Then we get

\[ M = \frac{N}{V} = \frac{n(0)}{V} + \frac{g_{3/2}(z)}{\lambda^3} \]

\[ M = \rho_0 + \frac{g_{3/2}(z)}{\lambda^3} \]

where \( \rho_0 = \frac{n(0)}{V} \) density of bosons in ground state

For a system with fixed \( M \), at higher \( T \) one can always choose \( z \) so that \( M = \frac{g_{3/2}(z)}{\lambda^3} \) and \( \rho_0 = 0 \).

But when \( T < T_c \) it is necessary to have \( \rho_0 > 0 \).

Using \( n(0) = \frac{z}{1-z} \) we can write above as

\[ M = \frac{z}{1-z} - \frac{1}{V} \frac{g_{3/2}(z)}{\lambda^3} \]

For \( T > T_c \) we will have a solution to the above for some fixed \( z < 1 \). In thermodynamic limit \( V \to \infty \), the first term will then vanish, i.e. the density of bosons in the ground state vanishes.

As \( V \to \infty \)

\[ \frac{z}{1-z} \frac{1}{V} \]

stays finite to give the additional needed density at \( T < T_c \):

\[ \frac{z}{1-z} \frac{1}{V} = \rho_0 = m - \frac{g_{3/2}(1)}{\lambda^3} \]

\[ m \to \rho_0 \]

\[ m \to \rho_0 \] and the first term \( \frac{z}{1-z} \frac{1}{V} \) diverges as \( z \to 1 \) and as \( V \to \infty \)
At \( T = 0 \), all bosons are in condensate.
At \( T > T_c \), all bosons are in the "normal state".
At \( 0 < T < T_c \), a macroscopic fraction of bosons are in the condensate, while the remaining fraction are in the normal state. Call it the "mixed state."
pressure - separate out ground state from sum as we saw we needed to do in computing \( \frac{N}{V} \)

\[
\frac{P}{k_B T} = \frac{1}{V} \ln \alpha = -\frac{1}{V} \sum_k \ln \left( 1 - \frac{e^{-\beta E(k)}}{2} \right)
\]

\[
\approx -\frac{1}{V} \ln \left( 1 - \frac{e^{-\beta \hbar^2 k^2/2m}}{2} \right)
\]

\[
= \frac{1}{V} \ln \left( \frac{1}{1 - \frac{e^{-\beta \hbar^2 k^2/2m}}{2}} \right)
\]

\[
\alpha = \left( \frac{\hbar^2}{2\pi m k_B T} \right)^{3/2}
\]

where \( g_{3/2}(z) \equiv \frac{1}{\Gamma(3/2)} \int_0^\infty dy \frac{y^{3/2}}{z^{-1}e^y - 1} \) as derived when we began our discussion of quantum gases.

Also recall the number of bosons occupying the ground state is

\[
n(0) = \frac{1}{z^{-1}e^{\beta E(0)} - 1} = \frac{1}{z^{-1} - 1} = \frac{z}{1 - z}
\]

So

\[
n(0) + 1 = \frac{z}{1 - z} + 1 = \frac{1}{1 - z}
\]

\[
\frac{P}{k_B T} = \frac{1}{V} \ln (n(0) + 1) + \frac{g_{3/2}(z)}{2^3}
\]

In the thermodynamic limit of \( V \to \infty \), the first term always vanishes as \( n(0) \ll N = mV \) and

\[
\lim_{V \to \infty} \left[ \frac{\ln (mV)}{V} \right] = 0
\]

So the condensate does not contribute to the pressure.

This is not surprising as particles in the condensate have \( k = 0 \) and hence carry no momentum. In the kinetic theory of gases, one sees that pressure arises from particles with finite momentum \( |p| > 0 \) hitting the walls of the container.
So \[ \frac{\Phi}{k_B T} = \frac{g_{5/2}(z)}{z^5} = g_{5/2}(z) \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} \]

\[ \Phi = g_{5/2}(z(T)) \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} (k_B T)^{5/2} \]

--- equation of state

for a system of fixed density \( m \), \( z \) must be chosen to be a function of \( T \) that gives the desired density \( m \).

\[ \begin{array}{c}
\text{Note} \quad g_{5/2}(z=1) = \zeta(5/2) = 1.342 \\
\text{is finite}
\end{array} \]

---

In the thermodynamic limit of \( N \to \infty \), \( z = 1 \) for \( T \leq T_c(m) \)

\[ \Rightarrow \Phi = g_{5/2}(1) \left( \frac{2\pi m}{\hbar^2} \right)^{3/2} (k_B T)^{5/2} \quad \text{for} \; T \leq T_c \]

--- critical temperature

\( T_c \) depends on the system's fixed density

---

Note: for \( T \leq T_c \), the pressure \( p \propto T^{5/2} \) is independent of the system density!

---

\[ \begin{array}{c}
\text{p vs T curves at constant density m}
\end{array} \]

\[ T_c(m) = \left( \frac{m}{2.16} \right)^{2/3} \frac{\hbar^2}{2\pi m k_B} \]
Define \( M_c(T) = \frac{2.912 \left( \frac{2\pi m k_B}{h^2} \right)^{3/2}}{4\pi^2} \) inverse of \( T_c(m) \).

\( M_c(T) \) is the critical density at a given \( T \).

- A system with \( m \geq M_c(T) \) will be in a
  - Bose condensed mixed state at temperature \( T \).

Phase diagram in \( \phi - T \) plane:

- Forbidden region above line \( \phi = \frac{5}{2} \alpha T \).
- Mixed state on line.
- Normal state \( M \leq M_c(T) \) below line.

Can also consider the transition in terms of \( p \) and \( \nu = \frac{\nu}{N} = \frac{1}{m} \) for various fixed \( T \).

At the transition \( \phi = T_c(m)^{5/2} \rightarrow T_c(m) \propto m^{-2/3} \)

\( \Rightarrow \) at the transition \( \phi \propto (m^{2/3})^{5/2} = m^{5/3} = \nu^{-5/3} \)

\( \Rightarrow \) below the transition, \( p \) is independent of density and hence independent of \( \nu \).

For fixed \( T \), the transition occurs when density \( m \)

- Exceeds \( M_c(T) \), or when \( \nu \) dips below \( \nu_c(T) = \frac{1}{M_c(T)} \)

\( \nu_c(T) \propto T^{-3/2} \).
Curves of \( p \) vs \( \nu \) at constant \( T \):

- \( T_1 < T_2 \)
- Mixed \( \nu < N_2 \)
- Normal \( \nu > N_2 \)

Thermodynamic functions:

Earlier we found \( \frac{E}{V} = \frac{3}{2} p \)

\[
\Rightarrow \frac{E}{N} = \frac{3}{2} p \frac{V}{N} = \frac{3}{2} p \nu = \frac{3}{2} \frac{k_B T \nu}{a^3} \frac{g_{5/2}(\nu)}{\nu}
\]

In above we regard \( \frac{E}{N} \) as a function of either \( \nu \) or \( \nu \). That is, we either determine \( \nu \) for a given \( \nu, T \) or we determine \( \nu \) needed for a given \( \nu, T \) (Recall \( \nu = e^{\beta m} \), \( \nu = \frac{V}{N} \) at \( N \) and \( \mu \) are conjugate variables).

Specific heat:

\[
\frac{C_V}{N k_B} = \left( \frac{\partial (E/N)}{\partial T} \right)_{\nu, N} = \frac{3}{2} \nu \frac{d}{dT} \left( \frac{T}{a^3} \right) g_{5/2}(\nu) + \frac{T}{a^3} \frac{\partial g_{5/2}(\nu)}{\partial \nu} \frac{d\nu}{dT}
\]
For $T \leq T_c$, $z = 1$ so \( \frac{d^2}{dT} = 0 \) and only 1st term remains

\[
\frac{T}{\lambda^3} \sim T^{3/2} \quad \text{so} \quad \frac{d}{dT} \left( \frac{T}{\lambda^3} \right) = \frac{3}{2} \left( \frac{T}{\lambda^3} \right) \frac{1}{T} = \frac{3}{2} \frac{1}{T^{3/2}}
\]

With $z = 1$ here for all $T \leq T_c$.

\[
C_V = \frac{3}{2} \sqrt{\left( \frac{3}{2} \frac{1}{\lambda^3} \right)} g_{3/2}(1) = \frac{15}{4} g_{3/2}(1) \frac{\sqrt{\pi}}{\lambda^3}
\]

\[
= \frac{15}{4} g_{3/2}(1) \sqrt{\left( \frac{2\pi m k_B T}{\hbar^2} \right)^{3/2}}
\]

Note, at $T_c$, $m = g_{3/2}(1)$, and $\nu = \frac{1}{m}$

\[
\frac{C_V(T_c)}{Nk_B} = \frac{15}{4} g_{3/2}(1) = \frac{15}{4} \left( \frac{1.34}{2.612} \right) = 1.925 < \frac{2}{3} \quad \text{this is smaller than the classical ideal gas value of } \frac{3}{2}
\]

So

\[
\frac{C_V}{Nk_B} = 1.925 \left( \frac{T}{T_c} \right)^{3/2} \quad T \leq T_c
\]

For $T > T_c$, $z$ varies with $T$ and we need to evaluate the 2nd term as well

1st term: solves $\frac{15}{4} g_{3/2}(z(T)) \frac{\sqrt{\pi}}{\lambda^3}$

2nd term: from Batavia Appendix D E8(10),

\[
z \frac{d}{dz} \left[ g_0(z) \right] = g_{-1}(z)
\]

\[
\Rightarrow \quad \frac{d}{dz} \frac{g_{3/2}}{d^2} = g_{3/2} \frac{1}{2} \frac{d^2}{dT}
\]
To find \( \frac{1}{2} \frac{d^2 \rho}{dT^2} \), consider our earlier result for the density when \( T > T_c \):

\[
m = \frac{g_{3/2}(\tau)}{\tau^3}
\]

determines \( z(\tau) \) for fixed \( n \)

For fixed \( n \)

\[
\frac{d}{dT} \left( \frac{1}{\tau^3} \right) g_{3/2} + \frac{1}{\tau^3} \frac{dg_{3/2}}{dz} \frac{dz}{dT}
\]

\[
o = \frac{3}{2} \frac{1}{\tau^3} g_{3/2} + \frac{1}{\tau^3} g_{3/2} z \frac{dz}{dT}
\]

\[
\Rightarrow \quad \frac{d^2 \rho}{dT^2} = -\frac{3}{2} \frac{g_{3/2}}{g_{1/2}} \frac{1}{\tau}
\]

\[
\frac{C_v}{Nk_B} = \frac{15}{4} \frac{g_{5/2}(\tau)}{\tau^3} + \frac{3}{2} \sqrt{\frac{T}{\tau^3}} g_{3/2}(\tau) \left( -\frac{3}{2} \right) \frac{g_{3/2}(\tau)}{g_{1/2}(\tau)} \frac{1}{\tau}
\]

\[
\text{use } \quad m = \frac{1}{\tau} = \frac{g_{3/2}(\tau)}{\tau^3} \Rightarrow \quad \frac{1}{\tau} = \frac{\sqrt{\tau}}{g_{1/2}(\tau)}
\]

\[
\frac{C_v}{Nk_B} = \frac{15}{4} \frac{g_{5/2}(\tau)}{g_{5/2}(\tau)} - \frac{9}{4} \frac{g_{3/2}(\tau)}{g_{1/2}(\tau)}
\]

\( T > T_c \)

Note \( g_{1/2}(1) = \frac{5}{e^2 - 1} \frac{1}{e^{1/2}} \to \infty \)

So as \( T \to T_c^+ \) from above, and \( z \to 1 \)

\[
\frac{C_v(T_c^+)}{Nk_B} = \frac{15}{4} \frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{9}{4} \frac{g_{3/2}(1)}{\infty} = \frac{15}{4} \frac{1.341}{2.612} = 1.925
\]

\[\Rightarrow \quad C_v \text{ is continuous at } T_c\]
Finally, we want to show that although $C_v$ is continuous at $T_c$, $rac{dC_v}{dT}$ is discontinuous.

For $T \leq T_c$

$$C_v = \frac{1.925}{Nk_B} \left( \frac{T}{T_c} \right)^{3/2}$$

$$\frac{d}{dT} \left( \frac{C_v}{Nk_B} \right) = \frac{3}{2} \left( 1.925 \right) \left( \frac{T}{T_c} \right) \frac{1}{T_c} = \frac{2.89}{T_c} \left( \frac{T}{T_c} \right)^{1/2}$$

so slope at $T_c^-$ (just below $T_c$)

\[ \frac{d}{dT} \left( \frac{C_v}{Nk_B} \right) = \frac{2.89}{T_c}, \quad T = T_c^- \]

For $T > T_c$

$$C_v = \frac{15}{4} \frac{g_{3/2}(z)}{g_{3/2}(z)} - \frac{9}{4} \frac{g_{5/2}(z)}{g_{5/2}(z)}$$

$$\frac{d}{dT} \left( \frac{C_v}{Nk_B} \right) = \frac{15}{4} \left( \frac{g_{3/2} \frac{d g_{3/2}}{dz} \frac{dz}{dT} - g_{5/2} \frac{d g_{5/2}}{dz} \frac{dz}{dT}}{(g_{3/2}(z))^2} \right)$$

$$- \frac{9}{4} \left( \frac{g_{5/2} \frac{d g_{5/2}}{dz} \frac{dz}{dT} - g_{3/2} \frac{d g_{3/2}}{dz} \frac{dz}{dT}}{(g_{3/2}(z))^2} \right)$$

$$= \frac{1}{2} \frac{dz}{dT} \left\{ \frac{2}{4} g_{3/2}^2 - \frac{g_{5/2} g_{3/2}}{g_{3/2}^2} \right\} - \frac{9}{4} \left( \frac{g_{5/2}^2 - g_{3/2} g_{5/2}}{g_{3/2}^2} \right)$$

Use $\frac{1}{2} \frac{dz}{dT} = -\frac{3}{2} \frac{g_{3/2}}{g_{3/2}} \frac{1}{T}$ as found earlier.
\[
\frac{d}{dT} \left( \frac{C_V}{N k_B} \right) = -\frac{3}{8 T} \frac{g_{3/2}}{g_{1/2}} \left\{ 15 \left( 1 - \frac{g_{5/2} g_{3/2}}{g_{3/2}^2} \right) - 9 \left( 1 - \frac{g_{3/2} g_{1/2}}{g_{1/2}^2} \right) \right\}
\]

Now as \( T \to T_c^+ \) from above, \( z \to 1 \), we have \( g_{5/2}(1) \) and \( g_{3/2}(1) \) are finite, but \( g_{1/2}(1) \) and \( g_{3/2}(1) \) \( \to \infty \)

\( \Rightarrow \) at \( T_c^+ \)

\[
\frac{d}{dT} \left( \frac{C_V}{N k_B} \right) = \frac{45}{8 T_c} \frac{g_{5/2}(1)}{g_{3/2}(1)} - \frac{27}{8 T_c} \frac{g_{3/2}(1)}{g_{1/2}(1)}
\]

Now from Pathria Appendix D (Eq. (16))

\[
g_{1/2}(1) = \lim_{a \to 0} \frac{\Gamma(1-v)}{a^{1-v}}
\]

So

\[
\frac{g_{1/2}(1)}{g_{3/2}(1)} = \lim_{a \to 0} \frac{\Gamma(3/2)}{\Gamma(1/2)} \left( \frac{a^{1/2}}{\Gamma(1/2)} \right)^3 = \frac{\Gamma(3/2)}{[\Gamma(1/2)]^3}
\]

\[
= \frac{1}{2 \pi^{3/2}} = \frac{1}{2 \pi} \quad \text{since} \quad \Gamma(\frac{1}{2}) = \sqrt{\pi} \quad \Gamma(\frac{3}{2}) = \frac{1}{2} \sqrt{\pi}
\]

\[
\frac{d}{dT} \left( \frac{C_V}{N k_B} \right) = \frac{45}{8} \frac{1.341}{2.612} \frac{1}{T_c} = \frac{27}{8} \frac{(2.612)^2}{2 \pi} \frac{1}{T_c}
\]

\[
= \frac{2.89}{T_c} - \frac{3.66}{T_c} = -0.77
\]

\[
\frac{d}{dT} \left( \frac{C_V}{N k_B} \right) = -0.77 \quad \frac{T}{T_c} \quad \text{slope of } C_V \text{ is discontinuous at } T_c.
\]
\( C_v \) has a coup at \( T_c \)

\[ \frac{C_V}{N k_B} \]

\( \frac{3}{2} \)

\( T \rightarrow T_c \)

\( T_c \)

\( T \rightarrow \infty \)

\( \frac{dC_V}{dT} \) goes to classical \( \frac{3}{2} \) as \( T \rightarrow \infty \)

\[ \frac{dC_V}{dT} > 0 \quad \text{for} \quad T = T_c^- \]
\[ \frac{dC_V}{dT} < 0 \quad \text{for} \quad T = T_c^+ \]

Entropy

For single species gas we had for Gibbs free energy

\( G = N \mu \)

Also \( G = E - TS + pV \) (since \( G \) is Legendre transform of \( E \) with respect to \( S \) and \( V \))

\[ \Rightarrow N \mu = E - TS + pV \]

or

\[ S = \frac{E + pV - N \mu}{T} \]

\[ S = \frac{E + pV - \mu}{N k_B T} + \frac{\mu}{k_B T} \]

we had earlier \( E = \frac{3}{2} pV \) \( \Rightarrow pV = \frac{2}{3} E \)

\[ \frac{S}{N k_B} = \frac{5}{3} \frac{E}{N k_B T} - \frac{\mu}{k_B T} \]
\[ Z = e^{\frac{M}{k_B T}}, \quad Z = 1 \quad \text{for} \quad T < T_c \]

We had earlier \[ \frac{\mathcal{E}}{N} = \frac{3}{2} \frac{k_B T}{\lambda^2} \frac{\nu}{g_{5/2}(Z)} \]

and \[ m = \frac{1}{\nu} = \frac{g_{3/2}(Z)}{\lambda^3} \quad \text{for} \quad T > T_c \]

\[ \Rightarrow \quad \frac{S}{Nk_B} = \frac{5}{2} \frac{\nu}{\lambda^2} g_{5/2}(Z) - \ln Z = \begin{cases} \frac{5}{2} \frac{g_{5/2}(Z)}{g_{3/2}(Z)}, & T > T_c \\ \frac{5}{2} \frac{\nu}{\lambda^2} g_{3/2}(1), & T \leq T_c \end{cases} \]

**Note:** For \( T \leq T_c \) we had that the density of the condensate, and a density \( m_0 = m - \frac{g_{3/2}(1)}{\lambda^3} \) in the normal state (i.e., the density of excited particles) \( \frac{\lambda^3}{\lambda^3} = m_0 \)

\[ \Rightarrow \quad \text{for} \quad T < T_c, \quad \frac{S}{Nk_B} = \frac{5}{2} \left( \frac{m_0}{m} \right) g_{5/2}(1) \rightarrow 0 \quad \text{as} \quad T \rightarrow 0 \]

We can imagine that each normal particle carries

entropy \( \frac{5}{2} k_B \frac{g_{5/2}(1)}{g_{3/2}(1)} \), The entropy at \( T < T_c \)

is just the value above entropy per "normal" particle times the fraction of normal particles.

\[ \Rightarrow \quad \text{normal particles carry the entropy condensate has zero entropy} \]

entropy difference per particle between normal state and condensed state is

\[ \Delta S = \frac{5}{2} k_B \frac{g_{5/2}(1)}{g_{3/2}(1)} \]
Latent heat of condensation

\[ L = T \Delta S = \frac{5}{2} k_B T \frac{g_{v/2}}{g_{s/2}}(1) \]

energy released upon converting one normal particle to one condensate particle.

\[ \Rightarrow \text{mixed phase is like coexistence region of a 1st order phase transition (like water} \rightarrow \text{ice)} \]

\[ \Rightarrow \text{"two fluid" model of mixed region} \]
The existence of Bose-Einstein condensation is particular to the dimensionality of the system. To see this, consider a general d-dimensional system. Then

$$\frac{1}{V} \sum_k n(\varepsilon_k) \to \alpha \int dk \frac{k^{d-1}}{e^{\beta^2 k^2/2m} - 1}$$

is largest when \( z \to 1 \), so consider this case

$$\alpha \int dk \frac{k^{d-1}}{e^{\beta^2 k^2/2m} - 1} \quad \text{let } y = \frac{\beta^2 k^2}{2m}$$

$$k = \sqrt{\frac{2m y}{\beta^2}}$$

$$dk = \frac{2m y}{\beta^2} \frac{dy}{2y}$$

again the most singular part of the integrand is as \( y \to 0 \)

For \( y^* < 1 \),

$$\int_0^{y^*} dy \frac{y^{d-1}}{e^{y} - 1} \propto \int_0^{y^*} dy \frac{y^{d-1}}{y} = \int_0^{y^*} y^{d-2}$$

The integral will converge at its lower limit \( y \to 0 \) only for \( \frac{d-2}{2} > -1 \) or \( d > 2 \)

For \( d \leq 2 \), the integral will diverge. Therefore, it will always be possible to find a \( z \) such that

$$m = \frac{N}{V} = \frac{1}{(2\pi)^d} \int \frac{dk}{2} \frac{1}{e^{\beta^2 k^2/2m} - 1}$$

\( \Rightarrow \) No Bose Einstein Condensation in two dimensions or below.