Superfluid 4He

Phase diagram

\[ P \]

2.26 atm - superfluid

\[ \text{liquid-gas transition} \]

\[ T_\lambda = 2.18^\circ K \]

\[ T \]

specific heat

\[ \frac{C_v}{Nk_B} \]

diverges at \( T_c \) where "\( \lambda \)" line is crossed

1. Shape of \( C_v \) reminds one of the cusp in \( C_v \) for Bose-Einstein condensation in ideal Bose gas, although here \( C_v \) diverges at \( T_c \)

2. In superfluid phase, some fraction \( \frac{\rho_s(T)}{\rho} \)
   \[ \rho_s = \text{superfluid density}, \quad \rho = \text{total density} \]
   of the fluid flows without any dissipation and carries no entropy
   - reminds one of the condensate of Bose-Einstein condensation
3) for $^4$He mass $m = 6.65 \times 10^{-24}$ g, and $^4$He fluid specific volume $V = 27.6$ cm$^3$/mole, one finds a Bose-Einstein condensation temperature of $T_c = 3.13$ K, not far from $T_A = 2.18$ K.

These observations suggested that superfluidity in $^4$He was related to Bose-Einstein condensation (BEC) of an ideal gas.

However

1) BEC transition line in $P$-$T$ plane, $P_c \propto T^{9/8}$ has a positive slope, whereas $T$-line of $^4$He has a negative slope.

3) As demonstrated by Landau, the condensate of BEC in an ideal Bose gas is NOT a superfluid.

Landau's argument

Consider a condensate (all particles in same state) of total mass $M$, flowing with velocity $\vec{v}$ down a pipe. It has total momentum $\vec{P} = MV$.

In the rest frame of the fluid, consider an excitation that is created with momentum $\vec{k}$ and energy $E(\vec{k})$. 

Back in the rest frame of the pipe, the change in energy due to the creation of the excitation is
(in the limit of large total mass $M \to \infty$)

$$
\Delta E = \frac{\left(\hat{p} + \vec{p}\right)^2}{2M} + \epsilon(\vec{p}) - \frac{\vec{p}^2}{2M}
$$

$$
= \frac{\epsilon \hat{p} \cdot \vec{p}}{2M} + \frac{\vec{p}^2}{2M} + \epsilon(\vec{p})
$$

$$
\Delta E = \epsilon(\vec{p}) + \frac{\vec{p} \cdot \vec{v}}{M} \quad \text{since} \quad \frac{\vec{p}}{M} = \vec{v}, \quad \text{and}
$$

$$
\frac{\vec{p}^2}{2M} \to 0 \quad \text{for large} \quad M
$$

If $\Delta E < 0$, then it will be energetically favorable to create such excitations of
momentum $\vec{p}$ — the motion of the fluid will therefore excite particles out of the condensate
and degrade the flow, i.e., the system will not display superfluidity.

$$
\Delta E < 0 \Rightarrow \epsilon(\vec{p}) < -\vec{p} \cdot \vec{v}
$$

for $\vec{p} \parallel -\vec{v}$, then $\epsilon(\vec{p}) < \frac{\vec{p} \cdot \vec{v}}{\vec{p}}$

For an ideal Bose gas, $\epsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$, so

$$
\epsilon(\vec{p}) = \frac{\vec{p}}{2m} \quad \text{will always be less than flow velocity} \quad \vec{v}, \quad \text{for sufficiently small} \quad \vec{p}.
So the condensate of an ideal Bose gas, flowing with any velocity no matter how small, will always excite particles out of the condensate into low-lying excited states, and hence will not be a superfluid.

However, from explicit measurement such as neutron scattering and specific heat, it was known that the low-lying excitation spectrum of \(^4\)He was not free-particle like, but rather as shown below:

![Graph showing the spectrum of energy \(E(p)\) vs. momentum \(p\).](image)

At small \(p\), the spectrum is linear:

\[
E(p) \approx Cp
\]

At higher \(p\) there is a dip:

\[
E(p) \approx A + \frac{(p-p_0)^2}{2m^*}
\]

Excitations in the low \(p\) linear region are phonons, or sound modes. Excitations near the dip were called "rotons."

\[
\frac{A}{k_B} = 8.7^\circ K
\]

\[
C = 2.4 \times 10^4 \text{ cm/s}
\]

\[
m^* = 0.16 m_{He}
\]

\[
p_0 = 1.9 \text{ A}^{-1}
\]
As the fluid flow velocity \( v \) is increased from zero, the condition \( \frac{\varepsilon(p)}{p} = v \) will first be reached at the velocity \( v^* \) shown below.

\[
\begin{align*}
\frac{\varepsilon(p)}{p} &= v \\
\text{v slope is } v^* &= \frac{\varepsilon(p_o)}{p_o} = \frac{\Delta}{p_o} \\
\text{\( v^* \approx 60 \text{ m/s} \)}
\end{align*}
\]

for \( v < v^* \), \( \frac{\varepsilon(p)}{p} > v \) for all values of \( p \), and so no particles will be excited out of the condensate by the flow. The fluid is now a superfluid!

Experimentally, while superfluidity does exist in \( ^4 \text{He} \) at low flow velocities, the critical velocity \( v_{\text{crit}} \) above which the flow becomes dissipative, is much lower than the \( v^* \) above, and is strongly dependent on the size and shape of the container. This is due to the onset of turbulence, accompanied by the proliferation of quantized vortices in the condensate flow.
The fact that the spectrum \( G(p) \) is linear at small \( p \), and not free particle-like, is due to the fact that the \(^4\)He atoms are not an ideal gas — in fact, the \(^4\)He atoms are strongly interacting. This should not be surprising — the \(^4\)He is a liquid and not a gas, particles collide with other gaseous nuclei to sound modes as the low-lying excitations, rather than free "ballistic" motion of individual atoms.

One can show that for an interacting Bose gas, even for weak interactions, the small \( p \) part of the excitation spectrum always is phonon-like, i.e. linear (see Pathria §10.3 or the book PHYS 509!). Superfluidity can only exist for interacting Bose systems.

Because interactions between \(^4\)He atoms are not weak, there is no good microscopic theory of superfluidity in \(^4\)He. Calculations can, however, be performed for weak interactions, which has lead people to look for superfluidity or BEC in weakly interacting Bose gases. This search was finally successful with the observation of BEC in magnetically trapped, laser cooled, dilute atomic gases (produced two Nobel prizes).
Bose-Einstein Condensation in laser cooled gases

Gases of alkali atoms Li, Na, K, Rb, Cs
- all have a single s-electron in outermost shell,
- important for trapping by laser cooling
- use isotopes such that total intrinsic spin of all electrons and nucleons add up to an integer \( \hbar \)
- atoms are bosons
- all have a net magnetic moment - used to confine dilute gas of atoms in a "magnetic trap"
- use "laser cooling" to get very low temperatures in low density gases, to try and see BEC

**magnetic trap \( \Rightarrow \) effective harmonic potential for atoms**

\[
V(r) = \frac{1}{2} mlw_0^2 r^2 \quad \text{with} \quad w_0 = \frac{\pi}{3} \times 100 \text{ Hz}
\]

- 1995: \( 10^3 \) atoms with \( T_c \approx 100 \text{ nK} \)
- 1999: \( 10^8 \) atoms with \( T_c \approx \mu \text{K} \)

**How was BEC in these systems observed?**

- energy levels of ideal (non-interacting)
  - bosons in harmonic trap

\[
E(n_x, n_y, n_z) = (n_x + n_y + n_z + \frac{3}{2}) \hbar w_0
\]

- \( n_x, n_y, n_z \) integers

**ground state condensate wavefunction**

\[
\psi_0(r) \sim e^{-r^2/2a^2} \quad \text{with} \quad a = \left( \frac{\hbar}{mw_0} \right)^{\frac{1}{2}}
\]

- \( a \approx 1 \mu \text{m} \) for current traps
Conclude that the spatial extent is

\[ \text{The spatial extent of the } n^{th} \text{ excited energy level is roughly} \]

\[ m\nu_0^2 \langle r^2 \rangle \sim E(n) \approx n\hbar\omega_0 \]

\[ \Rightarrow \langle r^2 \rangle \sim \frac{n\hbar}{m\nu_0} \quad \text{or} \quad \sqrt{\langle r^2 \rangle} = \left( \frac{n\hbar}{m\nu_0} \right)^{1/2} \]

For \( k_B T \gg \hbar\omega_0 \), the atoms are excited up to level \( n \sim \frac{k_B T}{\hbar\omega_0} \)

\[ \Rightarrow \text{spatial extent of the normal component of the gas is} \]

\[ R \sim \left( \frac{n\hbar}{m\nu_0} \right)^{1/2} \sim \left( \frac{\hbar k_B T}{\hbar m\nu_0^2} \right)^{1/2} = \left( \frac{k_B T}{m\nu_0^2} \right)^{1/2} \]

\[ R \sim a\left( \frac{k_B T}{\hbar\omega_0} \right)^{1/2} \gg a \]

If \( T \leq T_c \), the BEC transition temperature, then for \( T \gg T_c \) one sees a more or less uniform cloud of atoms with radius \( R \sim a\left( \frac{k_B T}{\hbar\omega_0} \right)^{1/2} \gg a \).

But when one cools to \( T < T_c \), one now has a finite fraction of the atoms condensed in the ground state, superimposed on the atomic cloud of radius \( R \); one sees the growth of a sharp peak in density at the center of cloud—the peak has a radius \( a < R \).
To find $T_c$, 

\[
m = m_0 + \frac{1}{e^{(n_x+n_y+n_z)kT/k_B T} - 1} \int_0^\infty \int_0^\infty \int_0^\infty \frac{1}{e^{x+y+z} - 1} \, dx 
\]

\[
= m_0 + \left(\frac{k_B T}{\hbar \omega_0}\right)^3 \int \mathcal{J}(3)
\]

at $T_c$, $m_0 < 0 \Rightarrow k_B T_c = \hbar \omega_0 \left(\frac{m}{\mathcal{J}(3)}\right)^{1/3}$

\[
\text{condensate density } \frac{m_0(T)}{m} = 1 - \left(\frac{T}{T_c}\right)^3
\]

\[
\mathcal{J}(3) = 1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \ldots
\]

different from ideal free gas due to presence of magnetic trapping potential.