(2) At \( h=0 \) on coexistence line from (**) with \( h=0 \) we have

\[
(1 - \frac{T}{T_c}) m + \left[ \frac{1}{3} \left( \frac{T}{T_c} \right)^3 + \frac{1}{3} \left( \frac{T}{T_c} \right) \right] m^3 = 0
\]

as \( T \to T_c^+ \), \( (1 - \frac{T}{T_c}) + \frac{1}{3} m^2 = 0 \)

\[
m = \pm \sqrt{3 \left( \frac{T_c - T}{T} \right)}
\]

Define \( t = \frac{T_c - T}{T_c} \)

\[m \propto \sqrt{3t} \propto t^{\frac{3}{2}} \quad \beta = \frac{3}{2}\]

(3) At \( h=0 \) on coexistence line as \( T \to T_c \)

from (**) 

\[
\frac{\partial h}{\partial m} = k_B T \left\{ (1 - \frac{T}{T_c}) + 3 \left[ (1 - \frac{T}{T_c}) \left( \frac{T_c}{T} \right)^2 + \frac{1}{3} \left( \frac{T}{T_c} \right)^3 \right] m^2 \right\}
\]

\[
= k_B T \left\{ (1 - \frac{T}{T_c}) + m^2 \right\} \quad \text{as } T \to T_c
\]

As \( T \to T_c^+ \) from above, \( m = 0 \)

\[\frac{\partial h}{\partial m} = k_B T \left( 1 - \frac{T}{T_c} \right) = k_B (T - T_c)\]

magnetic susceptibility \[\frac{\partial m}{\partial \chi} = \frac{1}{k_B (T - T_c)} \propto \frac{1}{|t|^{\gamma}} \quad \gamma = 1\]

Note: at high temp \( T \gg T_c \), \( \chi \propto \frac{1}{T} \) just like Curie paramagnetism. Hence we say the \( T > T_c \) phase is paramagnetic.
As $T \rightarrow T_c^-$ from below, $m^2 = 3 \left( \frac{T_c - T}{T} \right)$

$\Rightarrow \frac{\partial h}{\partial m} = k_b T \left( (1 - \frac{T_c}{T}) + 3 \left( \frac{T_c - T}{T} \right) \right)$

$= 2 k_b (T_c - T)$

$\frac{\partial m}{\partial \lambda} = \lambda = \frac{1}{2 k_b (T_c - T)} \propto \frac{1}{|\lambda|}$, $\lambda = 1$

Also, \( \lim_{T \to T_c} \left( \frac{\lambda T_c}{\lambda T} \right) = \frac{2 k_b (T_c - T)}{k_b (T - T_c)} = 2 \quad \leftarrow \text{amplitude ratio} \)

Our results here are identical to what we found for the van der Waals model of liquid-gas transition, if we make the identifications

\[ \alpha \leftrightarrow \delta \rho - \delta \rho(T), \text{ distance from coexistence curve} \]

\[ \mu \leftrightarrow \delta \nu - \delta \nu(T) \]

Free energy \( f(m,T) = f(0,T) = \int_0^m f(m') dm' \quad \text{as } T \to T_c \)

\( \Rightarrow f(m,T) - f(0,T) = k_b T \left\{ \frac{1}{2} \left(1 - \frac{T_c}{T} \right) m^2 + \frac{1}{12} m^4 \right\} \)

\[ g(h=0,T) = \min_m f(m,T) \Rightarrow \min \text{ of } f \text{ gives equilibrium state} \]
A closer look

\[ h = k_B T \left\{ \frac{1}{2} \left( 1 - \frac{T_c}{T} \right) m + \frac{1}{3} m^3 \right\} \]

For \( T < T_c \) we know that above the \( h(m) \) curve cannot be valid for \(-m_0 \leq m \leq +m_0\). This is the coexistence region where \( h = 0 \). For \( T > T_c \), the correct \( h(m) \) curve is

Such a "correction" based on our physical understanding is called the "Maxwell construction." Originally done in connection with the van der Waals theory of the liquid to gas phase transition.
If we use the above $h(m)$ for $T < T_c$, then to compute $f(m,T)$, then instead of

\[ f \]

\[-m_0 \quad a \quad +m_0 \]

we get

\[ f_0 \]

\[-m_0 \quad 0 \quad +m_0 \]

\[ \leq f(m) \text{ with Maxwell construction} \]

Note: this can be thought of as if we take the top curve and replace it by its convex envelope. The top curve cannot be physically correct since $f(m)$ must be convex in $m$. Only the lower curve is convex.

Using the above corrected $f(m)$, we can compute

\[ g(h,T) = \min_m \left[ f(m,T) - m \cdot h \right] \]
\[ g(h) = \min_m \left[ f(m) - mh \right] \quad \text{then results in} \]

\[ \frac{dg}{dh} = -m \quad \text{is discontinuous at} \quad h = 0 \]

\[ \Rightarrow g(h) \quad \text{has a cusp-like maximum at} \quad h = 0 \]
Note: The mean field approx is exact in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

\[ H = -J \sum_{i,j} s_i s_j - h \sum_i s_i \]

\[ = -J \sum_i s_i \left( \sum_j s_j \right) - h \sum_i s_i \]

\[ = -J \sum_i s_i N \mu - h \sum_i s_i \]

\[ H = \left( -\frac{J}{2} m + h \right) \sum_i s_i \]

where we took \( J = \frac{1}{N} \sum_i \mu \). In infinite range coupling model, need to take coupling \( J \propto \frac{1}{N} \) so that total energy scales with \( E \propto N \) as desired.

In the above, \( m[s_i] = \frac{1}{N} \sum_i s_i \) depends on the config \( \{s_i\} \), however it is the same for every spin \( s_i \).
Landau's Theory of phase transitions

Ising model

order parameter $m$ - \[
\begin{cases}
\text{zero} & T > T_c \\
\text{non-zero} & T < T_c
\end{cases}
\]

ordering field $h$

applying $h \neq 0$ reduces symmetry of Hamiltonian, induces $m \neq 0$.

For ordering field $h = 0$, Hamiltonian has higher symmetry. A finite order parameter $m \neq 0$ breaks this symmetry.

For $h = 0$, 2nd order phase transition at $T_c$ such that $m = 0$ for $T > T_c$ \Rightarrow thermodynamic state has full symmetry of $H$. When $T < T_c$ the order parameter becomes finite $m \neq 0$ \Rightarrow thermodynamic state breaks symmetry of $H$. \text{Spontaneous symmetry breaking}

For $T < T_c$, varying the ordering field $h$ through zero results in a discontinuous jump in the order parameter - 1st order transition line

At the 2nd order transition at $T_c$, that ends the 1st order transition line, $m$ goes to zero continuously as $T \to T_c$. 
For liquid-gas transition

(see Lecture 26
Spring '05 course)

1st order line does not
draw any particular
symmetry with respect
to the natural thermo
variables \( T, p \)

crossing the 1st order transition line.

there is a discontinuity in the specific
volume \( \nu = \frac{1}{p} \) \( p \) is the particle
density

\[ \Delta \nu = \nu_{\text{gas}} - \nu_{\text{liquid}} \to 0 \] as

approach the critical end point \( C \).

Define \( \nu_c = \) specific volume of system at the

critical point \( C \). Dashed line in phase
diagram is the critical isochore, i.e

\[ \nu(T, p) = \nu_c \].

define order parameter \( s_{\nu} = \nu - \nu_c \)
define ordering field \( s_p \) as distance from

critical isochore
Landau Methodology

1. Given a physical system with a phase transition, first identify the order parameter \( m \), a quantity that vanishes in the disordered phase and is non-zero in the ordered phase. Often this can be the hard part! Then identify the corresponding ordering field \( h \) that couples linearly to \( m \).

2. Consider a free energy \( f \) that depends on \( T, m, \) and \( h \) such that the Gibbs free energy is given by

\[
g(h, T) = \min_m \left[ f(m, T, h) \right]
\]

We can view the function \( f(m, T, h) \) in two different ways:

(i) It is the free energy for fixed \( T \) and \( h \) when \( m \) is constrained to some value. But the thermodynamic state is when the constraint is removed and \( m \) relaxes to the value that minimizes \( f(m, T, h) \). The minimum value of \( m \) then determines \( g(h, T) \).
\[ \text{or} \]
\[ f(m, T, h) = f'(m, T) - m h \]
where \( f'(m, T) \) is the usual Helmholtz free energy as function of \( m \) ad \( T \), and the \(-m h\) term is what we add to make the Legendre transformation from \( f'(m, T) \) to \( g(h, T) \).

Near the 2nd order critical point where \( m \) is small we can expand \( f(m, T, h) \) in a Taylor series in \( m \), keeping all terms that are consistent with the symmetries of the physical problem.

For the Ising Model
we require by symmetry that
\[ f(m, T, h = 0) = f(-m, T, h = 0) \]
hence we expand
\[ f(m, T, h) = f_0 + a m^2 + b m^4 + \ldots - m h \]
we add powers of \( m \) here due to linear coupling of order parameter to symmetry
For the liquid–gas transition

we expand

\[ f(\delta V, T, \delta p) = f_0 + a \delta V^2 + b \delta V^3 + c \delta V^4 + \cdots \]

new no symmetry
to rule out a
linear coupling
of order parameter
\( \delta V^3 \) term.

However, we can make a transformation

\[ \delta V = \delta V_0 + \epsilon \]

where \( \delta V_0 \) is a suitable constant chosen to make the coefficient of \( \delta V^3 \) vanish. This will in general result in an additional linear \( \delta V \) term, but that can be absorbed into a redefinition of the ordering field \( \delta p \).

The result will be

\[ f = f_0 + a' \epsilon^2 + b' \epsilon^4 - \delta p' \epsilon \]

with exactly the same form as the Ising model.

\( \Rightarrow \) Although the liquid–gas transition lacks the symmetry of the Ising model, and although the Ising model at liquid–gas transition took life completely different physical models, we expect them to have the same critical behavior \( \Rightarrow \) universality.