1) [30 points total] Consider a classical gas of $N$ identical, non-relativistic, non-interacting atoms confined to a box of volume $V$ in equilibrium at a temperature $T$. Each atom can be considered as a point particle, however it has an internal degree of freedom that can be in one of two possible states with energies $\epsilon_0 = 0$ and $\epsilon_1 > 0$ respectively.

a) [10 pts] Prove that the equation of state for the gas remains the familiar ideal gas law.

b) [10 pts] Find the chemical potential of the gas.

c) [10 pts] Find the specific heat per particle at constant volume of the gas and sketch it as a function of temperature.

2) [35 points total] Consider a non-interacting gas of $N$ identical non-relativistic spin $\frac{1}{2}$ fermions, confined to move in a one-dimensional system of length $L$.

a) [12 pts] The density of states, $g(\epsilon)$, is defined as the number of single particle states with energy $\epsilon$ per unit energy per unit length. Compute $g(\epsilon)$ for the gas.

b) [11 pts] Compute the Fermi energy of the gas as a function of its density $n = N/L$.

c) [12 pts] Compute the pressure of the gas at $T = 0$ as a function of its density $n$.

3) [35 points total] Consider a non-interacting gas of $N$ identical spin zero bosons. Suppose that the energy-momentum relationship for these particles is given by $\epsilon(p) = A|p|^s$, for some fixed positive numbers $A$ and $s$. The dimensionality of the gas is the number $d$, i.e. the “volume” of the gas is $V = L^d$, for a system of length $L$. In the following parts, we are considering behavior in the thermodynamic limit of $V \to \infty$.

a) [15 pts] For what values of $s$ and $d$ will there exist Bose-Einstein condensation at sufficiently low temperature?

b) [5 pts] For the case that there is Bose-Einstein condensation, write an expression that gives how the condensate density $n_0$ depends on the temperature $T$, the condensation temperature $T_c$, and the total density $n = N/V$.

c) [10 pts] Show that the pressure $p$ is related to the energy density $E/V$ by

$$p = \frac{sE}{dV}$$

d) [5 pts] Can a gas of photons in $d = 3$ undergo Bose-Einstein condensation?