Quantum Many-particle Systems

N identical particles described by a wavefunction

\[ \psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N) \quad \vec{r}_i = \text{position particle } i \]

\[ = \psi(1, 2, \ldots, N) \quad s_i = \text{spin of particle } i \]

Indistinguishable particles \( \Rightarrow \) prob distribution \( |\psi|^2 \) should be symmetric under interchange of any pair of coordinates:

\[ |\psi(1, \ldots, i, \ldots, j, \ldots, N)|^2 = |\psi(1, \ldots, j, \ldots, i, \ldots, N)|^2 \]

\[ \Rightarrow \text{two possible symmetries for } \psi \]

1) \( \psi \) is symmetric under pair interchange:

\[ \psi(1, \ldots, i, \ldots, j, \ldots, N) = \psi(1, \ldots, j, \ldots, i, \ldots, N) \]

2) \( \psi \) is antisymmetric under pair interchange:

\[ \psi(1, \ldots, i, \ldots, j, \ldots, N) = -\psi(1, \ldots, j, \ldots, i, \ldots, N) \]

(1) Bose-Einstein statistics - particle called "bosons"

(2) Fermi-Dirac statistics - particle called "fermions"

For a general permutation \( \Pi \) that interchanges any number of pairs of particles

1) BE \( \Rightarrow \) \( \Pi \psi = \psi \)

2) FD \( \Rightarrow \) \( \Pi \psi = (-1)^p \psi \)

where \( p = \# \text{ pair interchanged} \)

\[ \begin{cases} +\psi & \text{for even permutation} \\ -\psi & \text{for odd permutation} \end{cases} \]
BE statistics are for particles with integer spin, \( s = 0, 1, 2, \ldots \)

FD statistics are for particles with half-integer spin, \( s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots \)

(proved by quantum field theory)

Consider non-interacting particles

\[
H(1, 2, 3, \ldots N) = H^{(1)}(1) + H^{(2)}(2) + \cdots H^{(N)}(N)
\]

\( \text{sum of single particle Hamiltonians} \)

\[
\Rightarrow \psi(1, 2, \ldots, N) = \phi_1(1) \phi_2(2) \cdots \phi_N(N)
\]

where \( \phi_i \) is an eigenstate of single particle \( H^{(i)} \)

with energy \( E_i \).

But \( \psi \) above does not have proper symmetry.

For BE

\[
\Psi = \frac{1}{\sqrt{N_p}} \sum_{P} \psi \quad \Rightarrow \quad \Psi = \phi_1 \phi_2 \cdots \phi_N \text{ as above}
\]

\( \sum \) over all permutations \( P \)

\( N_p = \# \text{ possible permutations of } N \text{ particles} = N! \)

For FD

\[
\Psi = \frac{1}{\sqrt{N_p}} \sum_{P} (-1)^P \psi
\]

You can verify that the above symmetrizing operators yield

\[
\begin{align*}
P_0 \Psi_{\text{BE}} &= \Psi_{\text{BE}} \quad \text{as desired} \\
P_0 \Psi_{\text{FD}} &= (-1)^P \Psi_{\text{FD}}
\end{align*}
\]
For $\Psi$ described by the $N$ single particle eigenstates

$$\Phi_{i_1} \Phi_{i_2} \cdots \Phi_{i_N},$$

the total energy is

$$E = \varepsilon_{i_1} + \varepsilon_{i_2} + \cdots + \varepsilon_{i_N} = \sum_j n_j \varepsilon_j,$$

where $n_j$ is the number of particles in state $\Phi_j$.

For PD statistics, $n_j = 0$ or 1 only possibilities.

This is because if $\Psi(1, 2, \ldots, N) = \Phi_1(1) \Phi_2(2) \Phi_3(3) \cdots \Phi_N(N)$

then when we construct

$$\Psi_{PD} = \frac{1}{\sqrt{N!}} \sum_{\pi} (-1)^\pi \Psi_{\pi}$$

then for every term in the sum $\Phi_1(i) \Phi_2(j) \Phi_3(k) \cdots \Phi_N(l)$

there must also be a term $(-1) \Phi_1(j) \Phi_2(i) \Phi_3(k) \cdots \Phi_N(l)$

so these cancel pair by pair

and we find $\Psi_{PD} = 0$

$\Rightarrow$ Pauli Exclusion Principle – no two particles can

occupy the same state or no two fermions can have

the same "quantum numbers".

For BE statistics there is no such restriction

and $n_j = 0, 1, 2, 3, \ldots$ any integer.