What behavior do we expect from Ising model?
For a given $h$, what is the resulting $m(T, h)$?

For $h > 0$, expect $m > 0$ as energetically favorable for spins to align parallel to $h$. For $h < 0$, similarly expect $m < 0$.

In general, $m(T, -h) = -m(T, h)$, since Hamiltonian has the symmetry $T[\hat{s}_i, h] = T[\hat{s}_i - h]$.

What if $h = 0$?

As $T \to \infty$ we expect each spin to be random so $m \to 0$. But even at finite $T$ we might expect $m = 0$ because of symmetry: $T[\hat{s}_i, 0] = T[\hat{s}_i - \hat{s}_i, 0]$ so a configuration $\{\hat{s}_i\}$ in the partition function sum will enter with the same weight as the configuration $\{-\hat{s}_i\}$ and so expect $\langle \hat{s}_i \rangle = 0$.

But at $T = 0$, the system has two degenerate ground states: all up or all down, with $m = \pm 1$. The ground state breaks the symmetry of the Hamiltonian.

More specifically: \[ \lim_{h \to 0^+} \lim_{T \to 0} m(T, h) = +1 \]

\[ \lim_{h \to 0^+} \quad \lim_{T \to 0} \quad m(T, h) = -1 \]
Can one have such a broken symmetry state at finite $T$?

\[ \lim_{h \to 0^+} m(T, h) = m > 0 \]
\[ \lim_{h \to 0^-} m(T, h) = m < 0 \]

For a finite size system, $N$ finite, the answer is NO!

For a finite size system, the energy $H[\Sigma_i \vec{S}_i]$ is always finite. The statistical weight of $\Sigma_i \vec{S}_i$ will always be equal to that of $\Sigma_i - \Sigma_i \vec{S}_i$ in a small $h$, as we take $h \to 0$.

However, in the thermodynamic limit $N \to \infty$, the answer can be yes! Now the energy of states with a finite $\Sigma_i \vec{S}_i$ will grow infinitely large as $N$.

The statistical weight of config $\Sigma_i \vec{S}_i$ can be infinitely different from that of $\Sigma_i - \Sigma_i \vec{S}_i$ in a small $h$, even if take $h \to 0$. ($\infty \times 0 \neq 0$)

$H[\Sigma_i \vec{S}_i] - H[\Sigma_i - \Sigma_i \vec{S}_i] \propto hN$ does not necessarily vanish as $h \to 0$.

It is possible that at finite $T$

\[ \lim_{h \to 0^+ \mid N \to \infty} [\lim_{N \to \infty} m(T, h)] = m > 0 \]
\[ \lim_{h \to 0^- \mid N \to \infty} [\lim_{N \to \infty} m(T, h)] = m < 0 \]

It is important to take the limits in the above order — it first take $N \to \infty$ in a finite $h$, and then take $h \to 0$. Reversing the limits ($h \to 0$ first, then $N \to \infty$) gives $m = 0$ by symmetry of $H$. 
If such broken symmetry states exist at finite $T$, then do they persist at all $T$ or do they disappear at a well defined $T_c$?

**Possibility of a phase transition**

\[ m_0(T) \quad h = 0 \]

\[ m \]

\[ T_c \quad T \]

\[ -m_0(T) \]

$T > T_c$, $m = 0$ - paramagnetic phase

$T \leq T_c$, $m = \pm m_0(T)$ - ferromagnetic phase

$m(T, 0)$ is singular at $T = T_c$

$T_c$ is ferromagnetic phase transition

The ordered state at $T \leq T_c$ is a state of spontaneously broken symmetry. In $h = 0$, the system will pick either the up or the down state to order in, breaking the symmetry of the Hamiltonian.

At finite $h$, expect $m(T, h)$ to behave like

$m(T, h)$ is smooth function of $T$ for $h \neq 0$. 

$T_c$ is ferromagnetic phase transition.
We said that to have a state of spontaneously broken symmetry at finite $T$ requires one needs to be in the thermodynamic limit $N \to \infty$.

Similarly, true singular phase transitions can only occur in the $N \to \infty$ limit. Proof as follows:

Partition function sum:

$$Z(T, h) = \sum_{\{S_i\}} e^{-\beta H[\{S_i\}]}$$

For finite system ($N$ finite) the number of configurations to sum over is $2^N$ is finite.

$Z$ is therefore the sum of a finite number of analytic functions ("analytic" here in the sense of complex function theory - has no singularities as vary $T, h$). As such, $Z$ must itself be an analytic function of $T$ and $h$.

$\Rightarrow$ $Z$ can have no singularities

$\Rightarrow$ no singularities in any thermodynamic quantities

$\Rightarrow$ no phase transitions.

Only in thermodynamic limit of $N \to \infty$ is $Z$ now the sum of an infinite number of analytic functions. Such an infinite sum need not be analytic, so phase transitions can exist.
Phase diagram in $h-T$ plane

- 1st order phase transition. As cross this line, $m(T,h)$ has a discontinuity, jump from $M_0(T)$ to $-M_0(T)$.

- Critical end point. $m(T,h)$ is continuous if cross $h=0$ line above $T_c$. We will see that $T_c$ corresponds to a 2nd order phase transition - jump in $m(T,h)$ vanishes continuously as approach $T_c$ from below.

Phase diagram in $m-T$ plane

- Coexistence curve. "Up" and "down" states at $h=0$ can coexist in equilibrium along this line.

"Forbidden region" - there is no homogeneous phase with $T$ and $m$ in this region.

"Phase separation region" - if cool a system with fixed $M$ into this region it will phase separate into domains of "up" and "down" with average magnetization $M$.

Many similarities to liquid-gas phase diagram.