1. Define the creation-annihilation operators by

\[ a^\dagger |n> = \sqrt{n+1} |n+1> \quad \text{for} \quad n = 0, 1, \cdots, \quad a|n> = \sqrt{n} |n-1> \quad \text{for} \quad n = 1, 2, \cdots, \]

(1)

and \(a|0> = 0\). Here the collection of vectors \(|n>, n = 0, 1, \cdots\) is an orthonormal basis. For any complex number \(z\), find an eigenvector for \(a\) with eigenvalue \(z\), as a linear combination \(\sum_{n=0}^{\infty} c_n(z)|n>\). Find the length of the eigenvector by evaluating the sum \(\sum_{0}^{\infty} |c_n(z)|^2\) and use that to find an eigenvector of unit length.

2. Consider the matrix

\[ A = \begin{pmatrix}
1 & -i & 0 \\
i & 1 & 0 \\
0 & 0 & -2
\end{pmatrix} \]

(2)

2.1 What are its eigenvalues and eigenvectors?

2.2 Find its resolvent \(R(z) = (A - z)^{-1}\) and verify that the positions of the poles are the eigenvalues.
Solutions

1. If we put

\[ a \sum_{n=0}^{\infty} c_n(z)|n >= z \sum_{n=0}^{\infty} c_n(z)|n > \]  

we get

\[ L.H.S. = \sum_{n=1}^{\infty} \sqrt{n} c_n(z)|n-1 > \]
\[ = \sum_{n=0}^{\infty} \sqrt{n+1} c_{n+1}(z)|n > \]
\[ = z \sum_{n=0}^{\infty} c_n(z)|n > . \] (4)

So

\[ \sqrt{n+1} c_{n+1}(z) = z c_n(z) \] (5)

Or,

\[ c_n(z) = \frac{z}{\sqrt{n}} c_{n-1}(z) \]
\[ = \frac{z}{\sqrt{n}} \frac{z}{\sqrt{n-1}} c_{n-2}(z) \]
\[ = \cdots \]
\[ = \frac{z}{\sqrt{n}} \frac{z}{\sqrt{n-1}} \cdots \frac{z}{\sqrt{1}} c_0(z) \] (6)

That is,

\[ c_n(z) = \frac{z^n}{\sqrt{n!}} c_0(z) . \] (7)

The square of the length of the vector is

\[ \sum_{0}^{\infty} |c_n(z)|^2 = |c_0(z)|^2 \sum_{0}^{\infty} \frac{|z|^{2n}}{n!} = |c_0(z)|^2 e^{|z|^2} \] (8)

Thus the length is finite for any complex number \( z \).

Setting the length to one gives

\[ c_0(z) = e^{-\frac{1}{2}|z|^2} . \] (9)

Thus any complex number is an eigenvalue of \( a \) with normalized eigenvector

\[ e^{-\frac{1}{2}|z|^2} \sum_{0}^{\infty} \frac{z^n}{\sqrt{n!}} |n > . \] (10)
2. The characteristic polynomial of the matrix is

\[
\begin{vmatrix}
1 - z & -i & 0 \\
i & 1 - z & 0 \\
0 & 0 & -2 - z \\
\end{vmatrix} = -(z+2)[(1-z)^2-1] = -(z+2)[z^2-2z] = -z(z-2)(z+2).
\] (11)

Thus the eigenvalues are 0, 2, -2. The eigenvector corresponding to -2 is easiest to find

\[
\begin{pmatrix}
1 & -i & 0 \\
i & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix} = 0 \Rightarrow u_1 = u_2 = 0.
\] (12)

For the eigenvalue 0,

\[
\begin{pmatrix}
1 & -i & 0 \\
i & 1 & 0 \\
0 & 0 & -2 \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix} = 0 \Rightarrow u_1 = iu_2, \ u_3 = 0.
\] (13)

And for the eigenvalue 2

\[
\begin{pmatrix}
-1 & -i & 0 \\
i & -1 & 0 \\
0 & 0 & -4 \\
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3 \\
\end{pmatrix} = 0 \Rightarrow u_1 = -iu_2, \ u_3 = 0.
\] (14)

Choosing the length of each eigenvector to be one gives

\[
\begin{pmatrix}
0 \\
0 \\
1 \sqrt{2} \\
\end{pmatrix}, \begin{pmatrix}1 \\ -i \end{pmatrix}, \begin{pmatrix}1 \\ i \end{pmatrix}
\] (15)

respectively as the eigenvectors of the eigenvalues -2, 0, 2.

The resolvent is the inverse

\[
R(z) = \begin{pmatrix}
1 - z & -i & 0 \\
i & 1 - z & 0 \\
0 & 0 & -2 - z \\
\end{pmatrix}^{-1}
\] (16)

Because it is block diagonal we can break the matrix up into a two by two matrix

\[
r(z) = \begin{pmatrix}
1 - z & -i \\
i & 1 - z \\
\end{pmatrix}^{-1}
\] (17)
and inverting just the number $-2 - z$. Now
\[ r(z) = \frac{1}{z(z - 2)} \begin{pmatrix} 1 - z & i \\ -i & 1 - z \end{pmatrix} \] (18)

Thus
\[ R(z) = \begin{pmatrix} \frac{1-z}{z(z-2)} & \frac{i}{z(z-2)} & 0 \\ -\frac{i}{z(z-2)} & \frac{1-z}{z(z-2)} & 0 \\ 0 & 0 & -\frac{1}{2+z} \end{pmatrix} \] (19)

The singularities are at the points $z = -2, 0, 2$ which are the eigenvalues of the matrix $A$. 