As $T \to T_c^-$ from below, $m^2 = 3 \left( \frac{T_c - T}{T} \right)$

$$\Rightarrow \frac{2 k}{2m} = k_B T \left( 1 - \frac{T_c}{T} \right) + 3 \left( \frac{T_c - T}{T} \right)$$

$$= 2k_B (T_c - T)$$

$$\frac{\partial m}{\partial h} = m^- = \frac{1}{2k_B (T_c - h)} \propto \frac{1}{|t|} \quad \gamma = 1$$

Also $\lim_{T \to T_c^-} \left[ \frac{x^+}{x^-} \right] = \frac{2k_B (T_c - h)}{k_B (T - T_c)} = 2 \quad \leftarrow$ amplitude ratio

Free energy $f(m, T) - f(0, T) = \int_0^m \rho(m') dm'$ as $T \to T_c$

$$\Rightarrow f(m, T) - f(0, T) = k_B T \left\{ \frac{1}{2} \left( 1 - \frac{T_c}{T} \right) m^2 + \frac{1}{12} m^4 \right\}$$

$$= a m^2 + b m^4$$

Coefficient of $m^2$ term vanishes at $T_c$, goes negative below $T_c \Rightarrow$ minima of $f(m, T)$ changes from $m = 0$ to $m = \pm m_0(T)$

$g(h=0, T) = \min_m f(m, T) \Rightarrow \min$ of $g$ gives equilibrium state
Specific heat at $h = 0$ along 1st order transition line

From (i) we have $m_o^2 = -a^2 / 2b$ \( T < T_c \), \( m_o^2 = 0 \) \( T > T_c \)

\[ g(h=0,T) = g(m_o,T) = g_0(T), \quad T > T_c \]

\[ = g_0(T) + a \left( \frac{-a}{2b} \right) + b \left( \frac{-a}{2b} \right)^2, \quad T < T_c \]

\[ T < T_c: \quad g(m_o,T) = g_0(T) - \frac{a^2}{2b} + \frac{a^2}{4b} = g_0(T) - \frac{a^2}{4b} \]

\[ = g_0(T) - \frac{a_o^2}{b_o} (T - T_c)^2 \quad a = a_o (T_c - T) \]

Specific heat

\[ \Delta = -\frac{\partial g}{\partial T} \quad \Rightarrow \quad C = T \left( \frac{\partial g}{\partial T} \right)_{h=0} = -T \frac{\partial^2 g}{\partial T^2} \]

\[ C = -T \frac{d^2 f}{dT^2} (m_o(T),T) \]

\[ = \begin{cases} 
- T \frac{d^2 g_0}{dT^2} & T > T_c \\
- T \frac{d^2 g_0}{dT^2} + T \frac{a_o^2}{2b_o} & T < T_c
\end{cases} \]

\[ \Rightarrow \quad C(T \rightarrow T_c^-) - C(T \rightarrow T_c^+) = \frac{T_c a_o^2}{2b_o} \]

Grip in specific heat at $T_c$
The piece \( \frac{a^2}{2T^2} \) is the non-singular piece of the specific heat, so it is the same as the "reference" free energy we used earlier when integrating the equation of state in the mean field or the van der Waals approx.

We can define a critical exponent \( \alpha \) for the specific heat by \( C \sim |t|^\alpha \), or

\[
\alpha = \lim_{t \to 0} \left[ \frac{\log C}{\log |t|} \right]
\]

For Landau theory this gives \( \alpha = 0 \).

**Summary:** Landau theory = mean field theory

\[
\begin{array}{c|c|c}
\hline
T = T_c, & h(m) \propto m^\delta & \delta = 3 \\
\hline
h = 0, & \chi(T) \propto \frac{1}{|t|^\gamma} & \gamma = 1 \\
\hline
h = 0, & C(T) \propto |t|^\alpha & \alpha = 0 \\
\hline
\end{array}
\]

Mean field critical exponents.

Exponent values at mean field appear as index of dimension \( d \).

From exact solution of 2D Ising model

\( \delta = 15 \), \( \beta = \frac{1}{8} \), \( \gamma = \frac{3}{4} \), \( \alpha = 0 \), \( \nu = 1 \)

\( C \propto \sin(T) \)
A closer look

\[ h = k_B T \left\{ \frac{1}{2} (1 - \frac{TC}{T}) m + \frac{1}{3} m^3 \right\} \]

For \( T < T_C \), we know that above \( h(m) \) curve cannot be valid for \(-m_0 \leq m \leq +m_0\). This is the coexistence region where \( h = 0 \). For \( T < T_C \), the correct \( h(m) \) curve is

Such a "correction" based on our physical understanding is called the "Maxwell construction" (originally done in connection with the van der Waals theory if the liquid to gas phase transition.)
If we use the above $h(m)$ for $T < T_c$, then to compute $f(m, T)$, then instead of

\[ f \]

\[
\begin{array}{c}
\text{m} \\
-\text{m}_0 \\
\text{m}_0
\end{array}
\]

we get

\[ f' \]

\[
\begin{array}{c}
\text{m} \\
-\text{m}_0 \\
\text{m}_0
\end{array}
\]

\[ f(m) \text{ with Maxwell construction} \]

Note: this can be thought of as if we take the top curve and replace it by its convex envelope. The top curve cannot be physically correct since $f'(m)$ must be convex in $m$. Only the lower curve is convex.

Using the above corrected $f'(m)$, we can compute

\[ g(h, T) = \min_m \left[ f'(m, T) - m E \right] \]
\[ g(h) = \min_m \left[ f(m) - mh \right] \] then results in

\[ T < T_c \]

\[ \frac{dg}{dh} = -m \] is discontinuous at \( h = 0 \)

\[ \Rightarrow g(h) \text{ has a cusp-like maximum at } h = 0 \]
Note: The mean field approx is exact in the limit that every spin interacts with every other spin (not just nearest neighbors). Then

\[ H = -\frac{J}{2} \sum_{i,j} s_i s_j - h \sum_i s_i \]

\[ = -\frac{J}{2} \sum_i s_i \left( \sum_j s_j \right) - h \sum_i s_i \]

\[ = -\frac{J}{2} \sum_i s_i N m - h \sum_i s_i \]

\[ H = -\left( \frac{J}{2} m + h \right) \sum_i s_i \]

where we took \( J = \frac{z J}{N} \). In infinite range coupling model, need to take coupling \( J \ll \frac{1}{N} \) so that total energy scales with \( E \propto N \) as desired.

In the above, \( m[ s_i ] = \frac{1}{N} \sum_i s_i \) depends on the config \{ \( s_i \) \}, however it is the same for every spin \( s_i \).
Ising model in 1-dimension

\[ h = 0 \quad \text{for singlet} \quad \frac{s_1 \quad s_2 \quad \ldots \quad s_N}{1 \quad 2 \quad \ldots \quad N} \]

\[ H = -J \sum_{i=1}^{N} s_i s_{i+1} \]

Define \[ \sigma_i = s_i s_{i+1} \quad \text{for} \quad i = 1, \ldots, N-1 \]

\[ \sigma_i = \pm 1 \]

\[ H = -J \sum_{i=1}^{N-1} \sigma_i \]

\[ s_i s_j = \prod_{i=1}^{j-1} \sigma_i = (s_1 s_2)(s_2 s_3) \cdots (s_{j-1} s_j) = s_1 s_2 s_3 \cdots s_{j-1} s_j = s_i s_j \]

For every set of \( \{ \sigma_i \}_{i=1}^{N-1} \), there are 2 possible spin configurations depending on whether \( s_1 = +1 \) or \(-1\)

For a given value of \( s_1 \), then

\[ s_j = \frac{1}{s_1} \prod_{i=1}^{j-1} \sigma_i \]

So

\[ Z = \sum_{\{ \sigma_i \}} e^{\beta J \sum_{i=1}^{N-1} \sigma_i} = 2 \sum_{\{ \sigma_i \}} e^{\beta J \sum_{j=1}^{N-1} \sigma_j} = 2 \prod_{j=1}^{N-1} 2 e^{\beta J \sigma_j} \]

\[ \text{two values for } s_1 \]

\[ Z = 2 \left[ \sum_{\sigma = \pm 1} e^{\beta J \sigma} \right]^{N-1} = 2 \left[ 2 \cosh \beta J \right]^{N-1} \]
Gibbs free energy

\[ G(k=0, T) = -k_B T \ln Z = -k_B T \ln 2 - k_B T(N-1) \ln (2 \cosh \beta J) \]

\[ \theta = \lim_{N \to \infty} \frac{G}{N} = -k_B T \ln (2 \cosh \beta J) \]

entropy \[ S = -\left( \frac{\partial G}{\partial T} \right)_{k=0} \]

specific heat \[ C = T \left( \frac{\partial S}{\partial T} \right)_{k=0} = -\frac{7/2 \beta J}{\partial T^2} \]

\[ S = k_B \ln (2 \cosh \beta J) + \frac{k_B T}{2 \cosh (\beta J)} \frac{2}{\partial T} \left( \cosh (\beta J) \right) \]

\[ = k_B \ln (2 \cosh \beta J) + k_B T \frac{\sinh (\beta J) T \partial \beta}{\cosh (\beta J)} \]

\[ = k_B \ln (2 \cosh \beta J) - \frac{J}{T} \tanh \beta J \]

\[ S = k_B \left[ \ln (2 \cosh \beta J) - \beta J \tanh \beta J \right] \]

At \( T \to \infty, \beta \to 0 \), \[ \cosh \beta J \approx 1 + \frac{1}{2} (\beta J)^2 \]
\[ \tanh (\beta J) \approx \beta J \]
\[ S \approx k_B \left[ \ln \left(1 + \beta J^2\right) - (\beta J)^2 \right] \]

\[ \approx k_B \ln 2 \]

At \( T \to 0, \beta \to \infty \), \[ \cosh \beta J \approx e^{\beta J} \]
\[ \tanh \left( \frac{1 - e^{-2\beta J}}{1 + e^{-2\beta J}} \right) \approx 1 - 2 e^{-2\beta J} \]
\[ S \approx k_B \left[ \ln e^{\beta J} - \beta J \left( 1 - 2 e^{-2\beta J} \right) \right] \approx \frac{J}{T} e^{-\beta J / k_B T} \]
\[ C = \frac{T \left( \frac{\partial A}{\partial T} \right)}{k_B T^2} = k_B T \left( \frac{-2J^5 e^{\beta J}}{2 \cosh \beta J} \frac{1}{k_B T^2} + \frac{J}{k_B T^2} \tanh \frac{\beta J}{2} \right) \]

\[ + \frac{\beta J^2}{k_B T^2} \frac{2}{\tanh \beta J} \frac{d}{d(\beta J)} \]

\[ = \frac{J^2}{k_B T^2} \frac{2}{\tanh \beta J} \left( \tanh \beta J \right) = \frac{J^2}{k_B T^2} \frac{1}{(\cosh \beta J)^2} \]

\[ C = k_B \left( \frac{\beta J}{\cosh \beta J} \right)^2 \]

as \( T \to \infty, \beta \to 0 \)

\[ C \approx k_B \left( \frac{J}{k_B T} \right)^2 \]

as \( T \to 0, \beta \to \infty \)

\[ C \approx k_B \left( \frac{J}{k_B T} \right)^2 \frac{e^{-2J/k_B T}}{T^2} \]

\[ \Rightarrow \text{No singularity at any finite } T. \]

\[ \Rightarrow \text{No phase transition at any finite } T. \]