1) [50 points total]

Consider a system in the canonical ensemble. Let \( U \equiv \langle E \rangle \) be the total average energy, and \( \Delta E \equiv E - U \) be the fluctuation away from this average. [Warning: only part (c) refers to an ideal gas; parts (a) and (b) refer to any general system.]

a) [15 pts] Derive the following expression between the specific heat at constant volume, \( C_V \), and fluctuations in the total energy \( E \),

\[
C_V = \frac{1}{k_B T^2} \left[ \langle E^2 \rangle - \langle E \rangle^2 \right] = \frac{1}{k_B T^2} \langle (\Delta E)^2 \rangle
\]  

(1)

b) [20 pts] Show that

\[
\langle (\Delta E)^3 \rangle = k_B^2 T^4 \left( \frac{\partial C_V}{\partial T} \right)_V + 2k_B^2 T^3 C_V
\]  

(2)

c) [15 pts] For a classical, non-relativistic, non-interacting ideal gas of \( N \) indistinguishable particles, show that,

\[
\langle \left( \frac{\Delta E}{U} \right)^2 \rangle = \frac{2}{3N}, \quad \text{and} \quad \langle \left( \frac{\Delta E}{U} \right)^3 \rangle = \frac{8}{9N^2}
\]  

(3)

2) [50 points total]

Consider a classical gas of \( N \) indistinguishable, non-interacting, particles with \textit{ultra-relativistic} energies, i.e. the energy - momentum relation of a particle is given by \( \epsilon(p) = |p|c \), with \( c \) the speed of light and \( p \) the particle’s momentum. The gas is in equilibrium at temperature \( T \), confined to a three dimensional box of volume \( V \).

a) [20 pts] Compute the \textit{canonical} partition function for this system.

b) [15 pts] Show that this system obeys the usual ideal gas law, \( pV = Nk_BT \).

c) [15 pts] Find the specific heat at constant \textit{pressure}, \( C_p \).