Image Charge Method

For simple geometries, one can try to obtain \( G_D \) or \( G_N \) by placing a set of "image charges" outside the volume of interest \( V \), i.e., on the "other side" of the system boundary surface \( S \). Because these image charges are outside \( V \), they contribute to the potential inside \( V \) according to \( \nabla^2 \phi_{image} = 0 \), as necessary. Choose location of image charges so that total \( \phi \) has desired boundary condition.

1) Charge in front of infinite grounded plane

\[ \begin{align*}
\nabla^2 \phi &= -4\pi g \delta(x) \delta(y) \delta(z-d) \\
\phi &= 0 \quad \text{for} \quad z = 0 \\
\phi &= 0 \\
\end{align*} \]

If we find a solution to above, it is the unique solution.

Solution: put fictitious image charge \(-g\) at \( z = -d\)

\[ \phi(x, y, z) = \frac{g}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{g}{\sqrt{x^2 + y^2 + (z+d)^2}} \]

\( \phi \) is potential from the real charge \( +g \) and the image \(-g\).

Above satisfies \( \phi(x, y, 0) = 0 \) as required.

Also, \( \nabla^2 \phi = -4\pi g \delta(r-d) - 4\pi g \delta(r+d) \)

\[ = -4\pi g \delta(r-d) \quad \text{for region} \quad z > 0 \]
Can now find $\vec{E}$ for $z=0$

$$\vec{E} = -\vec{\nabla} \phi$$

In particular $E_z = -\frac{\partial \phi}{\partial z} = \frac{q}{4\pi} \int \frac{\left(\frac{1}{2}\right) \frac{z}{\sqrt{x^2+y^2+(z-d)^2}}}{\left[\sqrt{x^2+y^2+(z-d)^2}\right]^{3/2}} - \left(\frac{1}{2}\right) \frac{z + d}{\sqrt{x^2+y^2+(z+d)^2}}^{3/2} \, dz$.

$$E_z = \frac{q}{4\pi} \int \left[ \frac{(z-d)}{\sqrt{x^2+y^2+(z-d)^2}}^{3/2} - \frac{(z+d)}{\sqrt{x^2+y^2+(z+d)^2}}^{3/2} \right] \, dz$$

We can use above to compute the surface charge density $\sigma(x,y)$ induced on the surface of the conducting plane. At conductor surface

$$-\frac{\partial \phi}{\partial n} = 4\pi \sigma$$

$$\Rightarrow \sigma = -\frac{1}{4\pi} \frac{2\phi}{\partial z} = \frac{1}{4\pi} E_z \big|_{(x,y,z=0)}$$

$$\sigma(x,y) = \frac{q}{4\pi} \int \left[ \frac{-d}{\sqrt{(x^2+y^2+d^2)^{3/2}}} - \frac{d}{\sqrt{(x^2+y^2+d^2)^{3/2}}} \right]$$

$$= -\frac{q}{2\pi} \frac{d}{\sqrt{(x^2+y^2+d^2)^{3/2}}} = \frac{-qd}{2\pi \sqrt{(x^2+y^2)^{3/2}} + 2\pi \sqrt{(x^2+y^2)^{3/2}}}$$

$\sigma$

$\frac{1}{r^3}$

$r_1 = \sqrt{x^2+y^2}$
Total induced charge \( q_{\text{induced}} \):

\[
q_{\text{induced}} = \iint_{s} \sigma(x,y) \, dxdy
\]

\[
= \frac{2\pi \int_{0}^{\infty} r_{1} \sigma(r_{1}) \, dr_{1}}{2\pi \left( r_{1}^{2} + d^{2} \right)^{3/2}}
\]

\[
= -gd \left[ \frac{-1}{\left( r_{1}^{2} + d^{2} \right)^{1/2}} \right]_{0}^{\infty}
\]

\[
= -gd \left[ 0 - \frac{1}{d} \right]
\]

\[
q_{\text{induced}} = -q \quad \text{induced charge = image charge}
\]

Force on charge \( q \) in front of conducting plane is due to the induced \( \sigma \). The E field of this \( \sigma \) is, for \( q > 0 \), the same as the E field of the image charge.

\[
\Rightarrow F = -\frac{q^{2}}{(2d)^{2}} \hat{\mathbf{z}} = -\frac{q^{2}}{4d^{2}} \hat{\mathbf{z}} \quad \text{(attractive)}
\]

Work done to move \( q \) into position from infinity is

\[
W = \int_{\infty}^{d} \mathbf{F} \cdot d\mathbf{r} = -\int_{0}^{d} F_{z} \, dz
\]

we must oppose electrostatic force \( \mathbf{F} \).
\[ W = \int_1^0 \frac{\frac{q^2}{4\pi \varepsilon_0}}{4\pi \varepsilon_0 r^2} \, dr = -\frac{q^2}{4d} \]

\[ W < 0 \Rightarrow \text{energy released} \]

\text{Note: } W \text{ above is not the electrostatic energy that would be present if the image charge were real, i.e., } \Phi_{\text{image}}(\vec{r} = d\hat{z}) = -\frac{q^2}{2d}.

One way to see why is to note that as } q \text{ is moved quasi-statically in towards the conductor plane, the image charge also must be moving to stay equidistant on the opposite side.
2) Point charge in front of a grounded ($\phi = 0$) conducting sphere.

Charge $q$ placed a distance $s$ from center of grounded conducting sphere of radius $R$.

Place image charge $q'$ inside sphere so that the combined $\phi$ from $q$ and $q'$ vanishes on surface of sphere.

By symmetry, $q'$ should lie on the same radial line as $q$ does. Call the distance $s'$ from the origin "a."

Potential at position $\mathbf{r}$ is

$$\phi(\mathbf{r}) = \frac{q}{|\mathbf{r} - s\mathbf{a}|} + \frac{q'}{|\mathbf{r} - a\mathbf{a}|}$$

$$= \frac{q}{(r^2 + s^2 - 2sr\cos\theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ra\cos\theta)^{1/2}}$$

Can we choose $q'$ and $a$ so that $\phi(r, \theta) = 0$ for all $\theta$?
\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} + \frac{q'}{(r^2 + a^2 - 2ar \cos \theta)^{1/2}} \]

\[ \text{make denominators look alike} \]

\[ r^2 + a^2 - 2ar \cos \theta = \frac{a}{s} \left( \frac{s}{a} r^2 + sa - 2sr \cos \theta \right) \]

If choose \( S_a = R^2 \), ie \( a = \frac{R^2}{s} \), then \( \frac{S_r^2}{a} = s^2 \)

and then the denominator of the 2nd term is

\[ \left[ \frac{R^2}{s^2} (s^2 + R^2 - 2sr \cos \theta) \right]^{1/2} = \frac{R}{s} \left[ 3s^2 + R^2 - 2sr \cos \theta \right]^{1/2} \]

Then

\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} + \frac{q'(S/R)}{(R^2 + s^2 - 2sr \cos \theta)^{1/2}} \]

So choose \( q'(S/R) = -q \) \Rightarrow \( q' = -q \frac{R}{s} \)

to get \( \phi(r, \theta) = 0 \)

Solution is

\[ \phi(r, \theta) = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{qR/s}{(r^2 + \frac{R^2}{s^2} - 2r \frac{R^2}{s^2} \cos \theta)^{1/2}} \]

\[ = \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} - \frac{q}{\left( \frac{s^2r^2 + R^2 - 2rs \cos \theta}{R^2} \right)^{1/2}} \]

Can get induced surface charge on sphere by

\[ 4\pi \sigma = \vec{E} \cdot \hat{n} = -\frac{\partial \phi}{\partial r} \bigg|_{r=R} \]

see Jackson Eq (2.5) for result
\[ \sigma(\theta) = -\frac{q}{4\pi RS} \frac{1 - (R/s)^2}{(1 + (R/s)^2 - 2(R/s)\cos\theta)^{3/2}} \]

\( \sigma(\theta) \) is greatest at \( \theta = 0 \), as one should expect.

Can integrate \( \sigma(\theta) \) to get total induced charge. One finds

\[
\pi \int_0^\pi d\theta \sin \theta R^2 \sigma(\theta) = q' = -\frac{qR^2}{s} \]

In general, total induced charge = sum of all nuage charges.

**Force of attraction of charge to sphere**

Force on \( q \) is due to electric field from induced charge \( \sigma \) which is the same as the electric field from the nuage charge \( q' \).

\[
\vec{F} = -\frac{qq'}{\epsilon} \frac{\hat{z}}{(s-a)^2} = -\frac{q^2(R/s)^2}{(s-R^2)^2} = -\frac{q^2R^2}{(s^2R^2)^2} \hat{z} \]

Close to the surface of the sphere, \( s \approx R \), so write \( s = R + d \) where \( d \ll R \). Then

\[
\vec{F} = -\frac{q^2Rs}{(s-R)^2(s+R)^2} = -\frac{q^2R(R+d)}{d^2(2R+d)^2} \approx -\frac{q^2}{4d^2} \]

get same result as for infinite flat grounded plane.

When \( q \) is so close to surface that \( d \ll R \), the charge does not "see" the curvature of the surface.
for from the surface, \( s \gg R \)

\[
F = \frac{g g' \hat{s}}{(s-a)^2} = -\frac{g^2 R s}{(s^2 - R^2)^2} \hat{s} = -\frac{g^2 R}{s^3} \hat{s}
\]

\[
F \sim \frac{1}{s^3}
\]

very different from flat plane
also different from point charge

Note: In proceeding two problems, what we found was a

\( \phi \) such that \( \nabla^2 \phi = -4\pi \delta (\mathbf{r} - \mathbf{r}_0) \), for a charge at \( \mathbf{r}_0 \)

and \( \phi = 0 \) on the boundary. Such a \( \phi \) is nothing

more than \( G_0 \) the corresponding Green function for

Dirichlet boundary conditions.

Suppose now that instead of a grounded sphere we
have a sphere with fixed net charge \( Q \).

We want to add new image charge to represent this case.

If we put \( g' = -g \frac{Q}{s} \) at \( a = R \) as before, the
boundary condition of \( \phi = \text{const on surface } \mathbf{r} = R \) is
met. But the net charge on the sphere is \( g' \) (the
induced charge) not the desired \( Q \). We therefore need
to add new image charge(s) of total charge \( Q - g' \)
(so total image charge is \( Q \)) in such a way that we
keep \( \phi \) constant on the surface of the sphere. The
way to do this is to put \( Q - g' \) at the origin!
Solution is:

\[ \phi(r, \theta) = \frac{q + qR/s}{r} - \frac{q}{(r^2 + s^2 - 2rs \cos \theta)^{1/2}} \]

The force on the charge \( q \) is due to the \( \vec{E} \) field of the images.

\[ \vec{F} = \vec{F}^\prime = \frac{q}{s^2} \frac{(q + qR/s) \hat{z}}{(s - a)^2} + \frac{q}{(s - a)^2} \frac{qR/s \hat{z}}{s^2} \]

\[ F = \frac{qA}{s^2} + \frac{q^2R/s}{s^2} - \frac{q^2R/s}{(s - R^2/s)^2} \]

\[ = \frac{qA}{s^2} + \frac{q^2R}{s^3} \left[ \frac{1}{s^3} - \frac{1}{s^3 (1 - \frac{R^2}{s^2})^2} \right] \]

\[ = \frac{qA}{s^2} + \frac{q^2R}{s^3} \left[ 1 - \frac{1}{(1 - \frac{R^2}{s^2})^2} \right] \]

\[ F = \frac{qA}{s^2} - \frac{q^2R^3}{s} \frac{2 - \frac{R^2}{s^2}}{(s^2 - R^2)^2} \]

For large \( s \gg R \) far from surface

\[ F \sim \frac{qA}{s^2} - \frac{2q^2R^3}{s^6} \]

leading term is first Coulomb force between \( q \) and \( A \) at origin

for \( A > 0 \), \( F \) is always repulsive for large enough \( s \)
For $s = R + d$, $d \ll R$ close to surface

\[
F = \frac{qQ}{(R+d)^2} - \frac{q^2R^3}{R+d} \frac{2 - \frac{R^2}{(R+d)^2}}{(R^2 + d^2 + 2Rd - R^2)^2} \\
\approx \frac{qQ}{R^2} - \frac{q^2R^3}{R} \frac{(2 - 1)}{4R^2d^2} \\
F \approx \frac{qQ}{R^2} - \frac{q^2R^3}{4d^2} \approx -\frac{q^2}{4d^2} \text{ for } d \text{ small enough}
\]

$F$ is always attractive for small enough $d$, and is equal to the force in front of a grounded plane, no matter what is the value of $Q$. This is because the image charge $Q'$ lies so much close to $Q$ than does the $Q-Q'$ at the origin, that it dominates the force.

The crossover from attractive to repulsive occurs at a distance $s$ that depends on $Q$. This distance is given by

\[
Q = \frac{r^3s}{g} \left( 2 - \frac{R^2}{g^2} \right) - \frac{r^3}{s} \left( 2 - \left( \frac{Rg}{s} \right)^2 \right) \frac{2 - \left( \frac{Rg}{s} \right)^2}{\left[ 1 - \left( \frac{Rg}{s} \right)^2 \right]^2}
\]

Let $x = \frac{Rg}{s} \in (0,1)$

\[
Q = \frac{x^3}{g} \left( 2 - x^2 \right) \frac{2 - x^2}{(1-x^2)^2}
\]

gives $5^{th}$ order polynomial in $x$ and analytic solution can be solved graphically.
For $A = 1$, crossover is at $\frac{R}{S} = 0.62$

$S = 1.6 R$

$A = 0.1$, crossover is at $\frac{R}{S} = 0.36$

$S = 2.8 R$