec{r}, \vec{r}') = -\frac{\Phi}{\sigma} \delta(\vec{r} - \vec{r}')
\]

But our earlier interpretation of \( G(\vec{r}, \vec{r}') \) was that it was potential at \( \vec{r} \) due to point source at \( \vec{r}' \), i.e.

\[
\nabla^2 G(\vec{r}, \vec{r}') = -\frac{\Phi}{\sigma} \delta(\vec{r} - \vec{r}') \quad \text{Note, for general surface } S', \ G(\vec{r}, \vec{r}') \text{ is not in general a function of } |\vec{r} - \vec{r}'| \text{ but depends on } \vec{r} \text{ and } \vec{r}' \text{ separately. But the equivalence of the two definitions of } G \text{ above is obtained by noting that one can prove the symmetry property}
\]

\[
G(\vec{r}, \vec{r}') = G(\vec{r}', \vec{r})
\]

for Dirichlet b.c., and one can impose it as an additional requirement for Neumann b.c.

(see Jackson, end section 1.10)