

# Cosmological Tensor Perturbations

Benjamin Ostergaard<sup>a</sup>

<sup>a</sup>Department of Physics and Astronomy, University of Rochester, Rochester NY, 14627

[bosterga@u.rochester.edu](mailto:bosterga@u.rochester.edu)

## Abstract

Previously, we calculated the effect of scalar perturbations,  $\Phi$  and  $\Psi$ , of the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric on Einstein's equations. We found that  $\Phi$  and  $\Psi$  are coupled. In this paper, I perturb the FLRW metric with a tensor perturbation and calculate the Ricci Tensor and Ricci Scalar to first-order. We find that the tensor perturbations are decoupled from the scalar perturbations and that tensor perturbations do not affect the Ricci scalar at first-order. This paper is adapted from Scott Dodelson's book *Modern Cosmology*.

## 1. Introduction

Previously, we derived the Boltzmann equations for particles in the early universe. Then we perturbed the metric with scalar perturbation functions  $\Phi$  and  $\Psi$  and calculated the first-order Einstein equations. We found that  $\Phi$  and  $\Psi$  are coupled i.e. we couldn't determine  $\Phi$  without also determining  $\Psi$ . We first remember that  $\Phi$  and  $\Psi$  are scalars and thus invariant under spatial coordinate transformations. Physically, this is because  $\Phi$  and  $\Psi$  arise from perturbations in the mass-energy density of the universe. We determine  $\Phi$  and  $\Psi$  by observing the large-scale structures of the universe and "rewind the clock". However we find that most models of cosmological structure formation require tensor perturbations to the metric which produce large-scale variations in the CMB among other effects.

By the Decomposition Theorem we find that scalar perturbations are decoupled from tensor perturbations so we can use all of our work from the previous sections without having to start over. Without loss of generality, we can write the perturbed metric as

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & a^2(t) + h_+ & h_x & 0 \\ 0 & h_x & a^2(t) - h_+ & 0 \\ 0 & 0 & 0 & a^2(t) \end{pmatrix} \quad (1)$$

Thus, tensor perturbations are described by two (small) functions  $h_+$  and  $h_x$ . We choose a coordinate system here with the  $z$ -axis to be the direction of the wave vector  $\mathbf{k}$ .  $h_+$  and  $h_x$  are components of the tensor

$$\mathbb{H}_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+ & h_x & 0 \\ 0 & h_x & -h_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad (2)$$

Note that  $\mathbb{H}_{\mu\nu}$  is symmetric, traceless, and divergenceless. Since we are in the Fourier domain, divergenceless means  $k^i \mathbb{H}_{ij} = k^j \mathbb{H}_{ij} = 0$ . What we have done so far is the only thought we have to put into this process. We now compute the Ricci Tensor and the Ricci Scalar by plugging and chugging and throwing out terms of  $O(2)$  or greater.

## 2. Christoffel Symbols for Tensor Perturbations

We start with  $\Gamma^0_{\alpha\beta}$ . Since Christoffel symbols are merely derivatives of the metric it is easy to see that

$$\Gamma^0_{00} = \Gamma^0_{i0} = 0 \quad (3)$$

since the metric is constant for those indices. All of the Christoffel symbols with two lower indices are

$$\Gamma^0_{ij} = \frac{1}{2} g_{ij,0} \quad (4)$$

We have used the notation where a comma followed by an index in the subscript indicates a partial derivative with respect to that index. We can write the spatial components of the metric as

$$g_{ij} = a^2 (\delta_{ij} + \mathbb{H}_{ij}) \quad (5)$$

Thus

$$g_{ij,0} = 2H g_{ij} + a^2 \mathbb{H}_{ij,0} \quad (6)$$

Note that  $H$  represents the Hubble rate and note the perturbation tensor. Substituting (6) into (4) gives

$$\Gamma^0_{ij} = H g_{ij} + \frac{a^2 \mathbb{H}_{ij,0}}{2} \quad (7)$$

Now we only need  $\Gamma^i_{0j}$  and  $\Gamma^i_{jk}$ . We find the first one to be

$$\Gamma^i_{0j} = \frac{g^{ik}}{2} g_{jk,0} = \frac{g^{ik}}{2} [2H g_{jk} + a^2 \mathbb{H}_{jk,0}] \quad (8)$$

But we note that we only want first-order terms. In theory we could wait until the end of the calculation to discard higher order terms, but we can make the algebra much easier if we just get rid of them now. Note that  $g^{ik} g_{jk} = \delta^i_j$ . Neglecting first-order terms we note that  $g^{ik} = \delta^{ik}/a^2$ . Thus

$$\Gamma^i_{0j} = H \delta_{ij} + \frac{1}{2} \mathbb{H}_{ij,0} \quad (9)$$

A little algebra will show

$$\Gamma^i_{jk} = \frac{i}{2} [k_k \mathbb{H}_{ij} + k_j \mathbb{H}_{ik} - k_i \mathbb{H}_{jk}] \quad (10)$$

### 3. Ricci Tensor and Ricci Scalar for Tensor Perturbations

Now that we have the Christoffel symbols it is merely a matter of algebra to construct the Ricci Tensor. We start with the time-time component

$$R_{00} = \Gamma^{\alpha}_{00,\alpha} - \Gamma^{\alpha}_{0\alpha,0} + \Gamma^{\alpha}_{\alpha\beta} \Gamma^{\beta}_{00} - \Gamma^{\alpha}_{\beta 0} \Gamma^{\beta}_{0\alpha} \quad (11)$$

This is a formidable equation but we have already calculated much of it. Since the Christoffel symbols with time-time lower indices vanish, the first and third terms in (11) also vanish. Additionally, the indices in the second and fourth terms in (11) must be spatial. Thus we are really calculating

$$R_{00} = -\Gamma^i_{0i,0} - \Gamma^i_{j0} \Gamma^j_{0i} \quad (12)$$

Thus all we need to do is to plug in (9) and take its time derivative. Thus, we get

$$R_{00} = -3 \frac{dH}{dt} - \frac{1}{2} \mathbb{H}_{ii,00} - (H \delta_{ij} + \frac{1}{2} \mathbb{H}_{ij,0})(H \delta_{ij} + \frac{1}{2} \mathbb{H}_{ij,0}) \quad (13)$$

Expanding the parentheses, omitting higher order terms, and noting that the perturbation tensor is traceless, (13) reduces to

$$R_{00} = \frac{-3}{a} \frac{d^2 a}{dt^2} \quad (14)$$

A much more manageable equation. We have seen this equation before. We found it when we calculated the Ricci Tensor for scalar perturbations and even before when we analyzed the unperturbed metric. Thus, at first-order, tensor perturbations do not affect the time-time component of the Ricci Tensor.

Now we have one last laborious task, calculating the spatial components of the Ricci Tensor

$$R_{ij} = \Gamma_{ij,\alpha}^\alpha - \Gamma_{i\alpha,j}^\alpha + \Gamma_{\alpha\beta}^\alpha \Gamma_{ij}^\beta - \Gamma_{\beta j}^\alpha \Gamma_{i\alpha}^\beta \quad (15)$$

It turns out to be convenient to consider the first two terms together. If we separate the time and space components, we get

$$\Gamma_{ij,\alpha}^\alpha - \Gamma_{i\alpha,j}^\alpha = \Gamma_{ij,0}^0 + \Gamma_{ij,k}^k - \Gamma_{ik,j}^k \quad (16)$$

Note that the second term on the left hand side is zero when  $\alpha = 0$ . On the right hand side, note that  $\Gamma_{ij}^0 = g_{ij,0}/2$ . Thus the first term can be written as  $g_{ij,00}/2$ . From (10) we can see that the last term vanishes. Thus, we end up with

$$\Gamma_{ij,\alpha}^\alpha - \Gamma_{i\alpha,j}^\alpha = \frac{g_{ij,00}}{2} + \frac{1}{2}[-k_i k_k \mathbb{H}_{jk} - k_j k_k \mathbb{H}_{ik} + k^2 \mathbb{H}_{ji}] \quad (17)$$

We now can take advantage of coordinate choice and note that  $\mathbf{k}$  lies in the  $z$ -direction. This means that anywhere we see an index on  $\mathbf{k}$ , we replace it with 3. This kills the first two terms in the bracket leaving

$$\Gamma_{ij,\alpha}^\alpha - \Gamma_{i\alpha,j}^\alpha = \frac{g_{ij,00}}{2} + \frac{k^2}{2} \mathbb{H}_{ji} \quad (18)$$

This is much more manageable but remember that it is only two terms in the Ricci tensor, we still have two more to calculate. We expand out the third term in (15)

$$\Gamma_{\alpha\beta}^\alpha \Gamma_{ij}^\beta = \Gamma_{k0}^k \Gamma_{ij}^0 + \Gamma_{kl}^k \Gamma_{ij}^l \quad (19)$$

In the second term the Christoffel symbols are first-order so their product is second order and is thus neglected. For the first Christoffel symbol in the first term we look at (9), set  $i = j$ ,

and sum. The first term in (9) becomes  $3H$  and the second term is first-order so its product with the other part of the term in (19) is neglected. Using (7) we get

$$\Gamma_{\alpha\beta}^{\alpha}\Gamma_{ij}^{\beta} = \frac{3}{2}H g_{ij,0} \quad (20)$$

Using similar methods, we compute the final term in (15)

$$\Gamma_{\beta j}^{\alpha}\Gamma_{i\alpha}^{\beta} = 2H^2 g_{ij} + 2a^2 H \mathbb{H}_{ij,0} \quad (21)$$

Putting it all together we get

$$R_{ij} = \frac{g_{ij,00}}{2} + \frac{k^2}{2}\mathbb{H}_{ji} + \frac{3}{2}H g_{ij,0} - 2H^2 g_{ij} - 2a^2 H \mathbb{H}_{ij,0} \quad (22)$$

We are not quite done yet. We must substitute in the time derivatives of the metric. First we evaluate the second time derivative using (6)

$$g_{ij,00} = 2g_{ij}\left(\frac{1}{a}\frac{da}{dt} + H^2\right) + 4a^2 H \mathbb{H}_{ij,0} + a^2 \mathbb{H}_{ij,00} \quad (23)$$

We then substitute (23) into (22) and we get

$$R_{ij} = g_{ij}\left(\frac{1}{a}\frac{da}{dt} + 2H^2\right) + \frac{3}{2}a^2 H \mathbb{H}_{ij,0} + \frac{a^2}{2}\mathbb{H}_{ij,00} + \frac{k^2}{2}\mathbb{H}_{ij} \quad (24)$$

It is easy to compute the Ricci Scalar

$$\mathbb{R} = g^{00}R_{00} + g^{ij}R_{ij} \quad (25)$$

Looking at the first-order part, we can ignore the first term which is zero-order. In the second term, contracting the metric gives zero-order terms for the terms in (24) proportional to the metric. Since all the other terms in (24) are first-order in the tensor perturbation, we take the zero-order part of the metric,  $g^{jk} = \delta^{jk}/a^2$ . This takes the trace of the tensor perturbation which is traceless. Thus, to first-order, tensor perturbations do not affect the Ricci Scalar.

#### 4. Summary

With the Ricci Tensor and the Ricci Scalar, it is relatively easy to calculate the Einstein Tensor. You can then show that tensor perturbations give rise to gravitational waves. You can also see that the tensor perturbations are decoupled from the scalar perturbations.

This derivation required a lot of menial algebra and neglecting higher order terms. This is useful the first time you see such a problem to learn the techniques of the computation, but becomes tedious math if you want to see higher order terms or choose a new metric. Since this section was mostly a mathematics exercise with almost no physics, it makes sense to enlist the help of computers to do the tedious algebra of calculating the Christoffel Symbols, Ricci Tensor and Scalar, and the Einstein Tensor. I am currently working on doing symbolically this with Wolfram Mathematica. Hopefully, users will be able to input any metric and have Einstein's equations pop out. This would allow users to focus on the physics and leave the tedious algebra to the computers.

## **5. References**

Dodelson, S. *Modern Cosmology*. Academic Press. 2003