\[\ln \chi = \frac{L_0 \lambda_{1-1}}{2\pi} \sum_{n=0}^{\infty} \int dk \ln \left[ 1 + z e^{-\beta \left( \frac{k^2}{2m} + \frac{\lambda}{4\pi} (n+\frac{1}{2}) \right)} \right] \]

Once we find \( \chi \), we can compute \( M \), the total dipole moment, as follows:

Total energy in magnetic field is \( E(B) = E(B=0) - MB \)

\[\Rightarrow M = -\frac{\partial E}{\partial B} = -\langle \frac{\partial H}{\partial B} \rangle \quad H \text{ is Hamiltonian} \]

Now \(-\frac{\partial H}{\partial B} = -\sum_{\alpha} e^{-\beta (H(\alpha) - \mu N_\alpha)} \frac{\partial H}{\partial B} \)

\[= \frac{1}{\beta} \sum_{\alpha} \frac{\partial}{\partial B} \ln \sum_{\alpha} e^{-\beta (H(\alpha) - \mu N_\alpha)} \]

\[= \frac{1}{\beta} \frac{\partial}{\partial B} \ln \sum_{\alpha} e^{-\beta (H(\alpha) - \mu N_\alpha)} \]

\[M = \frac{1}{\beta} \frac{\partial}{\partial B} \ln \chi \quad \text{or using grand potential} \]

\[\Sigma = -k_B T \ln \chi \]

\[\Rightarrow M = -\frac{\partial \Sigma}{\partial B} \]
\[ V = \frac{4}{3} \pi \left( \frac{l_1 l_2 l_3}{2} \right) \]

\[ \ln Z = \frac{V}{2 \pi^2} \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} dk_3 \ln \left[ 1 + e^{-\beta \left( \frac{\hbar^2 k_3^2}{2m} + \hbar \omega_c (n + \frac{1}{2}) \right)} \right] \]

Define function \( h(x) = \int_{-\infty}^{\infty} e^{\beta \left( \frac{\hbar^2 k_3^2}{2m} - x \right)} \)

Then

\[ \ln Z = \frac{V}{2 \pi^2} \sum_{n=0}^{\infty} h(\mu - \hbar \omega_c (n + \frac{1}{2})) \]

Consider the limit of very weak magnetic field \( \hbar \omega_c \ll k_B T \)

In this case many Landau levels occupied. We might think to replace \( \sum \) by \( \int \), but it turns out that this would remove all dependence on \( B \). To do better we need to use Euler summation formula (Patria 8.2 eq (44))

\[ \sum_{n=0}^{\infty} f(n + \frac{1}{2}) = \int_{0}^{\infty} f(x) dx + \frac{1}{24} f'(0) \]

Apply to the above

\[ \ln Z = \frac{V}{2 \pi^2} \int_{0}^{\infty} dk_3 h(\mu - \hbar \omega_c k_3) + \frac{V g}{2 \pi^2} \frac{1}{24} (-\hbar \omega_c) \frac{d f(\mu)}{d \mu} \]

\[ = \frac{V}{2 \pi^2} \int_{0}^{\infty} dy h(y) \int_{-\infty}^{\infty} dy' h(y') \left( \frac{1}{\hbar \omega_c} - \hbar \omega_c \frac{d f(\mu)}{d \mu} \right) \]

Use \( \Phi_0 = \frac{\hbar c}{e} \quad \omega_c = \frac{eB}{mc} \)
\[ \ln \mathcal{Z} = \frac{V}{2\pi} \frac{Z B e}{kT} \left[ \int_{-\infty}^{\mu} dy \, h(y) - \frac{(h \omega c)^2}{24} \frac{d^4 h (\mu)}{d \mu} \right] \]

\[ = \frac{V}{2\pi} \frac{Z B e}{kT} \frac{mc}{\hbar} \left[ \int_{-\infty}^{\mu} dy \, h(y) - \frac{(h \omega c)^2}{24} \frac{d^4 h (\mu)}{d \mu} \right] \]

\[ = \frac{V m}{\hbar^2} \left[ \int_{-\infty}^{\mu} dy \, h(y) - \frac{(h \omega c)^2}{24} \frac{d^4 h (\mu)}{d \mu} \right] \]

Grand potential

\[ \Sigma(T, V, \mu, B) = -k_B T \ln \mathcal{Z} = -k_B T V m \left[ \int_{-\infty}^{\mu} dy \, h(y) - \frac{(h \omega c)^2}{24} \frac{d^4 h (\mu)}{d \mu} \right] \]

\[ \text{under}\ \text{of}\ \mathcal{B} \]

1st term gives

\[ \mathcal{Z} (T, V, \mu, 0) = -k_B T V m \left( \int_{-\infty}^{\mu} dy \, h(y) \right) \]

Now make

\[ -N = \left( \frac{\partial \Sigma}{\partial \mu} \right)_{T, V, B=0} = -k_B T V m \left( \int_{-\infty}^{\mu} dy \, h(y) \right) \]

\[ -\left( \frac{\partial N}{\partial \mu} \right)_{T, V} = \left( \frac{\partial^2 \Sigma}{\partial \mu^2} \right)_{T, V} = -k_B T V m \frac{d^4 h (\mu)}{d \mu} \]

Combine to get

\[ \Sigma(T, V, \mu, B) = \Sigma(T, V, \mu, 0) + \frac{(h \omega c)^2}{24} \left( \frac{\partial N}{\partial \mu} \right)_{T, V} \]
\[ \Sigma (T, v, M, B) = \Sigma (T, v, M, 0) + \left( \frac{\hbar e B}{m_c} \right)^2 \frac{1}{24} \left( \frac{\partial N}{\partial M} \right)_{T, V} \]

\[ M_B = \frac{e^2}{2mc} \]

\[ \Sigma (T, v, M, B) = \Sigma (T, v, M, 0) + \frac{1}{6} M_B B^2 \left( \frac{\partial N}{\partial M} \right)_{T, V} \]

Nour \[ \frac{\partial N}{\partial M} = \frac{2}{\partial M} \left[ V \int d\varepsilon \: g(\varepsilon) \: f(\varepsilon, M) \right] \]

\[ = V \int d\varepsilon \: g(\varepsilon) \: \frac{\partial f}{\partial \varepsilon} \]

\[ = V \int d\varepsilon \: g(\varepsilon) \left( \frac{\partial f}{\partial \varepsilon} \right) \]

\[ \propto V g(\varepsilon_F) \to \text{lowest order in Sommerfeld expansion} \]

\[ \Sigma (T, v, M, B) = \Sigma (T, v, M, 0) + \frac{V}{6} \mu_B^2 g(\varepsilon_F) B^2 \]

magnetization

\[ M = -\frac{\partial \Sigma}{\partial B} = -\frac{V}{3} \mu_B^2 g(\varepsilon_F) B \]

magnetic susceptibility

\[ \chi_L = \frac{\partial (M/N)}{\partial B} = -\frac{1}{3} \mu_B^2 g(\varepsilon_F) <0 \Rightarrow \text{diamagnetic} \]

\[ \chi_L = -\frac{1}{3} \chi_p \]

\[ \chi_p = \mu_B^2 g(\varepsilon_F) \]
Total magnetic susceptibility for a free electron gas is

\[ \chi_{\text{tot}} = \chi_p + \chi_L = \frac{2}{3} \chi_p \]

For electrons in metal (as opposed to free electrons), \( \chi_p = M_B g(\varepsilon_F) \) comes from interaction with electron spin.

\[ M_B = \frac{\hbar e}{2mc} \]

\( m \) is rest mass of electron.

\( \chi_L \) comes from orbital motion of electrons near Fermi energy. For such electrons, the energy spectrum is

\[ \varepsilon(k) \approx \frac{\hbar^2 k^2}{2m^*} \]

where \( m^* \) is the effective mass of motion in the periodic potential of the ions (take \( \text{P521} \)).

The \( M_B \) in \( \chi_L \) should therefore really be \( M_B^* = \frac{\hbar e}{2m^*c} \)

Then \( \chi_L = -\frac{1}{3} \left( \frac{m}{m^*} \right) \chi_p \)

We derived \( \chi_p \) and \( \chi_L \) by separately considering effects of spin and orbital motion. One could get the same results by combining the derivations into a single one that includes both effects.

Note that \( \chi_L = -\frac{1}{3} M_B^2 g(\varepsilon_F) \)

\[ g(\varepsilon_F) = \frac{3}{2} \frac{m}{\varepsilon_F} \]

\[ = -\frac{1}{3} \left( \frac{\hbar e}{2mc} \right)^2 \frac{3}{2} \frac{m}{\varepsilon_F} \]
Note: Landau diamagnetism is a purely quantum mechanical effect — does not exist classically.

Classical N particle partition function:

\[ Q_N = \frac{Q_1^N}{N!} \]

where

\[ Q_1 = \int \frac{d^3r}{\hbar^2} \int \frac{d^3p}{\hbar^3} e^{-\beta \mathcal{H}} \]

\[ = \int d^3r \int d^3p \ e^{-\beta \left( \frac{1}{2m} (\vec{p} + e \vec{A}(r))^2 \right)} \]

just substitute \( \vec{p}' = \vec{p} + e \vec{A}(r) \) to get

\[ Q_1 = \int d^3r \int d^3p' \ e^{-\beta \left( \frac{\vec{p}'^2}{2m} \right)} \]

Same as partition function with \( B = 0 \)!

So \( Q_1 \) is independent of \( B \)

\[ \Rightarrow \chi = -\frac{1}{N} \frac{\partial^2 \Sigma}{\partial B^2} = 0 \]

Orbital motion gives no magnetization

\[ M = -\frac{\partial \Sigma}{\partial B} = 0 \]

Bohr–VanLeeuwen Theorem
Amusing aside:

The classical result \( M = 0 \) may seem confusing if one considers that the classical electron in a uniform \( B \) undergoes a circular motion \( \Rightarrow \) electron is effectively a current loop \( \Rightarrow \) should have an orbital magnetic moment from classical \( \mathbf{F} \times \mathbf{i} \) (where \( \mathbf{i} \) is electric current).

Each electron goes around in a circular orbit and so the contributions from all electrons should add and give \( M \neq 0 \).

Argument fails when one considers electrons traveling close to the finite boundaries of the system.

[Diagram showing clockwise orbits in interior and counter-clockwise orbits on the surface.

Moments from the interior orbits and moments from skipping states exactly cancel. Proof: For any fixed \( \mathbf{i} \), at any point \( \mathbf{F} \), we get contributions to current from electrons going in opposite directions. These always cancel.

True even near boundary.

When we average over all electron orbits the resulting average current at any point \( \mathbf{F} \) in the system vanishes.

\( \Rightarrow \) no magnetic moment.

Sometimes it is important to consider in detail what happens at the boundaries!