Eigenvector expansion for Green Functions

Suppose \( D \) is some linear differential operator, for example \( \nabla^2 \).

Solutions to the equation

\[ D \psi(\vec{r}) = -4\pi f(\vec{r}) \]

can be solved if one knows the Green function, which is the solution to the problem with a point source

\[ DG(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}') \]

operates on \( f \).

Then

\[ \psi(\vec{r}) = \int d^3r' G(\vec{r}, \vec{r}') f(\vec{r}') \]

is solution.

If we need to solve for \( \psi \) subject to certain boundary conditions, then we can always add to the Green function a \( \phi(\vec{r}) \) such that \( D\phi(\vec{r}) = 0 \) in the desired region and then choose \( \phi \) accordingly as we did for Dirichlet or Neumann b.c. for \( \nabla^2 \).

One way to find \( G(\vec{r}, \vec{r}') \) is to find the eigenvalues and eigenfunctions of \( D \).

\[ D \psi_n(\vec{r}) = \lambda_n \psi_n(\vec{r}) \]

\[ \uparrow \quad \text{eigenfunction} \quad \uparrow \quad \text{eigenvalue} \]
Depending on the problem, the spectrum of eigenvalues might be discrete or might be continuous.

Note: When we solved Laplace's equation by separation of variables method, what we wound up doing was solving the eigenvalue problem for the Laplacian in spherical coordinates, radial, \( \theta \), and \( \phi \) pieces of the differential operator.

In many cases (you would have to prove this for the particular operator \( \Delta \)) the eigenfunctions \( \Psi_n(\vec{r}) \) form an orthogonal and complete set of basis functions over the region of interest (i.e. in the volume in which we are seeking a solution)

\[
\text{orthogonal} \Rightarrow \int_V d^3r \, \Psi_n^*(\vec{r}) \Psi_m(\vec{r}) = \delta_{m,n} \\
\text{complete} \Rightarrow f(\vec{r}) = \sum_n a_n \Psi_n(\vec{r}).
\]

Any function \( f \) can be expanded in a linear combination of the \( \Psi_n \).

The expansion coefficients \( a_n \) are obtained by

\[
\int_V d^3r \, f(\vec{r}) \Psi_m^*(\vec{r}) = \sum_n a_n \int_V d^3r \, \Psi_m^*(\vec{r}) \Psi_n(\vec{r}) = \sum_n a_n \delta_{m,n}
\]

So

\[
A_m = \int_V d^3r \, f(\vec{r}) \Psi_m^*(\vec{r}) \quad \text{"Fourier" coefficient for basis } \Psi_n
\]
In particular, the function $\delta(\vec{r} - \vec{r}')$ can be expanded as

$$\delta(\vec{r} - \vec{r}') = \sum_n a_n \psi_n(\vec{r})$$

where

$$a_n = \int d^3r ~ \delta(\vec{r} - \vec{r}') \psi_n^*(\vec{r}) = \psi_n^*(\vec{r}')$$

assuming $\vec{r}' \in V$.

So we have

$$\delta(\vec{r} - \vec{r}') = \sum_n \psi_n^*(\vec{r}') \psi_n(\vec{r})$$

Now we can solve for the Green function $G(\vec{r}, \vec{r}')$ as a function of $\vec{r}$, $\vec{r}'$.

$$G(\vec{r}, \vec{r}') = \sum_n a_n \psi_n(\vec{r})$$

Now use

$$\Delta G(\vec{r}, \vec{r}') = -4\pi \delta(\vec{r} - \vec{r}')$$

since $\Delta$ is linear.

$$\Rightarrow \sum_n a_n \Delta \psi_n(\vec{r}) = \sum_n a_n \lambda_n \psi_n(\vec{r}) = -4\pi \sum_n \psi_n^*(\vec{r}') \psi_n(\vec{r})$$

$$\Rightarrow \sum_n \left[ a_n \lambda_n + 4\pi \psi_n^*(\vec{r}') \right] \psi_n(\vec{r}) = 0$$

If a series in a set of basis functions vanishes, then each coefficient in the series must vanish.

$$\Rightarrow a_n = -\frac{4\pi \psi_n^*(\vec{r}')}{\lambda_n}$$

$$G(\vec{r}, \vec{r}') = -4\pi \sum_n \left[ \frac{\psi_n^*(\vec{r}') \psi_n(\vec{r})}{\lambda_n} \right]$$
Example: $\nabla^2$ in rectangular coordinate, $V = \text{all space}$

$\nabla^2 \psi(r) = -k^2 \psi(r)$

All the eigenvalues $\lambda = -k^2$

eigenfunctions are then $\psi_n = e^{i k \cdot r}$

cHECK $\nabla \psi = i k \cdot r e^{i k \cdot r}$

$\nabla^2 \psi = \nabla \cdot (\nabla \psi) = (i k \cdot r)(i k \cdot r) e^{i k \cdot r} = -k^2 \psi$

Normalize $\psi$ for orthogonality condition

$$\psi_k(r) = \frac{1}{(2\pi)^{3/2}} e^{i k \cdot r}$$

$$\int d^3r \frac{\psi^*_k(r') \psi_k(r)}{k} = \int d^3r \frac{1}{(2\pi)^3} e^{-i k' \cdot r} e^{i k \cdot r}$$

$$= \int d^3r \frac{e^{i (k-k') \cdot r}}{(2\pi)^3} = \delta(k-k')$$

$$\Rightarrow G(r, r') = -4\pi \int d^2k \frac{e^{i k \cdot (r-r')}}{(-k^2)} = \int d^3k \frac{4\pi}{k^2} \frac{e^{i k \cdot (r-r')}}{|r-r'|}$$

Now we already know that the Green function for this problem

$$G(r, r') = \frac{1}{|r-r'|}$$

So from this we see that the Fourier transform of

$$\frac{1}{|r-r'|} \propto \frac{4\pi}{k^2}$$
Example: Green's function for Dirichlet problem inside rectangular box \( x \in [0, a], y \in [0, b], z \in [0, c] \)

We are looking for eigenfunction \( \psi \)

\[ \nabla^2 \psi = \lambda \psi \]

with \( \psi = 0 \) on boundaries of the rectangular box.

Solutions are

\[ \psi_{lmn} = \frac{8}{abc} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{l\pi z}{c} \right) \]

with eigenvalues \( \lambda_{lmn} = -\pi^2 \left( \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right) \), \( l, m, n = 1, 2, \ldots \)

Check normalization for yourselves!

\[ G(F, F') = -4\pi \sum_{lmn=1}^{\infty} \frac{8}{abc} \frac{\sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{l\pi z}{c} \right)}{-\pi^2 \left( \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \times \sin \left( \frac{m\pi \xi}{a} \right) \sin \left( \frac{n\pi \eta}{b} \right) \]

\[ G(x, z') = \frac{32}{\pi abc} \sum_{lmn=1}^{\infty} \frac{\sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) \sin \left( \frac{l\pi z}{c} \right) \sin \left( \frac{m\pi \xi}{a} \right) \sin \left( \frac{n\pi \eta}{b} \right) \sin \left( \frac{l\pi \zeta}{c} \right) \sin \left( \frac{m\pi \xi'}{a} \right) \sin \left( \frac{n\pi \eta'}{b} \right) \sin \left( \frac{l\pi \zeta'}{c} \right)}{-\pi^2 \left( \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} \right)} \]

Note that in this case, \( G(F, F') \) is not a function of \( \bar{z} - \bar{z}' \), the boundary treats the translational invariance.
\[ \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \frac{4\pi \mathbf{j}}{c} \quad \text{Ampere's Law (statics only!)} \]
\[ \Rightarrow \nabla \times (\nabla \times \mathbf{A}) = \frac{4\pi \mathbf{j}}{c} \]
\[ \text{can write } \nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \]

where by \( \nabla^2 \mathbf{A} \) we mean \( (\nabla^2 A_x) \hat{x} + (\nabla^2 A_y) \hat{y} + (\nabla^2 A_z) \hat{z} \)
\( \nabla^2 \mathbf{A} \) only has a single expression in Cartesian coords

If tried to write it in spherical coords, for example,

one has
\[
\nabla^2 \mathbf{A} = \nabla^2 (A_x \hat{x} + A_\theta \hat{\theta} + A_\phi \hat{\phi}) \\
= (\nabla^2 A_x) \hat{x} + A_x (\nabla^2 \hat{x}) + (\nabla^2 A_\theta) \hat{\theta} + A_\theta (\nabla^2 \hat{\theta}) \\
+ (\nabla^2 A_\phi) \hat{\phi} + A_\phi (\nabla^2 \hat{\phi})
\]

one must not forget to take the derivatives of \( \hat{x}, \hat{\theta}, \hat{\phi} \)
since they vary with position!

for example, \( \hat{x} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \)

one could compute \( \nabla^2 \hat{x} \) by applying \( \nabla^2 \) in spherical coords
to each piece and summing up. Get a mess!

If work in Coulomb gauge, with \( \nabla \cdot \mathbf{A} = 0 \), then
\[
\nabla \times (\nabla \times \mathbf{A}) = -\nabla^2 \mathbf{A} = \frac{4\pi \mathbf{j}}{c} \quad \text{Poisson's equation!}
\]
Many of the same methods used to solve for electrostatic $\mathbf{E}$ can therefore be applied to solve for magnetostatic $\mathbf{A}$. But vector nature of $\mathbf{A}$ makes for complications!

For simple geometries, one can do the Coulomb-like integral

$$\mathbf{A}(\mathbf{r}) = \frac{1}{2\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \, d^3 r'$$

three equations for $A_x, A_y, A_z$!

for localized current sources $\mathbf{J}(\mathbf{r}) \to 0$ as $r \to \infty$

**Multiple expansion** - magnetic dipole moment

For a general treatment, analogous to how we did multiple expansion for electrostatics, one can use vector spherical harmonics - see Jackson Chpt. 9.

Here we do a more straightforward approach, but only up to magnetic dipole term.

For $r \gg r'$, approx

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} \approx \frac{1}{r} \left( 1 - \frac{2 \mathbf{r} \cdot \mathbf{r}'}{r^2} + \frac{(\mathbf{r}')^2}{r^2} + \cdots \right) = \frac{1}{r} + \frac{\mathbf{r} \cdot \mathbf{r}'}{r^3} + \cdots.$$