The calculation

Let \( g(\epsilon) \) be the density of states when \( B=0 \).

When \( B \neq 0 \), the density of states for \( \uparrow \) and \( \downarrow \) electrons are

\[
\begin{align*}
g_\uparrow(\epsilon + \mu_B B) & = \frac{1}{2} g(\epsilon) \Rightarrow g_\uparrow(\epsilon) = \frac{1}{2} g(\epsilon - \mu_B B) \\
g_\downarrow(\epsilon - \mu_B B) & = \frac{1}{2} g(\epsilon) \Rightarrow g_\downarrow(\epsilon) = \frac{1}{2} g(\epsilon + \mu_B B)
\end{align*}
\]

The density of \( \uparrow \) and \( \downarrow \) electrons is then

\[
m_\pm = \int_{-\infty}^{\infty} d\epsilon \ g_\pm(\epsilon) f(\epsilon, \mu(B))
\]

where

\[
f(\epsilon, \mu(B)) = \frac{1}{e^{(\epsilon - \mu(B))/k_B T} + 1}
\]

\( \mu(B) \) is the chemical potential — it might depend on \( B \).

We will consider only the case that

\[
\mu_B B < \mu(B) \approx E_F
\]

\( \mu_B B \) << \( \mu(B) \approx E_F \)

...spin interaction is small compared to \( E_F \).
First we will show:

\[ \mu(B) \approx \mu(B=0) \left[ 1 + O \left( \frac{MBB}{E_F} \right)^2 \right] \]

Consider total density of electrons

\[ m = n_+ + n_- = \int_{-\infty}^{\infty} d\epsilon \quad f(\epsilon, \mu(B)) \left[ g_+(\epsilon) + g_-(\epsilon) \right] \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} d\epsilon \quad f(\epsilon, \mu(B)) \left[ g(\epsilon - MB) + g(\epsilon + MB) \right] \]

\[ = \frac{1}{2} \int_{-\infty}^{\infty} d\epsilon \quad g(\epsilon) \left[ f(\epsilon + MB, \mu(B)) + f(\epsilon - MB, \mu(B)) \right] \]

Expand for small \( \frac{MB}{\mu} \ll 1 \)

\[ m = \frac{1}{2} \int_{-\infty}^{\infty} d\epsilon \quad g(\epsilon) \left[ f(\epsilon, \mu) - \frac{df}{d\mu} MB + f(\epsilon, \mu) + \frac{df}{d\mu} MB \right] \]

\[ = \int_{-\infty}^{\infty} d\epsilon \quad g(\epsilon) \cdot f(\epsilon, \mu) \]

Now since \( m \) does not change when one applies \( B > 0 \), and we have \( m = \int_{-\infty}^{\infty} d\epsilon \quad g(\epsilon) \cdot f(\epsilon, \mu(B=0)) \) when \( B = 0 \),

\[ \Rightarrow \mu(B) = \mu(B=0) \]

Corrections come from next order in the expansion \( \frac{df}{d\mu} (MBB) \) and one order \( \left( \frac{MBB}{\mu} \right)^2 \).
Now we compute

\[ M = \frac{-e B (m_+ - m_-)}{V} = \mu_B (m_- - m_+) \]

\[ M = \mu_B \int_{-\infty}^{\infty} f(\epsilon, \mu) \left[ g_-(\epsilon) - g_+(\epsilon) \right] \]

\[ = \mu_B \int_{-\infty}^{\infty} f(\epsilon, \mu) \left[ \frac{1}{2} g(\epsilon + \mu B) - \frac{1}{2} g(\epsilon - \mu B) \right] \]

\[ = \frac{1}{2} \mu_B \int_{-\infty}^{\infty} g(\epsilon) \left[ f(\epsilon, \mu + \mu B) - f(\epsilon, \mu - \mu B) \right] \quad \text{as before} \]

\[ \exp \left[ f(\epsilon, \mu \pm \mu B) \right] = f(\epsilon, \mu) \pm \frac{df}{d\mu} \mu B \]

\[ M = \frac{1}{2} \mu_B \int_{-\infty}^{\infty} g(\epsilon) \left[ 2 \frac{df}{d\mu} \mu B \right] \]

\[ = \mu_B^2 B \int_{-\infty}^{\infty} g(\epsilon) \left( -\frac{df}{d\epsilon} \right) \quad \text{since } \frac{df}{d\mu} = -\frac{df}{d\epsilon} \]

To lowest order in temperature \(-\frac{df}{d\epsilon} \propto \delta(\epsilon - \mu)\) with \(\mu = \epsilon_F\)

\[ \frac{M}{V} = \mu_B^2 B g(\epsilon_F) \]

\[ \text{magnetic susceptibility } \chi = \frac{2M}{MV} \]

\[ \text{Pauli susceptibility } \chi_p = \mu_B^2 g(\epsilon_F) \]

\[ \text{density of states at } \epsilon_F \]

\[ \chi_p \approx \frac{3}{2} \frac{m}{\epsilon_F} \]

\[ \Rightarrow \chi_p = \mu_B^2 \frac{3}{2} \frac{m}{\epsilon_F} \]

\[ \chi_p > 0 \Rightarrow \text{paramagnetic} \]
Compare this to classical result. Average magnetization of a single spin is:

\[ <m> = \mu_B \tanh (\beta \mu_B B) \]

\[ \frac{M}{V} = \frac{<m>}{N} = \mu_B m \tanh (\beta \mu_B B) \]

\[ \chi = \frac{d(M/V)}{dB} \]

At low \( T \to 0 \), \( \tanh (\beta \mu_B B) \to 0 \), \( \frac{M}{V} \to \mu_B m \) (all spins aligned).

Compare to quantum case:

\[ \frac{M}{V} = \frac{3}{2} \frac{m}{e_F} \mu_B^2 B \]

Smaller than classical result by factor \( \frac{3}{2} \frac{\mu_B^2}{e_F} \ll 1 \)

At high \( T \) (\( \beta \to 0 \)), \( \tanh (\beta \mu_B B) \to \beta \mu_B B \)

\[ \frac{M}{V} = \frac{\mu_B^2 B m}{k_B T} \]

\[ \chi = \frac{\mu_B^2 m}{k_B T} \sim \frac{1}{T} \]

Compare to quantum case - at room temp finite \( T \) corrections remain negligible and still

\[ \chi_p = \frac{\mu_B^2 m}{k_B T} \frac{3}{2} \frac{m}{e_F} \text{ instead of } \frac{1}{T} \]

Smaller than classical by factor \( \frac{3}{2} \frac{(kB T)}{e_F} \ll 1 \)
Ideal Bose Gas

Bose-Einstein Condensation

Bose occupation function

\[ n(\varepsilon) = \frac{1}{z^{-1}e^{\beta \varepsilon} - 1} \]

We had for the density of an ideal (non-interacting) Bose gas

\[ N = \frac{1}{V} \sum_k \frac{1}{z^{-1}e^{\beta \varepsilon(k)} - 1} = \frac{1}{(2\pi)^3} \int_0^\infty dk \frac{4\pi k^2}{z^{-1}e^{\beta k^2/2m} - 1} \]

recall, we need \( z \leq 1 \) for the occupation number

at \( \varepsilon(k=0) = 0 \) to remain positive \( m(0) > 0 \)

\[ m(0) = \frac{1}{z^{-1} - 1} = \frac{z}{1 - z} \Rightarrow z \leq 1 \]

\( z = e^{\beta M} \Rightarrow M \leq 0 \)

Substitute variables \( y = \frac{k^2}{2m} \Rightarrow k = \sqrt{\frac{2my}{\beta \hbar^2}} \)

\[ dk = \frac{2my}{\beta \hbar^2} dy \]

\[ \Rightarrow N = \left( \frac{2m}{\beta \hbar^2} \right)^{3/2} \frac{4\pi}{(2\pi)^3} \frac{1}{2} \int_0^\infty dy \frac{y^{1/2}}{z^{-1}e^{y} - 1} \]

\[ \frac{N}{V} = \frac{1}{\lambda^3} \tilde{g}_{3/2}(z) \]

where \( \lambda = \left( \frac{\hbar^2}{2\pi m k_B T} \right)^{1/2} \)

thermal wavelength

\[ \tilde{g}_{3/2}(z) = \frac{z}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{z^{-1}e^{y} - 1} \]
Consider the function

$$g_{3/2}(z) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{z^{-1} e^y - 1} = z + \frac{z^2}{2^{3/2}} + \frac{z^3}{3^{3/2}} + \cdots$$

$g_{3/2}(z)$ is a monotonic increasing function of $z$ for $z \leq 1$.

As $z \to 1$, $g_{3/2}(z)$ approaches a finite constant

$$g_{3/2}(1) = 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \cdots \approx 5(3/2) \approx 2.612$$

The Riemann zeta function.

We can see that $g_{3/2}(1)$ is finite as follows:

$$g_{3/2}(1) = \frac{2}{\sqrt{\pi}} \int_0^\infty dy \frac{y^{1/2}}{e^y - 1}$$

as $y \to \infty$ the integral converges. Integral is largest at small $y$ (recall small $y$ corresponds to low energy where $m(\xi)$ is largest).

For small $y$ we can approximate

$$\frac{y^{1/2}}{e^y - 1} \approx \frac{y^{1/2}}{y}$$

$$\int_0^{y^*} \frac{y^{1/2}}{e^y - 1} dy \approx \int_0^{y^*} \frac{y^{1/2}}{y} dy = \left[ \frac{y^{3/2}}{3/2} \right]_0^{y^*}$$

So we see the integral also converges at its lower limit $y \to 0$.

![Graph of $g_{3/2}(z)$ vs $z$]
So we conclude

\[ n = \frac{N}{V} = \frac{\sqrt[3]{3/2} \langle z \rangle}{\lambda^3} \leq \frac{\sqrt[3]{3/2} \langle 1 \rangle}{\lambda^3} = \frac{2.612}{\lambda^3} = 2.612 \left( \frac{2\pi mk_B T}{\hbar^2} \right)^{3/2} \]

But we now have a contradiction!

For a system with fixed density of bosons \( n \), as \( T \) decreases we will eventually get to a temperature below which the above inequality is violated!

The temperature is

\[ T_c = \left( \frac{m}{2.16} \right)^{2/3} \frac{\hbar^2}{2\pi mk_B} \]

Solution to the paradox:

When we made the approx \( \frac{1}{V} \sum_k \rightarrow \frac{1}{(2\pi)^3} \int dk \ 4\pi k^2 \)

we gave a weight \( \frac{4\pi k^2}{(2\pi)^3} \) to states with wavevector \( |k| \).

This gives zero weight to the state \( |k=0\rangle \), i.e. to the ground state. But as \( T \) decreases, more and more bosons will occupy the ground state, as it has the lowest energy. Thus when we apply the sum over an integral, we should treat the ground state separately.

\[ \frac{1}{V} \sum_k n(E(k)) \approx \frac{n(0)}{\sqrt{V}} + \frac{1}{(2\pi)^3} \int_0^\infty dk \ 4\pi k^2 \ n(E(k)) \]

The ground state with occupation \( n(0) \).

This term is important when \( n(0)/V \) stays finite as \( V \rightarrow \infty \), i.e. a macroscopic fraction of bosons occupy the ground state.
Then we get

\[ M = \frac{N}{V} = \frac{n_0}{V} + \frac{g_{3/2}(Z)}{\lambda^3} \]

\[ M = M_0 + \frac{g_{3/2}(Z)}{\lambda^3} \]

where \( M_0 = \frac{n_0}{V} \) density of bosons in ground state

For a system with fixed \( m \), at higher T one can always choose \( Z \) so that \( m = \frac{g_{3/2}(Z)}{\lambda^3} \) and \( M_0 = 0 \).

But when \( T < T_c \) it is necessary to have \( M_0 > 0 \).

Using \( n_0 = \frac{Z}{1-Z} \) we can write above as

\[ M = \frac{Z}{1-Z} \frac{1}{V} + \frac{g_{3/2}(Z)}{\lambda^3} \]

For \( T > T_c \), we will have a solution to the above for some fixed \( Z < 1 \). In thermodynamic limit \( V \to \infty \), the first term will then vanish, i.e. the density of bosons in the ground state vanishes.

As \( T \to T_c \), \( Z \to 1 \) and the first term \( \left( \frac{Z}{1-Z} \right) \left( \frac{1}{V} \right) \) stays finite to give the additional needed density at \( T < T_c \):

\[ \frac{Z}{1-Z} \frac{1}{V} = M_0 = m - \frac{g_{3/2}(1)}{\lambda^3} \]
To defines the Bose-Einstein transition temperature below which the system develops a finite density of particles in the ground state $m_0$. $m_0$ is also called the condensate density. The particles in the ground state are called the condensate.

$Z(T) \to 1$ as $T \to T_c$, $Z(T) = 1$ for $T \leq T_c$, $\mu(T) = 0$.

For $T \leq T_c$,

$$m_0(T) = m - 9^{3/2}(1) \frac{\lambda^3}{3^3} = m - 2.612 \left( \frac{2\pi m k_b T}{\hbar^2} \right)^{3/2}$$

$$m_0(T) = m \left( 1 - \left( \frac{T}{T_c} \right)^{3/2} \right)$$

The condensate density vanishes continuously as $T \to T_c$ from below.

At $T = 0$, all bosons are in condensate.

At $T > T_c$, all bosons are in the "normal state".

At $0 < T < T_c$, a macroscopic fraction of bosons are in the condensate, while the remaining fraction are in the normal state, called the "mixed state".