The Fundamental of Cosmology

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1 Introduction

The purpose of this independent study is to familiarize ourselves with the fundamental concepts of cosmology. This paper is based on my lecture for the Kapitza Society. Here, I briefly discuss the spacetime geometry and the expansion of the Universe. In Section 2, I will introduce the Robertson-Walker metric, which describes an isotropic and homogeneous universe. In Section 3, I will discuss cosmological redshift, Hubble's law, and the Hubble constant.

2 Spacetime Geometry

2.1 The isotropic and homogeneous universe

As the early Universe expanded into its current state from a high-density and high-temperature plasma gas, it left an electromagnetic imprint known as the cosmic microwave background (CMB). Observations of the CMB radiation, which carry information about the Universe at its early state, suggest the Universe is isotropic at a sufficiently large scale[1]: it is the same in all directions. Indeed, there is no reason to think we, the Solar System, and the Milky Way are in a special position of the Universe. It should be added the Universe is isotropic only to observers moving at low speed, i.e. the average speed of a typical galaxy in their neighborhood. For example, if an observer moves at a speed comparable to the speed of light, they will see objects displaced towards the direction of motion[2]. Because the Universe is the same in all direction at all points, it follows the Universe must also be homogeneous: it is the same in all location. Isotropy and homogeneity ensure the same observational evidence is available by looking in any direction and to observers at different locations.

[1] For some, this claim might seem counter-intuitive: when we look at the sky, we do see different pictures in different directions. It is true the Universe is not isotropic at the scale at which the human eyes can resolve, or even at the galactic scale. The Universe, however, is isotropic at distances larger than about 300 million light-years.

[2] This phenomenon is known as stellar aberration.

In this paper, we will use the picture of an isotropic and homogeneous universe to build the mathematical tools necessary to describe the spacetime geometry of the Universe.

2.2 The Robertson-Walker metric

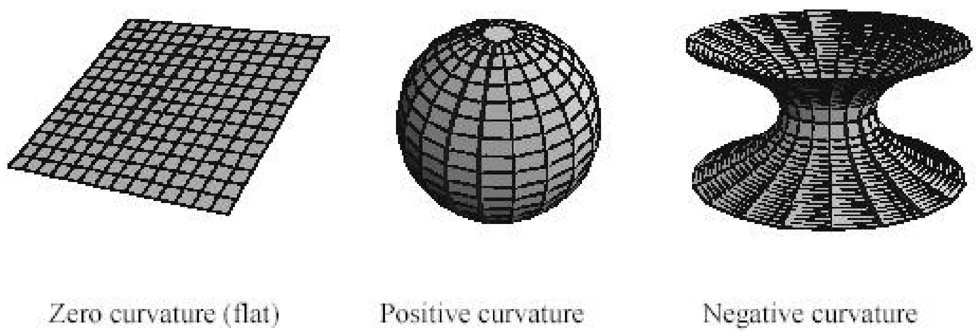

Figure 1: An illustration of the possible curvatures of spacetime.

In general relativity, the spacetime geometry is described by the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ where $g_{\mu\nu}$ is a metric tensor. Considering the geometry of a three-dimensional isotropic and homogeneous space, one may show only three metrics satisfy this condition:

$$ds^2 = a^2 dx^2, \tag{1}$$

$$ds^2 = a^2(d\mathbf{x}^2 + dz^2), \quad z^2 + \mathbf{x}^2 = 1, \tag{2}$$

$$ds^2 = a^2(d\mathbf{x}^2 - dz^2), \quad z^2 - \mathbf{x}^2 = 1, \tag{3}$$

where a is a constant ($a^2 > 0$ so that $ds^2 > 0$ at $\mathbf{x}^2$). Here, the first line element describes flat space and is invariant under three-dimensional rotations and translations. The second line element describes the geometry of a four-dimensional spherical surface in four-dimensional Euclidean space with a radius a and is invariant under four-dimensional rotations. The third line element describes the geometry of a four-dimensional hyperspherical surface in four-dimensional pseduo-Euclidean space with radius a and is invariant under four-dimensional pseduo-rotations.

Eliminating dz in Eq. 2 and Eq. 3 using the relation $zdz = \mp\mathbf{x}\cdot d\mathbf{x}$, we may rewrite the equations for the line element as:

$$ds^2 = a^2\left[d\mathbf{x}^2 + k\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1-k\mathbf{x}^2}\right], \text{ where } k = \begin{cases} +1 & \text{spherical} \\ -1 & \text{hyperspherical} \\ 0 & \text{Euclidean} \end{cases} \tag{4}$$

Here, k is also known as the curvature constant. Figure 1 shows a visualization of spacetime for the three possible curvature. To obtain the spacetime element from the above equation, we make a a function of the time t and include time into the expression:

$$ds^2 = g_{\mu\nu}dx^\mu dx^2\nu = -dt^2 + a^2(t)\left[d\mathbf{x}^2 + k\frac{(\mathbf{x}\cdot d\mathbf{x})^2}{1-k\mathbf{x}^2}\right] \tag{5}$$

This is known as the *Robertson-Walker metric.* $a(t)$ is known as the *Robertson-Walker scale factor*. The metric can be written as:

$$g_{ij} = a^2(t)(\delta_{ij} + k\frac{x^i x^j}{1 - k\mathbf{x}^2}), \quad g_{i0} = 0, \quad g_{00} = -1 \tag{6}$$

where i, j run from 1 to 3. An disadvantage of using the Cartesian coordinates x^i to describe spacetime is that the metric is not diagonalized. To get rid of the non-diagonal terms, we transform the line element into spherical coordinate:

$$d\mathbf{x}^2 = dr^2 + r^2 d\Omega \tag{7}$$

$$\Rightarrow \boxed{ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1 - k\mathbf{r}^2} + r^2 d\Omega\right]} \tag{8}$$

where $d\Omega$ is the solid angle ($d\Omega \equiv d\theta^2 + \sin^2\theta d\phi^2$). The metric is now diagonal:

$$g_{rr} = \frac{a^2(t)}{1 - kr^2}, \; g_{\theta\theta} = a^2(t)r^2, \; g_{\phi\phi} = a^2(t)r^2\sin^2\theta, \; g_{00} = -1 \tag{9}$$

From Eq. 8, r is known as the *comoving distance*, which factors out the expansion of the Universe. The comoving distance between two non-moving objects is the same for all time. Unlike the comoving distance, the *proper distance* is the distance an observer would see at a specific point in time and is given by:

$$d(r,t) = \int_0^r \sqrt{g_{rr}} dr = a(t)\int_0^r \frac{dr}{\sqrt{1 - kr^2}} = a(t) \times \begin{cases} \sin^{-1} r & k = +1 \\ \sinh^{-1} r & k = -1 \\ r & k = 0 \end{cases} \tag{10}$$

Note that at small distances ($r \ll 1$), the proper distance is simply the proper distance in a flat universe $d = a(t)r$. Figure 2 illustrates the comoving distance and the proper distance for two non-moving objects as the Universe expands.

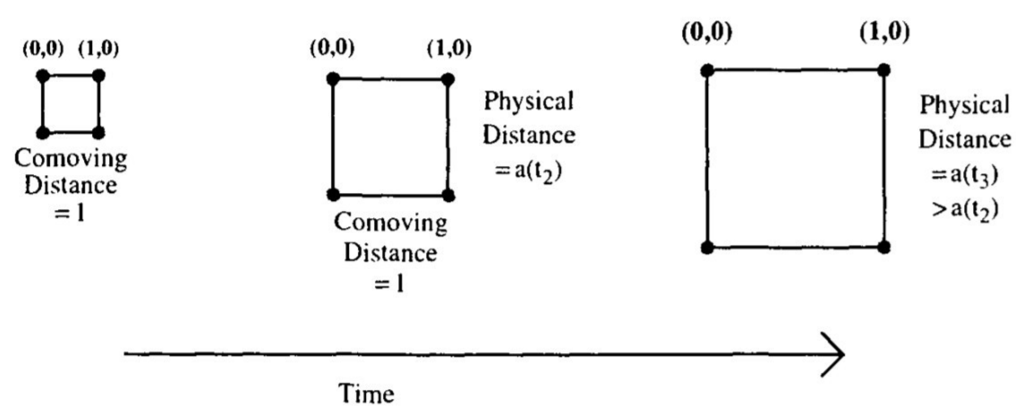

Figure 2: The comoving distance v. the proper distance.

3 The expanding Universe

3.1 The cosmological redshift

The Robertson-Walker scale factor $a(t)$ describes the evolution of the Universe. The Universe is expanding if $a(t)$ is increasing and shrinking if $a(t)$ is decreasing. In addition, the time derivative of $a(t)$ carries information about the rate of the expansion. Therefore, to determine the fate of the Universe (e.g. whether the Universe will expand into infinity or collapse into a singularity), it is important to measure $a(t)$ and $\dot{a}(t)$.

Assuming a source at a comoving distance r_1 at time t_1 emitting a light signal to Earth at the origin $r = 0$ at a time $t_0 > t_1$. A radial light ray has $d\Omega = 0$ and follows a null geodesic, which has the line element $ds^2 = 0$. From Eq. 8, we may have:

$$0 = -dt^2 + a^2(t)\frac{dr^2}{1 - kr^2} \tag{11}$$

$$\Rightarrow dt = \pm a(t)\frac{dr}{\sqrt{1 - kr^2}} \tag{12}$$

$$\Rightarrow \frac{dt}{a(t)} = \frac{-dr}{\sqrt{1 - kr^2}} \tag{13}$$

Here we choose the negative sign because the light is travels from the source to the origin (so r is decreasing). Let the wavelength and period of the light be $\delta r \ll 1$ and $\delta t \ll 1$. The equation becomes

$$\frac{\delta t}{a(t)} = \frac{\delta r}{\sqrt{1 - kr^2}} \quad \forall t \tag{14}$$

The right-hand side is constant because r is constant. Therefore, we arrive at the relation

$$\frac{\delta t_1}{a(t_1)} = \frac{\delta t_0}{a(t_0)} \Rightarrow \frac{\delta t_1}{\delta t_0} = \frac{a(t_1)}{a(t_0)} = \frac{\lambda_1}{\lambda_0} \tag{15}$$

where λ is the wavelength of the light. Here, λ_1 is the emitted wavelength and λ_0 is the observed wavelength. If $\lambda_1 > \lambda_0$, then $a(t_1) > a(t_0)$ and $a(t)$ is decreasing (as a reminder, $t_0 > t_1$). If $\lambda_0 > \lambda_1$, then $a(t_1) < a(t_0)$ and $a(t)$ is increasing. We define ratio between the observed wavelength and the emitted wavelength as:

$$1 + z \equiv \frac{a(t_0)}{a(t_1)} = \frac{\lambda_0}{\lambda_1} \Rightarrow z = \frac{\lambda_0 - \lambda_1}{\lambda_1} \tag{16}$$

If $z > 0$, $a(t)$ is increasing and the light is *redshifted*. If $z < 0$, $a(t)$ is decreasing and the light is *blueshifted*. Based on the observations of various emission and absorption lines from galaxy spectra, we found light from all galaxies is redshifted. The Universe is indeed expanding. For this reason, in astronomy, z is commonly known as the $redshift$. Note, however, that galaxies do not always recede away from us. For example, M31, also known as the Andromeda Galaxy, are moving towards the Milky Way at a speed of about 225 km/s[3].

[3] We need not to worry, however. The two galaxies are expected to collide in 4 billion years, a timescale much longer than any civilization has ever existed.

Indeed, there are roughly 100 observable blueshifted galaxies. This is due to the *peculiar motion* of galaxies created by the gravitational attraction between nearby galaxies. At such a small scale, the peculiar motion is much more dominant than the apparent motion due to the expansion.

3.2 The Hubble's law

Historically, although the expansion of the Universe was first suggested by Einstein's field equations, Edwin Hubble was the first to show this. In 1929, using the observed distances and radial velocities (which could be determined from redshifts) of galaxies, Hubble found not only galaxies are moving away from us, the further the galaxy the greater the receding velocity is. Hubble's law describes the distance-velocity relation of galaxies:

$$v = H_0 d \tag{17}$$

where H_0 is the *Hubble constant*. From Eq. 17, it is worth noting that if a non-moving object is further than $d_c = c/H_0$, then it will appear to be moving faster the speed of light. Indeed, the Universe is expanding at a speed faster than light. This, however, does not violate Einstein's special relativity because the limitation only applies to objects with nonzero mass, not the expansion of spacetime. To obtain an expression for the Hubble constant, we expand $a(t)$ in a power series:

$$a(t) = a(t_0)\left[1 + \frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0}(t - t_0) + ...\right] \tag{18}$$

$$\Rightarrow 1 = (1+z)\left[1 + \frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0}(t - t_0) + ...\right] \tag{19}$$

$$\Rightarrow 1 = 1 + z + \frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0}(t - t_0) + ... \tag{20}$$

$$\Rightarrow z \approx \frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0}(t_0 - t) \tag{21}$$

To the first order, the redshift is approximately v/c, with c is the speed of light. Moreover, for nearby galaxies, the proper distance is roughly $d = c(t_0 - t_1)$. We thus obtain the expression for the Hubble constant:

$$v = \left(\frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0}\right) d = H_0 d \tag{22}$$

$$\Rightarrow \boxed{H_0 = \frac{1}{a(t_0)}\frac{da}{dt}\bigg|_{t_0} = \frac{\dot{a}(t_0)}{a(t_0)}} \tag{23}$$

The Hubble constant measures the current expansion rate of the Universe with respect to the current size of the Universe, and has a dimension of velocity over distance. It is not a constant as its value will change as time passed. To measure the Hubble constant at any time t, $H \equiv \dot{a}(t)/a(t)$, we differentiate Eq. 16 with respect to t_0:

$$\frac{dz}{dt_0} = \frac{\dot{a}(t_0)}{a(t_1)} - \frac{a(t_0)\dot{a}(t_1)}{a^2(t_1)}\frac{dt_1}{dt_0} = \frac{a(t_0)}{a(t_1)}\left[\frac{\dot{a}(t_0)}{a(t_0)} - \frac{\dot{a}(t_1)}{a(t_1)}\frac{dt_1}{dt_0}\right] = (1+z)\left(H_0 - H(t_1)\frac{dt_1}{dt_0}\right) \quad (24)$$

With $dt_1/dt_0 = 1/(1+z)$, the equation becomes:

$$\frac{dz}{dt_0} = (1+z)H_0 - H(t_1) \ \Rightarrow H(t_1) = (1+z)H_0 - \frac{dz}{dt_0} \quad (25)$$

3.3 The Hubble constant

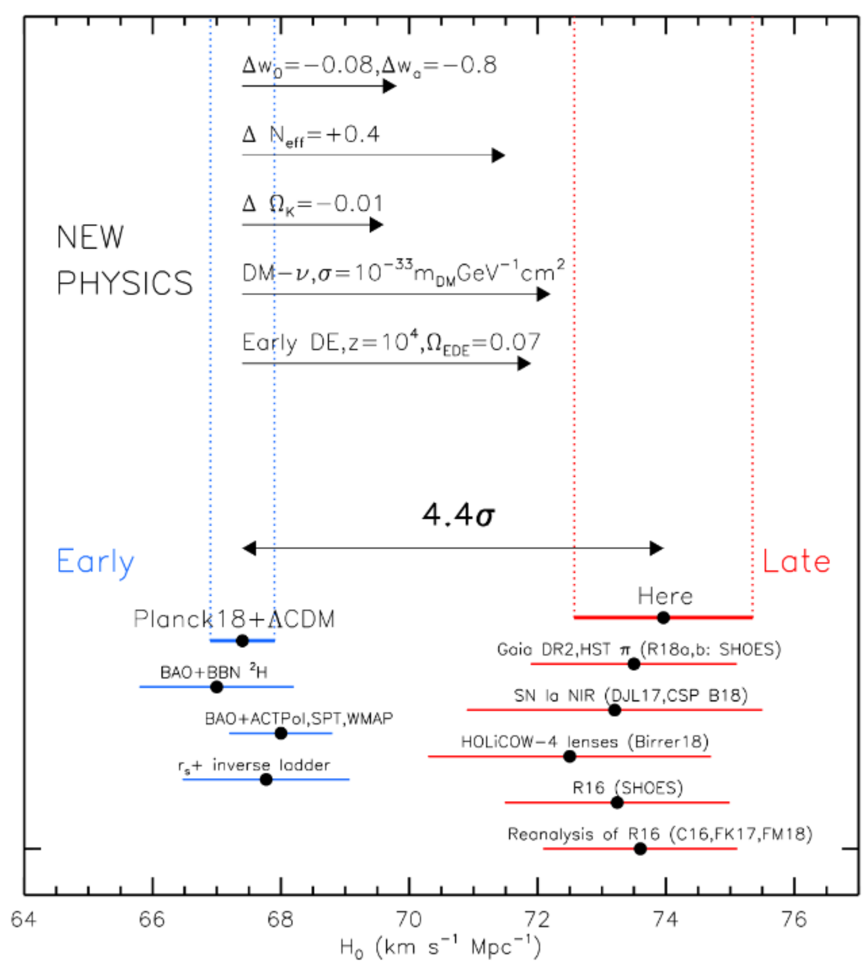

Figure 3: The measurements of the Hubble constant from different experiments. At the top, the arrows show how much the WMAP/Planck value would shift if we adjust the corresponding parameters in the ΛCDM model. Figure from [1].

The measurement of the Hubble constant has been a controversial topic in astronomy. The most straight forward way is to use Hubble's law (Eq. 17). The redshift (and receding velocity) can be obtained easily from spectroscopy. The distance, however, is much harder to measure. For nearby objects, one may use *geometric parallax*[4], or the period-luminosity relationship of Cepheid variable stars[5]. For distant objects, one may measure the distance from Type Ia supernovae[6]. Given the redshift and distance, one may then measure the Hubble constant. A recent measurement of the Hubble constant on March 18, 2019, using Type Ia supernovae technique from the SHOES (Supernovae, HO, for the Equation of State of Dark energy) collaboration using the Hubble Space Telescope yields $H_0 = 74.03 \pm 1.42$ (km/s)/Mpc [1].

Alternatively, because the Hubble constant is imprinted in the CMB, one may measure the Hubble constant from the CMB *assuming* a cosmological model. Assuming the ΛCDM model, the WMAP (Wilkinson Microwave Anisotropy Probe) and Planck satellites, whose mission is to accurately measure the CMB, measures the Hubble constant by characterizing the detailed structure of the CMB fluctuations. The Hubble constant from the WMAP/Planck data is $H_0 = 69.32 \pm 0.80$ (km/s)/Mpc [2].

Unfortunately, the WMAP/Planck value disagrees with the SHOES value by approximate 5 standard deviations. This discrepancy could potentially be because while WMAP/Planck utilizes data from the CMB from the early Universe, while SHOES uses supernovae data from the nearby, thus older, Universe. Indeed, independent measurements of the Hubble constant on the large scale disagree with those on the small scale. This suggests the possibility of new physics. It could be that the ΛCDM model is incomplete. It could also be that our understanding of Type Ia supernovae is incomplete. The true value of the Hubble constant is still an ongoing debate among cosmologists. Figure 3 shows measurements of the Hubble constant for different experiments and the adjustments to the parameters of the ΛCDM model needed to make the two measurements statistically consistent.

In 2016, the discovery of gravitational waves by the LIGO/Virgo collaboration opens up a new way to look into the Universe. On August 17, 2017, LIGO/Virgo detected a gravitational-wave signal from a binary neutron star merger, GW170817. At about 1.7 seconds after, the Fermi Gamma-ray Space Telescope detected a short gamma-ray burst, GRB 170817A. Interestingly, these two events appeared to have come from the same binary neutron star merger, i.e. GRB 170817A is the electromagnetic counterpart of GW170817. This event not only gives rise to a new era of multi-messenger astronomy but more importantly offers a new way to measure the Hubble constant. As mentioned above, the receding velocity can be measured easily from the spectrum of the merger's host galaxy. The gravitational-wave signal, in turn, carries the information about the distance. By comparing the amplitude of the observed waveform with that of the template waveform generated from numerical relativity, one may obtain the distance. In a recent study, physicists from the University of Chicago found the Hubble constant obtained from the GW170817 to be $H_0 = 70.0^{+12.0}_{-8.00}$ [3].

[4] Parallax is apparent displacement of an object when viewed along two different lines of sight. Objects on the sky appear to be shifting as the Earth orbits the Sun. Knowing the orbital radius of Earth, one may measure the distance to the object from its angular displacement.

[5] Cepheid variable stars are red giant stars whose luminosity fluctuates periodically. One may calculate the intrinsic luminosity of the star from its period. By comparing the intrinsic and observed luminosity, one may measure the distance to the star and its host galaxy.

[6] Type Ia supernovae have standardized peak luminosity because the masses of the white dwarfs creating them are constant. In addition, Type Ia supernovae are one of the brightest objects (they can outshine entire galaxies and radiate more energy than the Sun will in its lifetime), making them relatively easy to observe even at large distances. By comparing the intrinsic and apparent luminosity, one may measure the distance.

The measurement error is large (about 17%) because there is only one data point (namely GW170817). To reconcile the disagreement regarding the value of the Hubble constant, more data points are needed. At the time of this paper, LIGO/Virgo recently started the third observation run O3 (April 1, 2019) with a significant increase in sensitivity. Hopefully, in the future, the number of neutron star mergers with an electromagnetic counterpart will increase significantly, giving scientists a more precise measurement of the expansion of the Universe.

4 Conclusion

In Section 2, we derived the Robertson-Walker metric describing an isotropic and homogeneous universe. We obtained an expression for the comoving distance and the proper distance between two non-moving objects. While the comoving distance factors out the expansion of the Universe and thus is constant at all time, the proper distance measured by an observer increases as the Universe expands (or decreases as the Universe shrinks). In Section 3, we derived an expression for cosmological redshift. We found the Universe is expanding because light rays from distant galaxies are redshifted. In addition, we derived Hubble's law, which relates the distance and receding velocity of an object as seen by an observer on Earth. We obtained an expression for the Hubble constant H_0 and the expansion rate $H(t)$ at any time t. Last but not least, we discussed ways to measure the Hubble constant and the ongoing debate about its real value between the data from the CMB and the data from Type Ia supernovae. With the discovery of gravitational wave in 2016, we obtained a new way to measure the Hubble constant.

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