Quantum Field Theory II

T. Nguyen
PHY 391 Independent Study Term Paper
Prof. S.G. Rajeev
University of Rochester

April 20, 2018

1 Introduction

The purpose of this independent study is to familiarize ourselves with the fundamental concepts of quantum field theory. In this paper, I discuss how the theory incorporates the interactions between quantum systems.

With the success of the free Klein-Gordon field theory, ultimately we want to introduce the field interactions into our picture. Any interaction will modify the Klein-Gordon equation and thus its Lagrange density. Consider, for example, when the field interacts with an external source $J(x)$. The Klein-Gordon equation and its Lagrange density are

$$(\partial_\mu \partial^\mu + m^2)\phi(x) = J(x),$$
$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2}\phi^2 + J\phi.$$

In a relativistic quantum field theory, we want to describe the self-interaction of the field and the interaction with other dynamical fields.

This paper is based on my lecture for the Kapitza Society. The second section will discuss the concept of symmetry and conservation. The third section will discuss the self-interacting ϕ^4 theory. Lastly, in the fourth section, I will briefly introduce the interaction (Dirac) picture and solve for the time evolution operator.

2 Noether's Theorem

2.1 Symmetries and Conservation Laws

In 1918, the mathematician Emmy Noether published what is now known as the first Noether's Theorem. The theorem informally states that for every continuous symmetry there exists a corresponding conservation law. For example, the conservation of energy is a result of a time symmetry, the conservation of linear momentum is a result of a plane symmetry, and the conservation of angular momentum is a result of a spherical symmetry.

The action of a system is defined as

$$S = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{1}$$

where $\mathcal{L}$ is the Lagrange density. A transformation

$$\begin{aligned} x^\mu &\to x'^\mu \\ \phi(x) &\to \phi'(x') \\ \partial_\mu \phi(x) &\to \partial'_\mu \phi'(x') \end{aligned} \tag{2}$$

is a symmetry transformation if the action is invariant under this transformation. In other words:

$$S = \int d^4x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) = \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)).$$

Consider the case of a continuous infinitesimal transformation,

$$\left| \frac{\partial x'}{\partial x} \right| = 1, \tag{3}$$

the symmetry condition requires the action to be invariant, or

$$\begin{aligned} \delta S &= \int d^4x' \mathcal{L}(\phi'(x'), \partial'_\mu \phi'(x')) - \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ &= \int d^4x \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \int d^4x \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ &= \int d^4x (\mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x))) \\ &= 0. \end{aligned} \tag{4}$$

From Gauss's Law, the action is invariant if the integrand equals to $\partial_\mu K^\mu$, where K^μ is any arbitrary constant four-vector. To further evaluate the integrand, we define an infinitesimal change in the field variable as

$$\delta\phi(x) = \phi'(x) - \phi(x). \tag{5}$$

Note that $\delta(\partial_\mu \phi(x)) = \partial_\mu \delta\phi(x)$. The integrand can now be evaluate:

$$\begin{aligned} \delta\mathcal{L} \quad &= \mathcal{L}(\phi'(x), \partial_\mu \phi'(x)) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \\ &= \delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \delta(\partial_\mu \phi) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \\ &= \delta\phi \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) + \partial_\mu (\delta\phi) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \\ &= \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \right) \\ &= \partial_\mu K^\mu, \\ \text{or, } \partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - K^\mu \right) &= 0. \end{aligned} \tag{6}$$

Here we have used the Euler-Lagrange equation, namely,

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{7}$$

We may define a current and a charge corresponding to the continuous symmetry transformation. If we let the current be

$$\boxed{J^\mu = \delta\phi \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - K^\mu} \tag{8}$$

then from Eq.(6), the current will be conserved:

$$\partial_\mu J^\mu = 0. \tag{9}$$

By definition, the charge is then

$$\boxed{Q = \int d^3x J^0(t, \mathbf{x})} \tag{10}$$

If we assume the field variable falls off asymptotically at the spatial infinity, or $\lim_{\mathbf{x}->\infty} \phi(t, \mathbf{x}) = 0$, the integral of $\boldsymbol{\nabla} \cdot J$ will vanish. As a result, the charge is conserved:

$$\begin{aligned} \frac{dQ}{dt} &= \int d^3x \partial_0 J^0 \\ &= \int d^3x (\partial_0 J^0 - \partial_i J^i) \\ &= \int d^3x \partial_\mu J^\mu \\ &= 0. \end{aligned} \tag{11}$$

The time independence of the charge operator means that it will commute with the total Hamiltonian, or $[Q, H] = 0$. Consequently, the charge operator and the Hamiltonian may have simultaneously eigenstates. Therefore, the charge operator is the generator of the infinitesimal symmetry transformation.

From Eq.(9) and Eq.(11), we have shown if there exists a continuous infinitesimal symmetry transformation, it is possible to define a conserved current and a conserved charge. On the other hand, if there exists a conserved charge, it will generate a continuous infinitesimal transformation which defines a symmetry. Finally, we note the current needs not be a vector, so the charge needs not be a scalar. For example, in the case of an infinitesimal global space-time translation, the current is a second-rank tensor

$$T^{\mu\nu} = (\partial^\nu \phi) \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} - \eta^{\mu\nu} \mathcal{L}, \tag{12}$$

where η is the flat metric tensor $(+, -, -, -)$. In this case, the conserved charge is the energy-momentum four-vector operator defined as

$$P^\mu = \int d^3x T^{0\mu}. \tag{13}$$

Each component of the energy-momentum four-vector generates an infinitesimal space-time translation in its direction (e.g. the time-component generates an infinitesimal translation in time).

2.2 Internal Symmetry Transformations

When the constant four-vector $K^\mu = 0$ in Eq.(9), the transformation becomes an internal symmetry transformation. This is a special class of transformation in which,

$$x'^\mu = x^\mu, \tag{14}$$

or the space-time coordinate does not transform. In other words, an internal symmetry transformation acts only on the field, leaving the Lagrange density and any physical result invariant.

An important example of internal symmetry transformation is gauge symmetries. In classical electrodynamics, consider the transformation

$$\begin{aligned} \mathbf{A} &\to \mathbf{A}' = \mathbf{A} + \nabla\lambda, \\ V &\to V' = V - \frac{\partial \lambda}{\partial t}, \end{aligned} \tag{15}$$

where $\mathbf{A}$, V are the vector and scalar potential, and λ is any arbitrary scalar function. This transformation, while changing the potential, leaves the electric and magnetic field unaffected. In quantum field theory, the gauge symmetry of a field variable is

$$\phi \to \phi' = e^{i\lambda}\phi. \tag{16}$$

If λ is a constant, the symmetry is called as a global gauge (phase) symmetry. If λ is a function of the space-time coordinate ($\lambda = \lambda(x)$), the symmetry is called a local gauge (phase) symmetry. We demand the Lagrange density $\mathcal{L}$ to be invariant under local gauge symmetry. This has important consequences in quantum field theory, but is beyond the scope of this paper.

3 Self-interacting ϕ^4 Theory

3.1 The Quartic Self-interacting Term

A field may interact with itself. In this section, we show for a scalar Klein-Gordon field, the only allowed self-interacting term is the quartic term ϕ^4. We would like the interacting Lagrange density to persevere some properties of the free Lagrange density. Firstly, we require the interacting Lagrange density to be invariant under Lorentz transformations and translation. Secondly, the interacting Lagrange density must be invariant under the discrete transformation, namely,

$$\phi(x) \to -\phi(x). \tag{17}$$

The second condition ensures the Hamiltonian is bounded from below.

The total Lagrange density is the sum of the free and interacting Lagrange density,

$$\mathcal{L} = \mathcal{L}_I + \mathcal{L}_0 \quad \text{,where } \mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2. \tag{18}$$

Here the free Lagrange density is derived from the free Klein-Gordon equation. We perform a dimensional analysis on the action S. Because the action S has the unit of angular momentum, in natural unit this means:

$$[S] = [\hbar] = [c] = 1. \tag{19}$$

Moreover, the canonical dimension of any variable x^μ can be expressed in powers of an arbitrary mass dimension [M]. Let

$$[x^\mu] = [M]^{-1}, \quad \text{and } [\partial^\mu] = [M]. \tag{20}$$

From the action of the free Lagrange density, it is obvious the field variable ϕ has dimension of $[M]$,

$$\begin{aligned} [S_0] &= \left[\int d^4x \left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2\right)\right] \\ &= [d^4x][\partial_\mu][\phi][\partial^\mu][\phi] \\ &= [M]^{-4}[M][\phi][M][\phi] = 1 \\ \Rightarrow [\phi] &= [M]. \end{aligned} \tag{21}$$

If we introduce the interacting Lagrange density of the form

$$\mathcal{L}_I = \frac{-\lambda}{n!}\phi^n, \tag{22}$$

where λ is a positive coupling constant, we may show that the unit of λ is

$$\begin{aligned} [S_I] &= \left[\frac{-\lambda}{n!}\int d^4x\phi^n\right] \\ &= [\lambda][d^4x][\phi]^n \\ &= [\lambda][M]^{-4}[M]^n = 1, \\ \Rightarrow [\lambda] &= [M]^{4-n}. \end{aligned} \tag{23}$$

If $n > 4$, the coupling constant has inverse dimensions of mass. As it turns out, this will lead to a divergent scattering amplitude, in which no meaningful physical results can be extracted. In other words, the theory is non-renormalizable. We thus restrict ourselves to $n \leq 4$. We immediately excludes interactions with $n = 1$ and $n = 3$ because these terms violate the discreet symmetry condition in Eq.(17) and lead to a Hamiltonian unbounded from below. Since the quadratic $n = 2$ associates with the mass, the only term left is the quartic term $n = 4$. Therefore, we may define the self-interacting Lagrange density as

$$\boxed{\mathcal{L}_I = \frac{-\lambda}{4!}\phi^4, \text{ where } \lambda > 0} \tag{24}$$

Note that the coupling constant has to be positive for the total Hamiltonian to be bounded from below, as we shall see in below.

3.2 The Coupling Constant

Let

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \dot{\phi}(x) \tag{25}$$

then the Hamiltonian density is

$$\begin{aligned}
\mathcal{H} &= \Pi(x)\dot{\phi} - \mathcal{L} \\
&= \dot{\phi}^2 - \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\nabla\phi\cdot\nabla\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right) \\
&= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4.
\end{aligned} \tag{26}$$

The Hamiltonian is

$$H = \int d^4x\mathcal{H} = \int d^4x\left(\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\nabla\phi\cdot\nabla\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4\right). \tag{27}$$

We see that if $\lambda > 0$, each term in the integrand is positive, so the Hamiltonian is positive definite and bounded from below. On the other hand, if $\lambda < 0$, we cannot ensure that the Hamiltonian is bounded from below: the system will have no meaningful vacuum state.

4 Interaction and Time Evolution Operator

4.1 The Schrödinger and Heisenberg Pictures

In quantum mechanics, we are ultimately interested in the observables, which are the expectation values of Hermitian operators, and not the state vectors of the system (since we cannot measure them). As a result, there can be multiple mathematical formalisms of the dynamics of a quantum system.

A familiar formalism is the Schrödinger picture, in which state vectors are time-dependent while operators are time-independent. In Dirac notation, the Schrödinger equation is written as

$$i\frac{d}{dt}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle, \tag{28}$$

where H is the total Hamiltonian of the system. Alternatively, we may define the state vectors to be time-independent and the operators to be time-dependent. This is known as the Heisenberg picture. The time evolution of an operator in the Heisenberg picture is given by

$$\frac{dO(t)}{dt} = i[H, O(t)] + e^{iHt}\frac{\partial O^{(s)}}{\partial t}e^{-iHt}. \tag{29}$$

Here $O^{(S)}$ is the same operator in the Schrödinger picture. The Schrödinger and the Heisenberg pictures are related by the unitary transformation

$$\left|\psi^{(S)}(t)\right\rangle = e^{-iHt}\left|\psi^{(H)}\right\rangle, \tag{30}$$

$$O^{(S)} = e^{-iHt}O^{(H)}(t)e^{iHt}. \tag{31}$$

We see that the total Hamiltonian is the same for both the Schrödinger and the Heisenberg pictures. More importantly, the observables in both pictures are the same for all time:

$$\begin{aligned}\langle O^{(S)}\rangle &= \langle\psi^{(S)}(t)|O^{(S)}|\psi^{(S)}(t)\rangle \\ &= \langle\psi^{(S)}(t)|e^{-iHt}O^{(H)}(t)|e^{iHt}\psi^{(S)}(t)\rangle \\ &= \langle\psi^{(H)}|O^{(H)}(t)|\psi^{(H)}\rangle \\ &= \langle O^{(H)}(t)\rangle\,.\end{aligned} \tag{32}$$

Note that both pictures coincide at $t = 0$ as the state vectors and operators of both pictures are equal. For example, the Klein-Gordon field variable is given by:

$$\phi^{(S)}(\vec{\mathbf{x}}) = e^{-iHt}\phi^{(H)}(\vec{\mathbf{x}},t)e^{iHt} = \phi^{(H)}(\vec{\mathbf{x}},0). \tag{33}$$

4.2 The Dirac Picture

To introduce interactions in relativistic quantum field theory, we need both the Schrödinger and the Heisenberg pictures. We define the state vectors and the operators of the interaction picture, also known as the Dirac picture, to be:

$$|\psi^{(D)}(t)\rangle = e^{iH_0^{(S)}t}\,|\psi^{(S)}(t)\rangle = e^{iH_0^{(S)}t}e^{-iHt}\,|\psi^{(H)}\rangle\,, \tag{34}$$

$$O^{(D)}(t) = e^{iH_0^{(S)}t}O^{(S)}e^{-iH_0^{(S)}t} = e^{iH_0^{(S)}t}e^{iHt}O^{(H)}(t)e^{-iHt}e^{-iH_0^{(S)}t}, \tag{35}$$

where H_0 is the free Hamiltonian. Here, we see that both the state vectors and the operators are time-dependent. Again, the three pictures coincide at $t = 0$:

$$\begin{aligned}|\psi^{(D)}(0)\rangle = |\psi^{(S)}(0)\rangle = |\psi^{(H)}\rangle\,, \\ O^{(D)}(0) = O^{(S)} = O^{(H)}(0).\end{aligned} \tag{36}$$

It is also important to point out that the Dirac free Hamiltonian is time-independent and equal to the Schrödinger free Hamiltonian:

$$H_0^{(D)}(t) = e^{iH_0^{(S)}t}H_0^{(S)}e^{-iH_0^{(S)}t} = e^{iH_0^{(S)}t}e^{-iH_0^{(S)}t}H_0^{(S)} = H_0^{(S)}. \tag{37}$$

Because the state vectors and the operators in the Dirac picture are time-dependent, the time evolution of both the state vectors and the operators will determine the time evolution of the system. The time evolution of a state vector is given by

$$\begin{aligned}i\frac{\partial\,|\psi^{(D)}(t)\rangle}{\partial t} &= i\frac{\partial}{\partial t}\left(e^{iH_0^{(S)}t}e^{-iHt}\,|\psi^{(H)}\rangle\right) \\ &= -H_0^{(S)}e^{-iH_0^{(S)}(t)}e^{-iHt}\,|\psi^{(H)}\rangle + e^{-iH_0^{(S)}(t)}He^{-iHt}\,|\psi^{(H)}\rangle \\ &= -H_0^{(D)}\,|\psi^{(D)}(t)\rangle + e^{-iH_0^{(S)}(t)}He^{-iH_0^{(S)}(t)}e^{iH_0^{(S)}(t)}e^{-iHt}\,|\psi^{(H)}\rangle \\ &= -H_0^{(D)}\,|\psi^{(D)}(t)\rangle + H^{(D)}\,|\psi^{(D)}(t)\rangle \\ &= \boxed{H_I^{(D)}(t)\,|\psi^{(D)}(t)\rangle}\end{aligned} \tag{38}$$

where H_I is the interacting Hamiltonian ($H = H_0 + H_I$). For an operator, the time evolution equation is:

$$\begin{aligned}\frac{\partial O^{(D)}}{\partial t} &= \frac{\partial}{\partial t}\left(e^{iH_0^{(S)}t} O^{(S)} e^{-iH_0^{(S)}t}\right) \\ &= iH_0^{(S)} O^{(I)} - iO^{(I)} H_0^{(S)} \\ &= \boxed{i[H_0^{(D)}, O^{(D)}]}\end{aligned} \tag{39}$$

Interestingly, the time evolution of a state vector is given by the Schrödinger equation of motion using the interacting Hamiltonian, while the time evolution of an operator is given by the Heisenberg equation of motion using the free Hamiltonian. Because the free Hamiltonian is time-independent (as shown above), the operator can have a plane wave expansion.

4.3 The Time Evolution Operator

In the Schrödinger picture, the time evolution operator is a unitary operator defined as:

$$\begin{aligned}|\psi(t)\rangle &= U(t, t_0) |\psi(t_0)\rangle, \\ \text{where } U(t, t_0) &= e^{-iH(t-t_0)}.\end{aligned} \tag{40}$$

In the interaction picture, we may define the time evolution operator similarly to Eq.(40). Then from the relation in Eq.(35)

$$U(t, t_0) = e^{iH_0^{(S)}t} e^{-iH(t-t_0)} e^{-iH_0^{(S)}t}. \tag{41}$$

Here we have dropped the superscript D for the Dirac picture since this is the only picture we will consider from now on. The operator satisfies the following properties:

$$U(t, t) = \mathbb{1}, \tag{42}$$

$$U(t_2, t_1) U(t_1, t_0) = U(t_2, t_0), \tag{43}$$

$$U^\dagger(t, t_0) = U^{-1}(t, t_0) = U(t_0, t). \tag{44}$$

Consider Eq.(38) and Eq.(40):

$$\begin{aligned} & i\frac{\partial |\psi(t)\rangle}{\partial t} = H_I(t) |\psi(t)\rangle \\ \Rightarrow\ & i\frac{\partial U(t, t_0) |\psi(t_o)\rangle}{\partial t} = H_I(t) U(t, t_0) |\psi(t_0)\rangle \\ \Rightarrow\ & i\frac{\partial U(t, t_0)}{\partial t} = H_I(t) U(t, t_0).\end{aligned} \tag{45}$$

It is now possible to solve for the time evolution operator:

$$
\begin{aligned}
U(t,t_0) =&U(t_0,t_0) - i\int_{t_0}^{t} dt_1 H_I(t_1)U(t_1,t_0)\\
=&\mathbb{1} - i\int_{t_0}^{t} dt_1 H_I(t_1)U(t_1,t_0)\\
=&\mathbb{1} - i\int_{t_0}^{t} dt_1 H_I(t_1)\left(\mathbb{1} - i\int_{t_0}^{t_1} dt_2 H_I(t_2)U(t_2,t_0)\right)\\
=&\mathbb{1} - i\int_{t_0}^{t} dt_1 H_I(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)\\
&+ ... + ...\\
&+ (-i)^n\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 ... \int_{t_0}^{t_{n-1}} dt_n H_I(t_1)...H_I(t_n)\\
&+ ...
\end{aligned}
\tag{46}
$$

To simplify the expression, consider the second-order term of Eq.(46):

$$
\begin{aligned}
&(-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2)\\
=&\frac{(-i)^2}{2}\left(\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_2\int_{t_0}^{t_2} dt_1 H_I(t_2)H_I(t_1)\right)\\
=&\frac{(-i)^2}{2}\left(\int_{t_0}^{t} dt_1\int_{t_0}^{t} dt_2 \theta(t_1-t_2)H_I(t_1)H_I(t_2) + \int_{t_0}^{t} dt_2\int_{t_0}^{t} dt_1 \theta(t_2-t_1)H_I(t_2)H_I(t_1)\right)\\
=&\frac{(-i)^2}{2}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2\left(\theta(t_1-t_2)H_I(t_1)H_I(t_2) + \theta(t_2-t_1)H_I(t_2)H_I(t_1)\right)\\
=&\frac{(-i)^2}{2!}\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 T(H_I(t_1)H_I(t_2))\\
=&\frac{(-i)^2}{2!}T\int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 H_I(t_1)H_I(t_2)
\end{aligned}
\tag{47}
$$

where θ is the step function ($\theta(x) = 1$ for $x \geq 0$ and 0 otherwise) and T is the time-ordering operator defined as:

$$
T(A(t_1), B(t_2)) = \theta(t_1 - t_2)A(t_1)B(t_2) + \theta(t_2 - t_1)B(t_2)A(t_1). \tag{48}
$$

Similarly, we can show that the nth-order term can be simplified into

$$
\frac{(-i)^n}{n!}T\int_{t_0}^{t} ... \int_{t_0}^{t} dt_1 ... dt_n H_I(t_1)...H_I(t_n). \tag{49}
$$

As a result, the time evolution operator is

$$
\begin{aligned}
U(t,t_0) = \mathbb{1} + (-i)\int_{t_0}^{t} dt H_I(t_1) \\
&+ \frac{(-i)^2}{2!} T \int_{t_0}^{t}\int_{t_0}^{t} dt_1 dt_2 H_I(t_1) H_I(t_2) \\
&+ ... + \frac{(-i)^n}{n!} T \int_{t_0}^{t} ... \int_{t_0}^{t} dt_1 ... dt_n H_I(t_1) ... H_I(t_n) \\
&+ ...
\end{aligned}
\tag{50}
$$

The expression on the right-hand side is the Taylor expansion of the exponential function. We arrive at the final expression for the time evolution operator in the Dirac picture:

$$
\boxed{U(t,t_0) = T\left(e^{-i\int_{t_0}^{t} dt' H_I(t')}\right)}
\tag{51}
$$

5 Conclusion

In this paper, we incorporate the concepts of symmetries and interactions into the scalar quantum field theory. We discovered from Noether Theorem, whenever there is a continuous symmetry presented, it is possible to define a conserved current and a conserved charge generating the symmetry transformation. Next, we showed for a scalar Klein-Gordon field, the only self-interacting term allowed is the quartic term ϕ^4 because it is the only term which would result in a renormalizable theory and a meaningful vacuum state. Last but not least, we introduced the quantum Dirac (interaction) picture and solved for the time evolution operator.

With this knowledge, we are able to build some basic field transformations and interactions, such as the space-time translation, scattering, etc. Although a complete picture of quantum field theory is still far-fetched (e.g. we have not considered particle spin into our picture), the interacting scalar field theory plays an important role in providing the basic foundations and describing spinless particles like the Higgs boson.

References

[1] A. Das, *Lectures on Quantum Field Theory.* World Scientific Publishing Company, 2008.

[2] F. Halzen and A. Martin, *Quarks and leptons: an introductory course in modern particle physics.* Wiley, 1984.