Example: Circular current loop in xy plane

\( r < R \)

For \( r > R \), \( \nabla \times \mathbf{B} = 0 \) \( \Rightarrow \mathbf{B} = -\nabla \Phi_M \)

where \( \nabla^2 \Phi_M = 0 \).

Try Legendre polynomial expansion for \( \Phi_M \)

\[ \Phi_M = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta) \quad (A_\ell \text{ terms vanish as want } R \to \infty \text{ as } r \to \infty) \]

\[ \mathbf{B} = -\nabla \Phi_M = - \nabla \Phi_M \mathbf{r} - \frac{1}{r} \frac{\partial \Phi_M}{\partial \theta} \mathbf{\hat{\theta}} \]

\[ = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} \left[ (\ell+1) \frac{B_\ell}{r^{\ell+2}} P_\ell(\cos \theta) \mathbf{\hat{r}} - \frac{B_\ell}{r^{\ell+2}} 2 P_\ell(\cos \theta) \mathbf{\hat{\theta}} \right] \]

Write \( \frac{\partial B_\ell}{\partial \theta} = \frac{\partial B_\ell}{\partial x} \frac{\partial x}{\partial \theta} = -\frac{\partial B_\ell}{\partial x} \sin \theta \quad x = \cos \theta \)

\[ = -P_\ell'(\cos \theta) \sin \theta \]

\[ \mathbf{B} = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+2}} \left[ (\ell+1) \frac{B_\ell}{r^{\ell+2}} P_\ell(\cos \theta) \mathbf{\hat{r}} + \frac{B_\ell}{r^{\ell+2}} \sin \theta P_\ell'(\cos \theta) \mathbf{\hat{\theta}} \right] \]

To determine the \( \mathbf{B} \), we compare with exact solution along \( \mathbf{\hat{\theta}} \) axis

\[ \mathbf{B}(\hat{\theta}, \hat{\phi}) = \sum_{\ell=0}^{\infty} \frac{(\ell+1)B_\ell}{r^{\ell+2}} \mathbf{\hat{r}} = \sum_{\ell=0}^{\infty} \frac{(\ell+1)B_\ell}{r^{\ell+2}} \mathbf{\hat{z}} \]

Since \( P_\ell'(1) = 1 \), \( \sin(0) = 0 \) and \( P_\ell'(1) \) finite, \( \mathbf{\hat{r}} = \mathbf{\hat{z}} \) when \( \theta = 0 \)
exact solution on \( \frac{3}{3} \) axis:

\[
\vec{A} = \int_{\frac{3}{3}}^{3} \frac{\vec{f}(\vec{r})}{\vec{r}} \Rightarrow \vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \int_{\frac{3}{3}}^{3} \frac{\vec{\nabla} \times \vec{f}(\vec{r}^{'})}{\vec{r}^{'}}
\]

\[
\vec{B} = -\int_{\frac{3}{3}}^{3} \frac{\vec{f}(\vec{r}^{'}) \times \vec{r}'}{\vec{r}^{'}} \quad \text{Biot-Savart law for magnetostatics}
\]

For our loop

\[
\begin{array}{c}
\hat{r} \times \hat{\phi} = \hat{z} \\
\hat{\phi} \times \hat{r} = \hat{z} \\
\hat{r} \times \hat{z} = \hat{\phi}
\end{array}
\]

\[
\vec{B}(\phi) = \frac{2\pi I}{c} \int_{0}^{2\pi} \frac{\hat{r}(\vec{r}^{'}) \hat{z}}{(\vec{r}^{'})^2 + \vec{r}^2} \\hat{\phi} \times \hat{z} \quad \text{ten integral to get}
\]

\[
\vec{B}(\phi) = \frac{4\pi I \hat{z} R^2}{c} \frac{2}{(\vec{r}^2 + \vec{r}^2)^{3/2}}
\]

To match Legendre polynomial expansion, do Taylor series expansion of above

\[
\vec{B}(\phi) = \frac{2\pi I \hat{z} R^2}{c} \frac{1}{\vec{r}^3} \left( 1 - \frac{3}{2} \frac{R^2}{\vec{r}^2} + \ldots \right)
\]

\[
= \frac{2\pi I \hat{z} R^2}{c} \left\{ \frac{1}{\vec{r}^3} - \frac{3}{2} \frac{R^2}{\vec{r}^3} + \ldots \right\}
\]

\[
\frac{2\hat{z}}{3} \left( \frac{B_0}{3^2} + \frac{2B_1}{3^3} + \frac{3B_2}{3^4} + \frac{4B_3}{3^5} + \ldots \right)
\]
\[ B_0 = 0, \ B_1 = \frac{\pi R^2 I}{c}, \ B_2 = 0, \ B_3 = -\frac{3}{4c} \pi R^2 I R^2 \]

So to order \( I = 0 \)

\[
\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 P_1(\cos \theta) \hat{r} + \sin \theta P_1'(\cos \theta) \hat{\theta}}{r^3} \right. \\
- \left. \frac{\left[ 3R^2 P_3(\cos \theta) \hat{r} + \frac{3}{4} R^2 \sin \theta P_2'(\cos \theta) \hat{\theta} \right]}{r^5} \right\} + \ldots
\]

\[ P_1(x) = x \Rightarrow P_1'(x) = 1 \]
\[ P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow P_3'(x) = \frac{3}{2}(15x^2 - 3) \]

\[
\vec{B}(\vec{r}) = \frac{\pi R^2 I}{c} \left\{ \frac{2 \cos \theta \hat{r} + \sin \theta \hat{\theta}}{r^3} \right. \\
- \left. \frac{\left[ \frac{3}{2} R^2 (5 \cos^3 \theta - 3 \cos \theta) \hat{r} + \frac{3}{8} R^2 \sin \theta (15 \cos^2 \theta - 3) \hat{\theta} \right]}{r^5} \right\} + \ldots
\]

\[ \frac{\pi R^2 I}{c} = m \] is the magnetic dipole moment of the loop

We see that the 1st term is just the magnetic dipole approx. The 2nd term is the magnetic quadrupole term. Could easily get higher order terms by this method.

\[ \text{Compare our result above to Jackson (5.40)} \]
- example current carrying infinite cylinder \( \text{radius} R \)

\[ \begin{array}{c}
\mathbf{E} = \mathbf{K} \hat{k} \\
(\text{i}) \quad \mathbf{E} = K \hat{x} \\
(\text{ii}) \quad \mathbf{E} = K \hat{y}
\end{array} \]

wire with surface current

solenoid

\[ \begin{array}{c}
(\text{ii}) \quad \mathbf{E} = K \hat{z} \\
2\pi RK = I \quad \text{total current}
\end{array} \]

"guess" + show it is correct

\[ \begin{aligned}
r > R & : \quad \phi_M = -\frac{4\pi RK \rho}{z} \\
r < R & : \quad \phi_M = 0
\end{aligned} \]

magnetic scalar potential \( \nabla^2 \phi_M = 0 \)

\[ \begin{aligned}
c > R & : \quad \mathbf{B} = -\nabla \phi_M = -\frac{1}{r} \frac{\partial}{\partial \rho} \left( \frac{2\pi I \hat{\phi}}{cr} \right) = \frac{2\pi I \hat{\phi}}{cr} \\
c < R & : \quad \mathbf{B} = 0
\end{aligned} \]

\[ \text{Faraday} - \frac{\mathbf{E}}{cr} = \frac{2\pi I \hat{\phi}}{cr} = \frac{4\pi K R \phi}{cr} = \frac{4\pi K x \hat{m}}{cr} \]

where \( \mathbf{m} = \hat{m} \)

as \( \frac{\hat{z} \times \hat{z}}{r} = \hat{\phi} \)

Note: \( \phi_M = -\frac{4\pi RK \rho}{z} \) is not single valued.

would not have found this using expansion

of separation of cooreads in other cooreads

\[ \begin{array}{c}
(\text{iii}) \quad \mathbf{E} = K \hat{z}
\end{array} \]

\[ \begin{aligned}
r > R & : \quad \phi_M = -B_1 \hat{z} \\
r < R & : \quad \phi_M = -B_2 \hat{z}
\end{aligned} \]

\[ \begin{aligned}
r > R & : \quad \mathbf{B} = -\nabla \phi_M = B_1 \hat{z} \\
r < R & : \quad \mathbf{B} = -\nabla \phi_M = B_2 \hat{z}
\end{aligned} \]
\[
\begin{align*}
\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = (B_1 - B_2) \frac{\hat{z}}{2} &= \frac{4\pi \mu_0 k}{2} \hat{z} \\
&= \frac{4\pi \mu_0 k}{2} (\hat{\phi} \times \hat{r}) \\
&= -\frac{4\pi \mu_0 k}{2} \frac{\hat{z}}{2}
\end{align*}
\]

If current in solenoid is only source of \( \vec{B} \), then expect \( B_1 = 0 \)

\[
\Rightarrow \vec{B}_2 = \frac{4\pi \mu_0 k}{2} \frac{\hat{z}}{3} \quad \text{familiar result}
\]
Symmetry under parity transformation

vector vs. pseudovector

\[ \ldots \]

\[ \begin{align*}
\vec{r} &= (x, y, z) & \rightarrow & & \left(-x, -y, -z\right) \\
\mathbf{P}(\vec{r}) &= -\vec{r} & \text{position } \vec{r} & \text{is odd under parity}
\end{align*} \]

Any vector-like quantity that is odd under \( \mathbf{P} \) is a vector.

\[ \ldots \]

Dot angles of vectors

position \( \vec{r} \)

velocity \( \vec{v} = \frac{d\vec{r}}{dt} \) since \( \vec{v} \) is vector \( t \) is scalar

acceleration \( \vec{a} = \frac{d\vec{v}}{dt} \)

Force \( \vec{F} = m\vec{a} \) since \( \vec{a} \) is vector \( m \) is scalar

momentum \( \vec{p} = m\vec{v} \) since \( \vec{v} \) is vector \( m \) is scalar

electric field \( \vec{E} = \vec{g} \cdot \vec{E} \) since \( \vec{E} \) is vector \( \vec{g} \) is scalar

\[ \mathbf{P}(\vec{g}) = -\vec{g} \]

current \( \vec{j} = \sum_i n_i \vec{u}_i \times (\vec{v}_i - \vec{v}_i(t)) \)
any vector-like quantity that is even under $P$ is a pseudovector

angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ since $\mathbf{r} \rightarrow -\mathbf{r}$ and $\mathbf{p} \rightarrow \mathbf{p}$, $\mathbf{L} \rightarrow -\mathbf{L}$ under $P$

$\mathbf{L}$ is even under $P$

magnetic field $\mathbf{F} = \mathbf{q} \mathbf{v} \times \mathbf{B}$ since $\mathbf{F}$ and $\mathbf{B}$ are vectors and $\mathbf{q}$ is scalar, $\mathbf{B}$ must be pseudovector. Cross product of any two vectors is a pseudovector

"\" vector and pseudovector is a vector

When solving for $\mathbf{E}$, it can only be made up of vectors that exist in the problem

When solving for $\mathbf{B}$, it can only be made up of pseudovectors that exist in the problem

\text{charged plane}

\text{surface current}

only directions in problem is normal $\hat{\mathbf{m}}$ $\hat{\mathbf{m}}$ is a vector $\mathbf{E} \times \hat{\mathbf{m}}$

only directions are the vectors $\hat{\mathbf{K}}$ and $\hat{\mathbf{m}}$. But $\mathbf{B}$ can only be made of pseudovectors

$\Rightarrow \mathbf{B} \propto (\hat{\mathbf{K}} \times \hat{\mathbf{m}})$
Maxwell's equations apply exactly to the true microscopic electric and magnetic fields that arise from all charges and currents.

\[ \nabla \cdot \mathbf{b} = 0, \quad \nabla \times \mathbf{\varepsilon} + \frac{1}{c} \frac{\partial \mathbf{\varepsilon}}{\partial t} = 0 \]

\[ \nabla \cdot \mathbf{\varepsilon} = 4\pi j, \quad \nabla \times \mathbf{\varepsilon} = \frac{4\pi}{c} \frac{\partial \mathbf{\varepsilon}}{\partial t} + \frac{1}{c} \frac{\partial \mathbf{\varepsilon}}{\partial t} \]

where \( \mathbf{\varepsilon} \) and \( \mathbf{b} \) are microscopic fields from total charge density \( j \) and current density \( \dot{j} \).

However, in most problems involving macroscopic objects, if we took \( j \) and \( \dot{j} \) to describe charge and current of each individual atom in a material, then they, and the resulting \( \mathbf{\varepsilon} \) and \( \mathbf{b} \), would be enormously complicated functions varying rapidly over distances \( \sim 1 \times 10^{-8} \) cm and times \( \sim 10^{-12} \) sec.

In classical EMT we are generally concerned with phenomena that vary extremely slowly compared to these length and time scales.
Rather than worry about the microscopic details of $g$ and $f$, as resulting $E$ and $b$, we want to describe phenomena in terms of averaged, smooth varying given averaged quantities that are smoothly varying at the atomic scale. This results in what are known as the macroscopic Maxwell equations.

Dielectric Materials

A dielectric material is an insulator. Electrons are bound to the ionic cores of the atoms. When no electric field is present, the averaged $\mathbf{p}$ in the dielectric vanishes! One might therefore think that electrodynamics in a dielectric is just due to whatever “extra” or “free” charge is added to the dielectric, however this is not true due to the phenomena of “polarization”.

\[ \mathbf{E} = 0 \quad \mathbf{E} > 0 \]

- electron cloud centered on ionic core
- dipole moment vanishes

- electron cloud centered on ionic core
- displaced \( \mathbf{d} \propto \mathbf{E} \)
- atom is “polarized” has dipole moment \( \mathbf{p} = \mathbf{q} \times \mathbf{g} \)

\[ \mathbf{p} = \alpha \mathbf{E} \]
Polarization density $\mathbf{P}(\mathbf{r}) = \sum_i \mathbf{p}_i \delta(\mathbf{r} - \mathbf{r}_i)$

dipole moment of atom $i$
at position $\mathbf{r}_i$

Polarization density $\mathbf{P}$ can give rise to regions of net charge – sometimes called "bound charge".

**Example**

\[ \mathbf{E} = 0 \quad \text{uniform} \quad \mathbf{E} \rightarrow \quad \mathbf{E} > 0 \]

in terms of averaged charge

For a non-uniform $\mathbf{E}$, atoms are more strongly polarized where $\mathbf{E}$ is largest.

\[ \mathbf{E} \rightarrow \quad \mathbf{E}_{\text{strong}} \quad \mathbf{E}_{\text{weak}} \]

\[ \mathbf{E} \rightarrow \quad \mathbf{E} > 0 \]

For uniform $\mathbf{P}$, build up surface charge $\sigma_s$

For nonuniform $\mathbf{P}$, also can build up vol charge density $\mathbf{J}.$