**Sommerfeld model of electrons in a conductor**

\[ \text{Fermi gas} - \text{high density/low temperature limit} \]

"Degenerate" Fermi gas

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Consider first \( T=0 \)

\[ \langle n(e) \rangle = \frac{1}{e^{\frac{e}{k(T=0)}} + 1} \]

as \( T \to 0 \) \( e^{\frac{e}{k(T=0)}} \to \)

\[ \begin{cases} \infty & e > \mu \\ 0 & e < \mu \end{cases} \]

\[ \Rightarrow \langle n(e) \rangle \to \begin{cases} 0 & e > \mu \\ 1 & e < \mu \end{cases} \]

\( \Rightarrow \) all states with \( e < \mu \) are filled, all states with \( e > \mu \) are empty. This is the \( T=0 \) ground state of the Fermi gas.

We therefore see that \( \mu(T=0) \) is the energy of the highest energy single particle state that is occupied in the ground state. One calls this energy the Fermi energy \( \epsilon_F = \mu(T=0) \)

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At \( T=0 \)

\[ N = \frac{g_s}{2} \sum_k \frac{1}{k^2} \text{ s.t. } k^2 \leq \epsilon_F \]

\[ = \frac{g_s}{2} \sqrt{\frac{4\pi}{(2\pi)^3}} \int_0^{k_F} k^2 \text{d}k = \frac{g_s}{2} \frac{\pi^3}{6} \epsilon_F \text{ where } \frac{k_F^2}{2m} = \epsilon_F \]

\[ m = \frac{N}{V} = \frac{g_s}{6\pi^2} k_F^3 = \frac{g_s}{6\pi^2} \left( \frac{2m\epsilon_F}{\pi^2} \right)^{3/2} \]

**---**

\( \epsilon_F = \frac{k_F^2}{2m} \left( \frac{6\pi^2 m}{g_s} \right)^{2/3} \quad k_F = \left( \frac{6\pi^2 m}{g_s} \right)^{1/3} \)

Relation between \( \mu(T=0) \) and density \( m = N/V \)
Now at finite $T$

region of energy where $\langle m \rangle$ differs from ground state ($T=0$) is a region of order $k_B T$ about $\mu$.

So the $T=0$ approx is good when $k_B T \ll \mu$

Since $\mu(0) = E_F$, we have

Using $\mu(0) = E_F$ we have

$$k_B T \ll \frac{\hbar^2}{2m} \left( \frac{6 \pi^2 n}{g_s} \right)^{3/2} \Rightarrow \frac{8 \pi m k_B T}{\hbar^2} \ll \frac{1}{4 \pi} \left( \frac{6 \pi^2 n}{g_s} \right)^{3/2}$$

$$\Rightarrow \frac{\hbar^2}{2m} \gg 4 \pi \left( \frac{g_s}{6 \pi^2 n} \right)^{3/2}$$

$$\Rightarrow \frac{m^3}{g_s} \gg \frac{4 \pi^{3/2}}{g_s} \Rightarrow g_s = \frac{4}{3 \sqrt{\pi}} g_s$$

So this is equivalent to a low $T$ or a high density limit $m^3 \gg 1$ — called the degenerate limit.

(just as the classical limit $k_B T \gg m^3 \ll 1$ was a high $T$, low density limit)

Fermi temperature $T_F = \frac{E_F}{k_B}$, Degenerate limit is $T \ll T_F$

For electrons in a metal, $T_F \approx 10^5$ K.

So electrons in a metal are always in the degenerate limit.
Energy in the degenerate limit \( T = 0 \)

\[
\frac{E}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon) \epsilon
\]

\[g(\epsilon) = C \sqrt{\frac{\epsilon}{V}}\]

with

\[C = \left( \frac{2\pi m}{h^2} \right) \frac{2^{3/2} \pi^{3/2}}{\sqrt{\pi}}\]

\[n = \frac{N}{V} = \int_0^{\epsilon_F} d\epsilon g(\epsilon)\]

\[
\Rightarrow \frac{E}{V} = C \int_0^{\epsilon_F} d\epsilon \epsilon^{3/2} = \frac{2}{3} C \epsilon_F^{5/2}
\]

\[
\Rightarrow \frac{E}{V} = \frac{3}{5} \frac{N}{V} \epsilon_F
\]

\[
E = \frac{3}{5} m \epsilon_F \quad \text{or} \quad \frac{E}{N} = \frac{3}{5} \epsilon_F
\]

\[\text{energy per volume} \quad \text{energy per particle}\]

Above gives \( T = 0 \) results. To get behavior at low \( T > 0 \),

or to get quantities such as \( C_v = \left( \frac{\partial E}{\partial T} \right)_V \), we need to

get the next order terms in a \( \log T \) expansion.

In general we need to do integrals of the form

\[
\int d\epsilon \frac{\bar{\phi}(\epsilon)}{e^{\beta \epsilon} + 1}
\]

\[
eq \int d\epsilon \bar{\phi}(\epsilon) m(\epsilon), \quad \bar{\phi}(\epsilon) \text{ some function}
\]

ex: to compute \( m \), \( \bar{\phi}(\epsilon) = g(\epsilon) \) to compute \( \frac{E}{V} \), \( \bar{\phi}(\epsilon) = g(\epsilon) \epsilon \)
\[ g(\varepsilon) = C \sqrt{\varepsilon} \]
\[ g(\varepsilon_F) = C \sqrt{\varepsilon_F} \]
\[ m = \frac{3}{2} C \varepsilon_F^{3/2} \]
\[ \Rightarrow C = \frac{3}{2} \frac{m}{\varepsilon_F^{3/2}} \]
\[ g(\varepsilon_F) = \frac{3}{2} m \varepsilon_F^{1/2} \frac{\varepsilon_F}{\varepsilon_F^{3/2}} = \frac{3}{2} \frac{m}{\varepsilon_F} \]
Transform variables to $X = \beta e$.

Then we want to do integrals of the form

$$
\Phi = \int_0^\infty dx \frac{\phi(x)}{e^{-x} + 1}
$$

$\phi(x)$ is any function of $x$.

For example, to get the "standard" function $f_n(z)$, we use $\phi(x) = \frac{1}{n(n-1)} x^{n-1}$

Define $\xi = \beta \mu = \ln z$

$$
\overline{\Phi} = \int_0^\infty dx \frac{\phi(x)}{e^{-x} + 1}
$$

Define $\psi(x) = \int_0^x \phi(x')dx'$, $f(x) = \frac{1}{[e^{-x} + 1]}$ semi-function

$$
\overline{\Phi} = \int_0^\infty dx \left( \frac{\partial \psi}{\partial x} \right) f(x) \quad \text{integrate by parts}
$$

$$
= \psi(x) f(x) \bigg|_0^\infty + \int_0^\infty dx \psi(x) \left( -\frac{2f}{\partial x} \right)
$$

$$
= \int_0^\infty dx \psi(x) \left( -\frac{2f}{\partial x} \right) \quad \text{since } \psi(0) = 0 \text{ and } f(\infty) = 0
$$

1st term vanishes

Now we use the fact that at low $T$, $\left( -\frac{\partial f}{\partial x} \right)$ is strongly peaked about $x = \xi$

Now we use the fact that at low $T$, $\left( -\frac{\partial f}{\partial x} \right)$ is strongly peaked about $x = \xi$
Expand \( \psi(x) \) about \( x = 5 \)

\[
\psi(x) = \sum_{n=0}^{\infty} \frac{d^n \psi}{dx^n} \bigg|_{x=5} \frac{(x-5)^n}{n!}
\]

\[
\Rightarrow \Phi = \sum_{n=0}^{\infty} \frac{d^n \psi}{dx^n} \bigg|_{x=5} \int_0^\infty dx \frac{(x-5)^n}{n!} \left( \frac{-2f}{2x} \right)
\]

Since \( \frac{-2f}{2x} \) is zero except for a region of order 1 about \( x = 5 \), we can replace the lower limit of the integral by \( -\infty \) without any noticeable change.

Then we can make a change of variables \( y = x - 5 \) and the integrals become

\[
\int_{-\infty}^{\infty} dy \frac{y^n}{n!} \left( \frac{-2f}{ay} \right) \quad \text{where} \quad f(y) = \frac{1}{e^y + 1}
\]

\[
\text{Now} \quad \frac{-2f}{ay} = \frac{e^y}{(e^y + 1)^2} = \frac{e^y}{e^{2y} + 2e^y + 1} = \frac{1}{e^y + 2 + e^{-y}}
\]

is symmetric about \( y = 0 \).

\Rightarrow all the integrals for \( m \) odd vanish!
To sum over only even terms, let $n = 2n$

$$\Phi = \sum_{n=0}^{\infty} \frac{d^{2n}\Psi}{dx^{2n}} \bigg|_{x=5} \frac{\int_{-\infty}^{\infty} y^{2n} \left( -\frac{\partial f}{\partial y} \right)^n dy}{(2n)!}$$

Let

$$a_n = \int_{-\infty}^{\infty} y^{2n} \left( -\frac{\partial f}{\partial y} \right)^n dy$$

$$a_0 = \int_{-\infty}^{\infty} \left( -\frac{\partial f}{\partial y} \right) dy = 1$$

The $a_n$ are just numbers that are computed.

They contain no system parameters whatsoever.

For $n \geq 1$ one can show

$$a_n = 2 \left( 1 - \frac{1}{2^{2n-1}} - \frac{1}{3^{2n-1}} - \frac{1}{4^{2n-1}} - \ldots \right)$$

$$= \left( 2 - \frac{1}{2^{2n-1}} \right) \zeta(2n)$$

where $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots$ is the Riemann zeta function.

In particular $a_1 = \frac{\pi^2}{6}$, $a_2 = \frac{7\pi^4}{360}$

$$\Phi = \sum_{n=0}^{\infty} a_n \frac{d^{2n}\Psi}{dx^{2n}} \bigg|_{x=5} = \psi(\xi) + \sum_{n=1}^{\infty} a_n \frac{d^{2n}\Psi}{dx^{2n}} \bigg|_{x=5}$$

We use $d\psi = \phi$ to finally get

$$\psi(x) = \int_{0}^{x} d\phi' \phi(x')$$

$$\Phi = \int_{0}^{\xi} dx \phi(x) + \sum_{n=1}^{\infty} a_n \frac{d^{2n}\phi}{dx^{2n}} \bigg|_{x=5}$$

$$= \int_{0}^{\xi} dx \phi(x) + \frac{\pi^2}{6} \frac{d\phi}{dx} \bigg|_{x=5} + \frac{7\pi^4}{360} \frac{d^3\phi}{dx^3} \bigg|_{x=5} + \ldots$$
This gives a power series in temperature.

To see this, transform back to the energy variable

\[ x = \beta \varepsilon, \quad \varepsilon = k_B T x \]

\[
\Phi = \int_0^\infty \frac{d\varepsilon}{Z - e^\varepsilon + 1} = k_B T \int_0^\infty \frac{dx}{Z - e^{k_B T x} + 1} = k_B T \int_0^\infty dx \phi(k_B T x) = \int_0^\infty d\varepsilon \phi(\varepsilon)
\]

\[
\frac{d\Phi}{dx} = \frac{d\Phi}{d\varepsilon} \frac{d\varepsilon}{dx} = \frac{d\Phi}{d\varepsilon} k_B T
\]

we get

\[
\Phi = \int_0^\infty d\varepsilon \phi(\varepsilon) m(\varepsilon)
\]

\[
\Phi = \int_0^\infty d\varepsilon \phi(\varepsilon) + \frac{\pi^2 (k_B T)^2}{6} \frac{d\phi}{d\varepsilon} \bigg|_{\varepsilon = \mu} + \frac{\pi^4 (k_B T)^4}{360} \frac{d^3 \phi}{d\varepsilon^3} \bigg|_{\varepsilon = \mu} + \ldots
\]

Example

\[ m = \frac{N}{V} = \int_0^\infty d\varepsilon \, g(\varepsilon) m(\varepsilon) \Rightarrow \phi(\varepsilon) = g(\varepsilon) \]

\[ m = \int_0^\mu d\varepsilon \, g(\varepsilon) + \frac{\pi^2 (k_B T)^2}{6} \frac{d\phi}{d\varepsilon} \bigg|_{\varepsilon = \mu} + \ldots \]

Now as \( T \to 0 \), \( \mu \to \varepsilon_F \) the free energy
\[ m = \int \frac{E_F}{E_F} d\epsilon \, g(\epsilon) + \int_0^{E_F} d\epsilon \, g(\epsilon) + \frac{\pi^2 (k_B T)^2}{6} \frac{dg}{d\epsilon} \bigg|_{\epsilon = \mu} \]

But \( E_F \) was determined by \( m = \int_0^{E_F} d\epsilon \, g(\epsilon) \)

\[ \Rightarrow \int_0^{E_F} d\epsilon \, g(\epsilon) = -\frac{\pi^2 (k_B T)^2}{6} \frac{dg}{d\epsilon} \bigg|_{\epsilon = \mu} \]

Since left hand side is \( O(k_B)^2 \) is small, we can approximate the right hand side as it is.

\[ \int_0^{E_F} d\epsilon \, g(\epsilon) \approx (\mu - E_F) g'(E_F) \]

\[ \Rightarrow (\mu - E_F) \approx -\frac{\pi^2 (k_B T)^2}{6} \frac{dg}{g'(E_F)} \bigg|_{\epsilon = \mu} \]

So \( \mu - E_F \approx O(k_B T)^2 \) is small, so to lowest order can evaluate \( \frac{dg}{d\epsilon} \) on right hand side at \( \epsilon = E_F \) instead of \( \epsilon = \mu \).

\[ \mu(T) \approx E_F - \frac{\pi^2 (k_B T)^2}{6} \frac{g'(E_F)}{g(E_F)} \]

shows that chemical potential \( \mu \) decreases from \( E_F \) by \( O(k_B T)^2 \) at low \( T \).
For free electrons where \( g(E) = C \sqrt{E} \)
\[ g'(E) = \frac{1}{2} C \frac{1}{\sqrt{E}} \]
\[
\mu(T) \approx E_F - \frac{\pi^2}{6} \left( k_B T \right)^2 \frac{1}{2E_F} = E_F - \frac{\pi^2}{12} \left( \frac{k_B T}{E_F} \right)^2
\]
\[
\mu(T) \approx E_F \left( 1 - \frac{1}{3} \left( \frac{\pi k_B T}{2E_F} \right)^2 \right) = E_F \left( 1 - \frac{1}{3} \left( \frac{\pi T}{2T_F} \right)^2 \right)
\]
Correction is small for metals at room temp as \( T_F \approx 10^4 \)K

\[ \text{2) energy} \quad \frac{E}{V} = \int_0^\infty dE \ g(E) \ e \ m(E) \rightarrow \phi(E) = g(E) \ e \]

\[
u = \frac{E}{V} = \int_0^E \ g(E) \ e + \frac{\pi^2}{6} (k_B T)^2 \left[ g(E) + \mu g'(E) \right]
\]
\[
= \int_0^{E_F} g(E) \ e + \int_0^\infty g(E) \ e + \frac{\pi^2}{6} (k_B T)^2 \left[ g(E) + \mu g'(E) \right]
\]
\[
= \frac{u(0) + (\mu - E_F) g(E_F) \ e_F + \pi^2 (k_B T)^2 \left[ g(E_F) + E_F \ g'(E_F) \right]}{6}
\]

\[
\text{ground state} \quad \approx (\mu - E_F) g(E_F) \ e_F \quad \text{replace } \mu \approx E_F
\]

\[
u(T) = \frac{u(0) + (\mu - E_F) g(E_F) \ e_F + \pi^2 (k_B T)^2 \left[ g(E_F) + E_F \ g'(E_F) \right]}{6}
\]

\[
\rightarrow \frac{u(T)}{6} = \frac{u(0) + \pi^2 (k_B T)^2 g(E_F)}{6}
\]

\[
u(T) = u(0) + \pi^2 (k_B T)^2 g(E_F)
\]
specific heat per volume

\[ C_V = \frac{C_V}{V} = \frac{1}{V} \left( \frac{\Delta E}{\Delta T} \right)_V = \frac{dU}{dT}_V \]

\[ C_V = \frac{\pi^2 k_B^2 T}{3} \left[ g(E_F) \right] \]

for free electrons we can write \[ g(E) = C \sqrt{E} \]

\[ m = \int_0^{E_F} dE g(E) = \frac{2}{3} C E^{3/2} \Rightarrow C = \frac{3}{2} \frac{m}{E^{3/2}} \]

\[ \Rightarrow g(E_F) = \frac{3}{2} \frac{m}{E_F^{3/2}} \cdot E_F^{1/2} = \frac{3}{2} \frac{m}{E_F} \quad \text{density of states at Fermi energy} \]

\[ C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) m k_B \]

or total specific heat \[ C_V = Vc_v, \quad mV = N \]

\[ C_V = \frac{\pi^2}{2} \left( \frac{k_B T}{E_F} \right) N k_B \]

\Rightarrow \text{specific heat due to Fermi gas of electrons in a conductor is } C_v \propto T \text{ at low temperatures.}

We already saw that specific heat due to ionic vibrations (phonons) in a solid went like \[ C_v \propto T^3 \]
at low temperatures (Debye model)

\Rightarrow \text{electronic contribution to } C_V \text{ dominates at sufficiently low } T.
Simple estimate of $C_V$

When increase temperature to $k_B T$, the electrons near the Fermi energy $E_F$ will increase their energy by an amount $\sim k_B T$. The number of such electrons, roughly per unit volume, is roughly

$$\frac{g(E_F)(k_B T)}{T}$$

energy interval about $E_F$ $g$

density of states $g$ states which increase get excited at $E_F$

$\Rightarrow$ increase in energy per unit volume is

$$\Delta U \sim \left( \frac{g(E_F) k_B T}{T} \right) \left( k_B T \right) \sim g(E_F) (k_B T)^2$$

$\Rightarrow C_V = \frac{d\Delta U}{dT} \sim g(E_F) k_B^2 T = \frac{3}{2} \frac{m}{e^2} k_B^2 T = \frac{3}{2} m k_B \frac{T^2}{E_F}$

The previous calculation gives the precise numerical coefficient
Electronic specific heat per volume

\[ C_V^{\text{elec}} = \frac{\pi^2}{2} \left( \frac{k_B T}{e_F} \right) N k_B \left( 1 + 0 \left( \frac{k_B T}{e_F} \right)^2 \right) \]

Compare to classical result \( C_V^{\text{classical}} = \frac{N k_B}{V} \).

The correct result for degenerate Fermi gas is a factor
\[ \frac{\pi^2}{2} \left( \frac{k_B T}{e_F} \right) = \frac{\pi^2}{2} \left( \frac{T}{T_F} \right) \]
smaller than classical result by factor \( \approx \frac{10^2}{10^4} \cdot 10^{-7} \) at room temperature.

Also, classical \( C_V \) is independent of \( T \), whereas Fermi gas result is \( \propto T \).

At low \( T \), the conic contribution to \( C_V \) is

\[ C_V^{\text{con}} = \frac{12 \pi^4}{5} \left( \frac{T}{\Theta_D} \right)^3 N k_B \]

\[ \frac{C^{\text{elec}}}{C^{\text{con}}} = \frac{\pi^2}{2} \left( \frac{k_B T}{e_F} \right) \frac{5}{12 \pi^4} \left( \frac{\Theta_D}{T} \right)^3 \approx \frac{5}{24 \pi^2} \left( \frac{\Theta_D}{T_F} \right)^2 \left( \frac{T}{T} \right)^2 \]

\( \approx 1 \) when \( T^* = \sqrt[3]{\frac{5}{24 \pi^2} \left( \frac{\Theta_D}{T_F} \right)^2} \Theta_D \approx 0.15 \left( \frac{\Theta_D}{T_F} \right)^{1/3} \Theta_D \)

for metals, \( T_F \approx 10^4 \text{K}, \Theta_D \approx 10^2 \text{K} \)

\[ T^* = 0.15 \sqrt{10^{-2}} \Theta_D \approx 0.015 \Theta_D \]

So conic contrib to \( C_V \) dominates over electronic contrib until \( T \approx 0.01 \Theta_D \) or at \( 0(1)^{\text{st}} \text{K} \). The electronic contrib dominates at lower temperatures.