$SU(2)$'s Double-Covering of $SO(3)$

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Abstract

We discuss properties of $SU(2)$, especially in relation to its double-covering of $SO(3)$. We first point out the double covering, then we show the isomorphism of the Lie algebras of $SU(2)$ and $SO(3)$. We then explicitly find the elements of $SU(2)$ and show how they map to rotations. We also discuss the pseudoreality of the fundamental representation of $SU(2)$, and the finally show that $U(1) \cong {}^{SU(2)}/_{\mathbb{Z}_N} \times U(1)$, discussing some of the subtleties in applications.

1 Introduction

$SU(2)$ is important in both physics and math. In physics, it originally became important as a tool for studying electron spin, and Heisenberg, reasoning by analogy, used it to predict the existence of isospin. Mathematically, it has many of interesting properties, especially its rather intricate relationship with $SO(3)$. Historically, physicists needed to master the mathematics of $SU(2)$ before moving to $SU(3)$ and beyond. Thus, just as when studying $SO(N)$ we generally begin with $SO(3)$, it's instructive to begin studying $SU(N)$ by considering the specific case of $SU(2)$, which is interesting in its own right.

2 $SU(2)$ is locally isomorphic to $SO(3)$

One of the most interesting properties of $SU(2)$ is its aforementioned relationship to $SO(3)$, namely that the two are *locally isomorphic*. Essentially, this means that, in a neighborhood of any $U \in SU(2)$, $SU(2)$ and $SO(3)$ are isomorphic. This manifests itself as an isomorphicm of their Lie algebras. As we shall see, this does not hold globally, ie not restricted to a small neighborhood.

To prove this local isomorphism, first consider that any 2-by-2 Hermitian traceless matrix X can be written as a linear combination of the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

We can state this more succinctly (and often more usefully) as $X = \vec{x} \cdot \vec{\sigma}$, where $\vec{x} = (x, y, z)$. Explicitly,

$$X = \begin{pmatrix} z & x - iy \\ x + iy & -z \end{pmatrix} \tag{2}$$

Pick any arbitrary element U of $SU(2)$, and consider $X' \equiv U^\dagger X U$. We want to show that we can associate some rotation R with this arbitrary U (ie, define a map that takes $U \to R$)

First, X' is Hermitian:

$$(X')^\dagger = (U^\dagger X U)^\dagger = U^\dagger X^\dagger U = U^\dagger X U = X'$$

since X is Hermitian by assumption. Also, X' is traceless:

$$\mathrm{tr}(X') = \mathrm{tr}\big(U^\dagger X U\big) = \mathrm{tr}(X) = 0$$

using the cyclicity of the trace.

Now, since X' is Hermitian and traceless, we can write $X' = \vec{x}' \cdot \vec{\sigma}$, just as we did with X. We see that $\vec{x}$ and $\vec{x}'$ are linearly related: taking $\vec{x} \to \lambda\vec{x}$ for a real number λ also takes $\vec{x}' \to \lambda\vec{x}'$:

$$X' = U^\dagger X U \leftrightarrow \vec{x}' \cdot \vec{\sigma} = U^\dagger(\vec{x} \cdot \vec{\sigma}$$

Now compute the determinant: notice that

$$\det(X) = -z^2 - (x - iy)(x + iy) = -(x^2 + y^2 + z^2) = -\vec{x}^2$$

for a general X. So,

$$\det(X') = -(\vec{x}')^2 = \det\big(U^\dagger X U\big) = \det\big(U^\dagger\big)\det(X)\det(U) = \det(X) = -\vec{x}^2$$

So $\vec{x}$ and $\vec{x}'$ are linearly related and have the same magnitude. Then by definition, they are related by a rotation! So we can associate a rotation R with U.

3 $SU(2)$ covers $SO(3)$ twice

$f : U \to R$ is actually 2-to-1, since $f(U) = f(-U)$: ie, $U^\dagger X U = (-U)^\dagger X(-U)$, so U and $-U$ are mapped to the same R. We say that $SU(2)$ *double-covers* $SO(3)$.

Since the map $U \to R$ is 2-to-1, not 1-to-1, $SO(3)$ and $SU(2)$ clearly can't be isomorphic groups. However, for a sufficiently small neighborhood around the identity, that neighborhood will include exactly one of U and $-U$, $\forall U \in SU(2)$. So *locally*, in a neighborhood of the identity, they are isomorphic.

There are hints of this in many other areas of the usual discussion around representations of $SU(2)$ and $SO(3)$. For example, $SU(2)$ has a three-dimensional representation. This is exactly its correspondence with $SO(3)$.

4 Properties of the Pauli Matrices

Since the Pauli matrices form the basis for the fundamental two-dimensional representation of $SU(2)$, understanding their behavior allows us to understand all of $SU(2)$. So let's study them in a bit more detail.

First, we will enumerate some of their properties. It turns out that:

$$\sigma_a^2 = I \qquad \forall a \tag{3}$$

Additionally, they *skew-commute*:

$$\sigma_a\sigma_b = -\sigma_b\sigma_a \qquad \forall a, b, a \neq b \tag{4}$$

We'll check this explicitly for $a = 1$, $b = 2$:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\sigma_3$$
$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\sigma_3$$

Notice the correspondence with σ_3! In fact, distinct Pauli matrices give the other with a factor of $\pm i$ in front.

We can summarize these two properties with the following equation:

$$\sigma_a \sigma_b = \delta_{ab} I + i\epsilon_{abc}\sigma_c \tag{5}$$

If $a = b$, the second term vanishes and we have I. If not, the first term vanishes, and we're left with $\pm i\sigma_c$.

Interchanging a and b, we also have

$$\sigma_b \sigma_a = \delta_{ba} I - i\epsilon_{abc}\sigma_c \tag{6}$$

We can then add and subtract (5) and (6) to get

$$\{\sigma_a, \sigma_b\} = 2\delta_{ab} \qquad \text{(add)} \tag{7}$$
$$[\sigma_a, \sigma_b] = 2i\epsilon_{abc}\sigma_c \quad \text{(subtract)} \tag{8}$$

Note the factors of 2! They are critical in what is to come.

Now, the Lie algebra for $SU(N)$ is given by

$$[T^a, T^b] = if^{abc}T^c \tag{9}$$

where the f^{abc}'s are the structure constants of the algebra. Dividing (8) by 2, we get

$$\left[\frac{\sigma_a}{2}, \frac{\sigma_b}{2}\right] = i\epsilon_{abc}\frac{\sigma_c}{2} \tag{10}$$

which matches up with (9) perfectly. Thus, the generators of the fundamental two-dimensional representation of $SU(2)$ are represented by the $\sigma_a/2$, with structure constants ϵ_{abc}.

Recall that the Lie algebra for $SO(3)$ is given by

$$[J_a, J_b] = i\epsilon_{abc}J_c \tag{11}$$

We can see that by identifying the T^a of the $SU(2)$ representation with the J_a, the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic; in other words, identical!

Given all of this, we don't need to do anything more to find the representations of $SU(2)$, as we can replicate exactly the steps for analyzing $SO(3)$. Just define $T^\pm = T^1 \pm T^2$, which has the commutation relations

$$[T^3, T^\pm] = \pm T \quad , \quad [T^+, T^-] = 2T^3$$

and continue from there. Just as with $SO(3)$, the representations will be $(2t+1)$-dimensional, $t = 0, {}^1\!/\!_2, 1, {}^3\!/\!_2, 2, \ldots$. This, in a sense, "explains" the intuitive strangeness of the two-dimensional representation of $SO(3)$; it's simply the fundamental representation of $SU(2)$.

5 The Group Elements of $SU(2)$

Every element of $SU(2)$ can be written as

$$U = e^{i\phi_a \sigma_a/2} = e^{i\vec{\phi}\cdot\vec{\sigma}/2} \tag{12}$$

where repeated indices are summed. Let's look at what these are in more detail.

We denote $\vec{\phi} = \phi\hat{\phi}$, where ϕ is the magnitude of $\vec{\phi}$ and $\hat{\phi}$ is a unit vector pointing in the same direction, and expand (12) in a Taylor series:

$$U = e^{i\vec{\phi}\cdot\vec{\sigma}/2} = \sum_{n=0}^{\infty} \frac{i^n}{n!}\left(\frac{\vec{\phi}\cdot\vec{\sigma}}{2}\right)^n \tag{13}$$

To figure this out, we need to work out what $(\vec{\phi}\cdot\vec{\sigma})^n$ is.

For arbitrary $\vec{u}$ and $\vec{v}$,

$$\begin{aligned}
(\vec{u}\cdot\vec{\sigma})(\vec{v}\cdot\vec{\sigma}) &= u^a v^b \sigma_a \sigma_b \\
&= u^a v^a (\delta_{ab} I + i\epsilon_{abc}\sigma_c) \\
&= (\vec{u}\cdot\vec{v}) I + i(\vec{u}\times\vec{v})\cdot\vec{\sigma}
\end{aligned}$$

So if $\vec{u} = \vec{v} = \vec{\phi}$, we get

$$(\vec{\phi}\cdot\vec{\sigma})^2 = \phi^2 I$$

Now we can split up (13) into even and odd powers:

$$\begin{aligned}
\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\left(\frac{\phi}{2}\right)^{2k}\right) I + i\left(\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!}\left(\frac{\phi}{2}\right)^{2k+1}\right)\hat{\phi}\cdot\vec{\sigma} \\
= \cos\left(\frac{\phi}{2}\right) I + i(\hat{\phi}\cdot\vec{\sigma})\sin\left(\frac{\phi}{2}\right)
\end{aligned}$$

Since we're free to align our coordinates, we'll align $\hat{z}$ with $\hat{\phi}$, giving us

$$\cos\left(\frac{\phi}{2}\right) I + i\sigma_3 \sin\left(\frac{\phi}{2}\right) \tag{14}$$

These are the elements of $SU(2)$.

6 Half-angles become Full-angles

In section 2, we showed that for $X' = \vec{x}'\cdot\vec{\sigma}$ and $X = \vec{x}\cdot\vec{\sigma}$ related by $X' = U^\dagger X U$ for some $U \in SU(2)$, then $\vec{x}$ and $\vec{x}'$ are related by a rotation. But which rotation is it? And how exactly do the half-angles in (14) become full angle rotations? To see all of this, it's instructive to compute the rotation relating $\vec{x}$ and $\vec{x}'$ explicitly.

We will compute $U^\dagger X U$ by brute force. Without loss of generality, let $\hat{\phi}$ point along $\hat{z}$, so we can use the form of U found in (14).

We'll first find $U^\dagger \sigma_a U$, and then use that to work out the result for a linear combination of σ_a. So,

$$U^\dagger \sigma_a U = \left[\cos\left(\frac{\phi}{2}\right) I - i\sigma_3 \sin\left(\frac{\phi}{2}\right)\right] \sigma_a \left[\cos\left(\frac{\phi}{2}\right) I + i\sigma_3 \sin\left(\frac{\phi}{2}\right)\right]$$

For $a = 3$, this becomes

$$= \cos^2\left(\frac{\phi}{2}\right)\sigma_3 + \sin^2\left(\frac{\phi}{2}\right) = \sigma_3$$

Otherwise, $a = 1, 2$. We first get the $\sin^2(\phi/2)$ coefficient:

$$\sigma_3 \sigma_a \sigma_3 = -\sigma_a \sigma_3 \sigma_3 = -\sigma_a$$

using (3)and (4). Then multiplying out, we get

$$\begin{aligned}
&\left[\cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right]\sigma_a - i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\sigma_3\sigma_a + i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)\sigma_a\sigma_3 \\
&\qquad = \left[\cos^2\left(\frac{\phi}{2}\right) - \sin^2\left(\frac{\phi}{2}\right)\right]\sigma_a - i\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)[\sigma_3, \sigma_a]
\end{aligned}$$

By (8), $[\sigma_3, \sigma_1] = 2i\sigma_2$ and $[\sigma_3, \sigma 2] = -2i\sigma_1$. Then using the identities $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$ and $2\sin(\theta)\cos(\theta) = \sin(\theta)$, we get

$$U^\dagger \sigma_1 U = \quad \sigma_1 \cos(\phi) + \sigma_2 \sin(\phi) \tag{15}$$

$$U^\dagger \sigma_2 U = -\sigma_1 \sin(\phi) + \sigma_2 \cos(\phi) \tag{16}$$

Now plug in X:

$$\begin{aligned}
X' = U^\dagger X U &= U^\dagger (x\sigma_1 + y\sigma_2 + z\sigma_3) U \\
&= (x\cos(\phi) - y\sin(\phi))\sigma_1 + (x\sin(\phi) + y\cos(\phi))\sigma_2 + z\sigma_3
\end{aligned}$$

So $x' = x\cos(\phi) - y\sin(\phi)$ and $y' = x\sin(\phi) + y\cos(\phi)$, exactly what we would expect for a rotation about $\hat{z}$ by an angle ϕ.

7 Quantum Mechanics and the Double Covering

Now for a striking fact!

As before, set $U(\phi) = e^{i\phi\sigma_3/2}$ which, as we just checked, leads to a rotation by an angle ϕ around $\hat{z}$. But notice:

$$U(2\pi) = e^{i2\pi\sigma_3/2} = e^{i\pi\sigma_3} = \begin{pmatrix} e^{i\pi} & 0 \\ 0 & e^{-i\pi} \end{pmatrix} = -I \tag{17}$$

So by the time ϕ has gone from 0 to 2π, U has only gone from I to $-I$! To reach I, ϕ needs to go all the way to 4π - two "total rotations".

Recall in section 3 that we showed $SU(2)$ double-covered $SO(3)$. This is the manifestation of that. In this case, I and $-I$ both map to the same rotation in $SO(3)$, and in general, we'll always have U and $-U$ mapping to the same rotation. More concretely

$$U(\phi + 2\pi) = e^{i(\phi+2\pi)\sigma_3/2} = e^{i\pi\sigma_3} e^{i\phi\sigma_3/2} = -U(\phi)$$

Since ϕ and $\phi+2\pi$ are the same rotation in $SO(3)$, these are two elements of $SU(2)$ that correspond to the same rotation.

All of this underlines (and relates back to) the mathematical statement that, while the Lie algebras of $SU(2)$ and $SO(3)$ are isomorphic, making them locally work the same, the groups themselves manifestly *do not*.

8 SU(2) and Tensors

$SO(3)$ has special properties that $SO(N)$ does not, in general, share. The situation is similar for $SU(2)$ and $SU(N)$. These properties are best explored using tensor representations.

The tensors that furnish representations of $SU(N)$ are precisely the traceless ones, notated $T^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_m}$. These should have definite symmetry properties under permutation of the upper and lower indices. For $SU(2)$, since ϵ_{ij} and ϵ^{ij} carry only two indices, we can in fact remove all of the lower indices from said traceless tensors.

It's easier to see this from a specific example; it easily generalizes. So consider the the traceless tensor T^{ijk}_{mn}. Then using the construction

$$\epsilon^{pm}\epsilon^{qn}T^{ijk}_{mn} = T^{pqijk}$$

we transform the lower indices into upper indices, just so! Thus we can get away with only considering traceless tensors with no lower indices.

But we can go further! In fact, it suffices to consider only tensors with the upper indices all symmetrized under interchange. We will show this by induction.

Suppose the statement holds for tensors with fewer than four indices, and suppose T^{ijkl} has no specific symmetry under interchange of i and k. Then we can construct the tensors

$$S^{ijkl} = T^{ijkl} + T^{kjil}$$
$$A^{ijkl} = T^{ijkl} - T^{kjil}$$

$\epsilon_{ik}A^{ijkl}$ is a tensor with two upper indices, which we need not worry about by our inductive hypothesis. And of course, S^{ijkl} is now symmetric on i and k.

Of course, you couldn't do any of this with, say, $SU(5)$. This simplicity is what makes $SU(2)$ special.

We can use this understanding to count the dimensions of representations. Take the representation furnished by $T^{i_1 i_2 \ldots i_m}$. Each i_l can take on the value 1 or 2, and the tensor is invariant under the interchange of any two indices. So we just count: $T^{11\ldots11}$, $T^{11\ldots12}$, $T^{11\ldots22}$, ... , $T^{12\ldots22}$, $T^{22\ldots22}$. So the number of 2's ranges from 0 to m, and thus there are $m+1$ independent objects (and thus the representation is $m+1$-dimensional). If we write $m = 2t$, $t = 0, 1/2, 1, 3/2, ...$, then it's $2t+1$-dimensional.

So irreducible representations of $SU(2)$ are indexed by t, which is either an integer or half-integer, such that the dimensions are $2t+1$. This is precisely what we expect from the local isomorphism with $SO(3)$, and in fact we could see this from our reflections in section 4. Good to know everything is consistent.

9 The Pseudoreality of $SU(2)$

Recall the purpose of upper and lower indices in tensor representations. For a tensor ψ, it's necessary to have ψ_i to stand in for ψ^{i*}. However, in the preceding discussion we were able to transform away all of the lower indices. It seems like ψ^i must be invariant under complex conjugation; ie, that it's real.

But this clearly can't be the case! After all, the group elements of $SU(2)$ in equation (14) are explicitly complex.

To understand this, we develop the concept of "pseudoreal" representations. We say that if, for a representation $D(g)$,

$$D(g)^* = SD(g)S^{-1} \tag{18}$$

for all $g \in G$ and for some similarity transformation S, then the representation is pseudoreal.

To show that the fundamental representation of $SU(2)$ is pseudoreal, we will explicitly find the S transformation. First, note that the fundamental representation is given, as always, by $D(g) = e^{i\vec{\phi}\cdot\vec{\sigma}/2}$, and the conjugate representation is given by $D(g)^* = e^{-i\vec{\phi}\cdot\vec{\sigma}^*/2}$.

Notice that the only Pauli matrix that's at all complex is σ_2. This suggests that σ_2 should be our attack vector (or rather, attack transformation). From our Pauli matrix identities,

$$\begin{aligned}
\sigma_2\sigma_1^*\sigma_2 &= \sigma_2\sigma_1\sigma_2 = -\sigma_1\sigma_2\sigma_2 = -\sigma_1 \\
\sigma_2\sigma_3^*\sigma_2 &= \sigma_2\sigma_3\sigma_2 = -\sigma_3\sigma_2\sigma_2 = -\sigma_3 \\
\sigma_2\sigma_2^*\sigma_2 &= -\sigma_2\sigma_2\sigma_2 = -\sigma_2
\end{aligned}$$

Generalizing, we have $\sigma_2\sigma_a^*\sigma_2 = -\sigma_a$. So,

$$\sigma_2\left(e^{i\vec{\phi}\cdot\vec{\sigma}/2}\right)^*\sigma_2 = \sigma_2\left(\sum_{k=0}^{\infty}\frac{i^k}{k!}\left(\frac{\vec{\phi}\cdot\vec{\sigma}}{2}\right)^k\right)^*\sigma_2 = e^{i\vec{\phi}\cdot\vec{\sigma}/2} \tag{19}$$

Thus, we can choose $S = S^{-1} = \sigma_2$, which proves that the fundamental representation of $SU(2)$ is, in fact, pseudoreal.

Note that since our requirements for S are simply that $S^\dagger S = I$ and equation (18), S is only determined up to a phase. If we instead use $S = i\sigma_2 = \left(\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right)$, we can see the connection between S and the tensor approach: namely, $S = \epsilon_{ij}$, which connects ψ^i to ψ_i.

10 $U(N)$ and $SU(N)$

Now that we've discussed a bit about $SU(2)$, we can make a few statements about $U(N)$, which is an important group for physics in its own right.

The definition of $U(N)$ is very similar to that of $O(N)$: all $N \times N$ matrices satisfying $U^\dagger U = I$. With $O(N)$, the analogous condition leads to the fact that $\det(O^T O) = \det(O^T)\det(O) = (\det(O))^2 = 1 \Rightarrow \det(O) = \pm 1$. With $U(N)$ on the other hand, we get something slightly different:

$$\det(U^\dagger U) = \det(U^\dagger)\det(U) = \det(U)^*\det(U) = |\det(U)|^2 = 1$$

which implies a continuum of possibilities: $\det(U) = e^{i\alpha}$, $0 \leq \alpha < 2\pi$. We pull out $SU(N)$ with the additional condition that $\det(U) = 1$.

$U(N)$ also has another important subgroup: that of all matrices of the form $Ie^{i\alpha}$. Since this is isomorphic to $U(1)$, we label it as such (slightly abusing notation). We can use these subgroups to better understand $U(N)$.

First, notice that the intersection of $SU(N)$ and $U(1)$ is nontrivial. The matrices of the form $Ie^{i\alpha/n}$ are included in both subgroups (since $\det\left(Ie^{i\alpha/n}\right) = \left(e^{i\alpha/n}\right)^n = 1$. These are really just the n^{th} roots of unity, which is isomorphic to $\mathbb{Z}_n$. Again with a slight abuse of notation, since strictly speaking $\mathbb{Z}_n \notin U(n)$, but it will do.

The remainder of this section will prove that

$$U(N) \cong {}^{SU(N)}/_{\mathbb{Z}_N} \times U(1) \tag{20}$$

and briefly discuss what this means in the context of Lie algebras.

We will prove this in two steps, by proving the following two isomorphisms:

$${}^{SU(N)}/_{\mathbb{Z}_N} \times U(1) \cong {}^{SU(N)\times U(1)}/_{\mathbb{Z}_N} \cong U(1) \tag{21}$$

which we will do using the First Isomorphism Theorem:

First Isomorphism Theorem. *Let G and H be groups, and let $\varphi : G \to H$ be a homomorphism. Then*

1. $\ker(\varphi) \trianglelefteq G$,
2. $\operatorname{Im}(\varphi) \leq H$, *and*
3. ${}^{G}/_{\ker(\varphi)} \cong \operatorname{Im}(\varphi)$

Let's prove the first isomorphism in (21) now, and then do the second.

Isomorphism 1. ${}^{SU(N)\times U(1)}/_{\mathbb{Z}_N} \cong {}^{SU(N)}/_{\mathbb{Z}_N} \times U(1)$

Proof. Let $\varphi : SU(N)\times U(1) \to {}^{SU(N)}/_{\mathbb{Z}_N} \times U(1)$ be a homomorphism, defined such that $\varphi(S, e^{i\alpha}) = (\overline{S}, e^{i\alpha})$, for $S \in SO(N)$, $\alpha \in [0, 2\pi)$, $\overline{S} \in {}^{SU(N)}/_{\mathbb{Z}_N}$. Note that this is only a homomorphism because $SU(N)$ is a normal subgroup of $U(N)$.

φ is onto since $\forall \overline{S} \in {}^{SU(N)}/_{\mathbb{Z}_N}$, it's represented by one of its elements $S \in SO(N)$. Thus, we can always find at least one element $(S, e^{i\alpha})$ that maps to $(\overline{S}, e^{i\alpha})$.

Furthermore, $\ker(\varphi) = \{(S, 1) \mid S \in \mathbb{Z}_N\}$, which is exactly the definition of $\mathbb{Z}_N \times \{1\} \cong \mathbb{Z}_N$.

Thus, by the First Isomorphism Theorem, ${}^{SU(N)\times U(1)}/_{\mathbb{Z}_n} \cong {}^{SU(N)}/_{\mathbb{Z}_N} \times U(1)$. □

Now, to conquer the second isomorphism in (21), we need the following lemma:

Lemma 1. *Every $U \in U(N)$ can be written $U = e^{i\alpha/N}S$, for some $\alpha \in [0, 2\pi)$ and some $S \in SU(N)$.*

Proof. Note that, since $U \in U(N)$, $\det(U) = e^{i\alpha}$ for some $\alpha \in [0, 2\pi)$. So write $U = e^{i\alpha/N}S$, where we let $S = e^{-i\alpha/N}U$. Clearly $e^{i\alpha/N} \in U(1)$, so we just need to show $S \in SU(N)$.

$\det(S) = \det\left(e^{-i\alpha/N}U\right) = e^{-i\alpha}\det(U) = 1$, and

$S^\dagger S = e^{i\alpha/N}U^\dagger e^{-i\alpha/N}U = I$

So we're done. □

Finally we're ready for the second isomorphism:

Isomorphism 2. ${}^{SU(N)\times U(1)}\!/\!_{\mathbb{Z}_N} \cong U(1)$

Proof. Let $\psi : SU(N) \times U(1) \to U(N)$ be a homomorphism, defined such that $\psi(S, e^{i\alpha}) = Se^{i\alpha}$.

By lemma (1), every $U \in U(N)$ can be written as $e^{i\alpha/N}S$ for some $S \in SU(N)$. Thus, for any $U \in U(N)$, there exists an element of $SO(N) \times U(1)$ such that $\psi(S, e^{i\alpha/N}) = e^{i\alpha/N}S = U$. Thus, ψ is onto.

Finally, $\ker(\psi) = \{(Ie^{-i\alpha}, e^{i\alpha})\} \cap SO(N) \times U(1) = \{(Ie^{-i\alpha/N}, e^{i\alpha/N})\}$, which can be seen to be isomorphic to $\mathbb{Z}_N$.

Thus, by the First Isomorphism Theorem, ${}^{SU(N)\times U(1)}\!/\!_{\mathbb{Z}_N} \cong U(1)$. $\square$

At the level of the Lie algebra, we can in fact make the further simplification that $U(N) \cong SU(N) \times U(1)$. While this isn't strictly a true statement, there doesn't end up being a practical distinction between $SU(N)$ and ${}^{SU(N)}\!/\!_{\mathbb{Z}_N}$ when talking about their Lie algebras.

To see why, write an element of $U(N)$ as $U = e^{iH} \approx I + iH$, where H is some matrix. Then we can see that H must be Hermitian: $U^\dagger U \approx (I - iH)(I + iH) \approx I - iH^\dagger + iH = I$, which means $H^\dagger = H$. This ends up being the only restriction on H (unlike with $SU(N)$, where we can find that $\mathrm{tr}\{H\} = 0$). Thus, the generators of the Lie algebra of $U(N)$ are the generators of the group of $N \times N$ Hermitian matrices *and the identity* I. Thus, the $\mathbb{Z}_N$ part doesn't need to be "removed". This is a very rough way of seeing it.

Ultimately, for typical applications in physics, the global properties of these sorts of groups don't matter at all. The theory of the strong, weak, and electromagnetic interactions is based on the group $SU(3) \times SU(2) \times U(1)$. For this group, we really only need worry about its Lie algebra — its *local*, not global, properties. Thus, the distinction doesn't end up being necessary, and we can use it in its simpler form.