CONTEMPORARY MATHEMATICS

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Mathematical Aspects of Quantization

Center for Mathematics at Notre Dame

Summer School and Conference
May 31–June 10, 2011
Notre Dame University, Notre Dame, Indiana

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We dedicate this volume in memory of Jean-Marie Souriau, in honor of his contributions to quantization.

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Preface

Quantization is an important topic in mathematics and physics. From the physics point of view, methods of quantization are procedures for building models for quantum mechanical systems from analogous and more intuitive classical mechanical systems, which provide strikingly precise experimental predictions. Much of the development of theoretical physics in the 20th century may be regarded as the process of refining quantization to give improved experimental predictions, and the search for a unified field theory is an attempt to quantize general relativity in a manner compatible with existing quantum theory. On the mathematics side, problems related to quantization and quantum mechanics was a strong motivation for the development of functional analysis, the representation theory of Lie groups, and spectral geometry. More recent developments with much current activity include geometric quantization, deformation quantization, and quantum analogues of various classical objects.

Geometric quantization seeks to give as natural as possible a procedure for associating a Hilbert space to each symplectic manifold satisfying an integrality condition. Geometric quantization was developed by Kirillov, Kostant, and Souriau in the 1960's, and was energized by the attempt to prove the "quantization commutes with reduction" conjecture of Guillemin and Sternberg in the 1980's and 1990's, which was completed in part by work of Meinrenken, and is continued through the study of its L^2-analogues. Much work has been done on the semiclassical asymptotics of geometric quantization, such as the proof by Bordemann, Meinrenken, and Schlichenmaier of a general asymptotic formula for Berezin–Toeplitz quantization on compact Kähler manifolds. Semiclassical analysis has resulted in applications in number theory, as in the work of Borthwick and Uribe on relative Poincare series.

The theory of deformation quantization seeks to deform the commutative algebra of functions on a Poisson manifold into a noncommutative algebra in which the semi-classical limit is given by the Poisson bracket of functions. Kontsevich's proof of his formality conjecture showed that every Poisson manifold has a star product on the formal level, and this work was one of the key results which earned him the Fields medal. This work was later reinterpreted by Tamarkin and related to path integrals by Cattaneo and Felder. Ideas from deformation quantization also play a central role in recent work of Costello giving a rigorous geometric construction of the Witten genus. Deformation quantization is used by Etingof and Ginzburg to give a better geometric understanding of the rational Cherednik algebra and its representations, as well as for other associative algebras.

The papers in this volume are based on talks given at the Center for Mathematics at Notre Dame program on quantization, which was held from May 31 to June 10 of 2011. The program consisted of a summer school on quantization, followed by a conference titled "Mathematical aspects of quantization". The papers by Berest–Samuelson, Dolgushev–Rogers, Lerman, and Meinrenken are based on talks given at the summer school. The paper by Berest and Samuelson begins with an elegant proof of properties of the Dunkl operators using a deformation of the de Rham complex, and continues to discuss the representation theory of the Cherednik algebra. The paper by Dolgushev and Rogers is concerned with the graph complex, which plays a key role in the Kontsevich formality conjecture, and results of Willwacher which relate the cohomology of the graph complex to the cohomology of the Gerstenhaber operad. Dolgushev and Rogers give a detailed and complete discussion of Willwacher's proof and the necessary background. The paper by Lerman gives a short introduction to geometric quantization, which we hope will make the subject more accessible to graduate students. The paper by Meinrenken gives a survey of the theory of group-valued moment maps and its applications to the moduli space of flat bundles on a surface, which was developed by Meinrenken together with his collaborators Alekseev, Malkin, and Woodward. The remaining papers are based on talks at the conference. Barron discusses interactions between quantization and automorphic forms. Berest, Chen, Eshmatov, and Ramadoss discuss derived versions of Poisson structures and their applications to Calabi–Yau algebras. Kar and Rajeev give an elementary explanation of renormalization. Schlichenmaier's paper gives a survey of Berezin–Toeplitz quantization and star products in the Kähler setting. Śniatycki gives a survey of his results concerning commutation of geometric quantization with algebraic reduction.

Jean-Marie Souriau, who was one of the pioneers in the theory of quantization, passed away on March 15, 2012, as this volume was being prepared. Souriau's 1966 paper, "Quantification géométrique," in Communications on Mathematical Physics was one of the seminal papers leading to the modern theory of geometric quantization. Souriau also made important contributions to the study of moment maps and coadjoint orbits, both of which are by now standard tools in the quantization toolbox. Souriau spent most of his career as Professor of Mathematics at the University of Provence in Marseille.

The Center for Mathematics at Notre Dame provided the resources to run the program in quantization, and we would like to thank Gregory Crawford, Dean of the College of Science at Notre Dame, for enabling us to establish this new center. We would also like to thank our colleagues at Notre Dame for helping organize the center, and especially thank Lisa Tranberg for her efficient organization of the conference. Finally, we would like to thank all of the participants in our quantization program for making the program such an interesting event.

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Dunkl operators and quasi-invariants of complex reflection groups

Yuri Berest and Peter Samuelson

Abstract. In these notes we give an introduction to representation theory of rational Cherednik algebras associated to complex reflection groups. We discuss applications of this theory in the study of finite-dimensional representations of the Hecke algebras and polynomial quasi-invariants of these groups.

1. Introduction

These are lecture notes of a minicourse given by the first author at the summer school on Quantization at the University of Notre Dame in June 2011. The notes were written up and expanded by the second author who took the liberty of adding a few interesting results and proofs from the literature. In a broad sense, our goal is to give an introduction to representation theory of rational Cherednik algebras and some of its recent applications. More specifically, we focus on the two concepts featuring in the title (Dunkl operators and quasi-invariants) and explain the relation between them. The course was originally designed for graduate students and non-experts in representation theory. In these notes, we tried to preserve an informal style, even at the expense of making imprecise claims and sacrificing rigor.

The interested reader may find more details and proofs in the following references. The original papers on representation theory of the rational Cherednik algebras are [**EG02b**], [**BEG03a, BEG03b**], [**DO03**] and [**GGOR03**]; surveys of various aspects of this theory can be found in [**Rou05**], [**Eti07**], [**Gor08**] and [**Gor10**]. The quasi-invariants of Coxeter (real reflection) groups first appeared in the work of O. Chalykh and A. Veselov [**CV90, CV93**] (see also [**VSC93**]); the link to the rational Cherednik algebras associated to these groups was established in [**BEG03a**]; various results and applications of Coxeter quasi-invariants can be found in [**EG02a**], [**FV02**], [**GW04**], [**GW06**], [**BM08**]; for a readable survey, we refer to [**ES03**].

The notion of a quasi-invariant for a general complex reflection group was introduced in [**BC11**]. This last paper extends the results of [**BEG03a**], unifies the proofs and gives new applications of quasi-invariants in representation theory and noncommutative algebra. It is the main reference for the present lectures.

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2. Lecture 1

2.1. Historical remarks. The theory of rational Cherednik algebras was historically motivated by developments in two different areas: integrable systems and multivariable special functions. In each of these areas, the classical representation theory of semisimple complex Lie algebras played a prominent role. On the integrable systems side, one should single out the work of M. Olshanetsky and A. Perelomov [**OP83**] who found a remarkable generalization of the Calogero-Moser integrable systems for an arbitrary semisimple Lie algebra. On the special functions side, the story began with the fundamental discovery of Dunkl operators [**Dun89**] that transformed much of the present day harmonic analysis. In 1991, G. Heckman [**Hec91**] noticed a relationship between the two constructions which is naturally explained by the theory of rational Cherednik algebras. The theory itself was developed by P. Etingof and V. Ginzburg in their seminal paper [**EG02b**]. In fact, this last paper introduced a more general class of algebras (the so-called symplectic reflection algebras) and studied the representation theory of these algebras at the 'quasi-classical' ($t = 0$) level[1]. At the 'quantum' ($t = 1$) level, the representation theory of the rational Cherednik algebras was developed in [**BEG03a, BEG03b**], [**DO03**] and [**GGOR03**], following the insightful suggestion by E. Opdam and R. Rouquier to model this theory parallel to the theory of universal enveloping algebras of semisimple complex Lie algebras.

In this first lecture, we briefly review the results of [**OP83**] and [**Dun89**] in their original setting and explain the link between these two papers discovered in [**Hec91**]. Then, after giving a necessary background on complex reflection groups, we introduce the Dunkl operators and sketch the proof of their commutativity following [**DO03**]. In the next lecture, we define the rational Cherednik algebras and show how Heckman's observation can be reinterpreted in the language of these algebras.

2.2. Calogero-Moser systems and the Dunkl operators.

2.2.1. *The quantum Calogero-Moser system.* Let $\mathfrak{h} = \mathbb{C}^n$, and let $\mathfrak{h}^{\mathrm{reg}}$ denote the complement (in $\mathfrak{h}$) of the union of hyperplanes $x_i - x_j = 0$ for $1 \le i \ne j \le n$. Write $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ for the ring of regular functions on $\mathfrak{h}^{\mathrm{reg}}$ (i.e., the rational functions on $\mathbb{C}^n$ with poles along $x_i - x_j = 0$). Consider the following differential operator acting on $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$

$$H = \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^2 - c(c+1)\sum_{i\neq j} \frac{1}{(x_i - x_j)^2} \ ,$$

where c is a complex parameter. This operator is called the quantum Calogero-Moser Hamiltonian: it can be viewed as the Schrödinger operator of the system of n quantum particles on the line with pairwise interaction proportional to $(x_i - x_j)^{-2}$. It turns out that H is part of a family of n partial differential operators $L_j : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \to \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ of the form

$$L_j := \sum_{i=1}^{n} \left(\frac{\partial}{\partial x_i}\right)^j + \text{lower order terms} \ ,$$

[1] As yet another precursor of the theory of Cherednik algebras, one should mention the beautiful paper [**Wil98**] which examines the link between the classical Calogero-Moser systems and solutions of the (infinite-dimensional) Kadomtsev-Petviashvili integrable system.

satisfying the properties:

(1) L_j is S_n-invariant (where S_n is the n-th symmetric group acting on $\mathbb{C}^n$ by permutations of coordinates),
(2) L_j is a homogeneous operator of degree $(-j)$,
(3) $L_2 = H$,
(4) $[L_j, L_k] = 0$, i.e. the L_j's commute.

The last property (commutativity) means that $\{L_j\}_{1\le j\le n}$ form a quantum integrable system which is called the (quantum) Calogero-Moser system.

2.2.2. *A root system generalization.* In [**OP83**], the authors constructed a family of commuting differential operators for a Lie algebra $\mathfrak{g}$ that specializes to the Calogero-Moser system when $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. Precisely, let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^*$ its dual vector space, W the corresponding Weyl group, $R \subset \mathfrak{h}^*$ a system of roots of $\mathfrak{g}$, and $R_+ \subset R$ a choice of positive roots (see, e.g., [**Hum78**] for definitions of these standard terms). To each positive root $\alpha \in R_+$ we assign a complex number (multiplicity) $c_\alpha \in \mathbb{C}$ so that two roots in the same orbit of W in R have the same multiplicities; in other words, we define a W-invariant function $c : R_+ \to \mathbb{C}$, which we write as $\alpha \mapsto c_\alpha$. Now, let $\mathfrak{h}^{\mathrm{reg}}$ denote the complement (in $\mathfrak{h}$) of the union of reflection hyperplanes of W. The operator H of the previous section generalizes to the differential operator $H_{\mathfrak{g}} : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \to \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$, which is defined by the formula

$$H_{\mathfrak{g}} := \Delta_{\mathfrak{h}} - \sum_{\alpha\in R_+} \frac{c_\alpha(c_\alpha+1)(\alpha,\alpha)}{(\alpha,x)^2} \ ,$$

where $\Delta_{\mathfrak{h}}$ stands for the usual Laplacian on $\mathfrak{h}$.

THEOREM 2.1 (see [**OP83**]). *For each $P \in \mathbb{C}[\mathfrak{h}^*]^W$, there is a differential operator*

$$L_P = P(\partial_\xi) + \textit{lower order terms}$$

acting on $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$, with lower order terms depending on c, such that

(1) *L_P is W-invariant with respect to the natural action of W on $\mathfrak{h}^{\mathrm{reg}}$*
(2) *L_P is homogeneous of degree $-\deg(P)$,*
(3) *$L_{|x|^2} = H_{\mathfrak{g}}$,*
(4) *$[L_P, L_Q] = 0$ for all $P, Q \in \mathbb{C}[\mathfrak{h}^*]^W$.*

The assignment $P \mapsto L_P$ defines an injective algebra homomorphism

$$\mathbb{C}[\mathfrak{h}^*]^W \hookrightarrow \mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W \ ,$$

where $\mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W$ is the ring of W-invariant differential operators on $\mathfrak{h}^{\mathrm{reg}}$.

In general, finding nontrivial families of commuting differential operators is a difficult problem, so it is natural to ask how to construct the operators L_P. An ingenious idea was proposed by G. Heckman [**Hec91**]. He observed that the operators L_P can be obtained by restricting the composites of certain first order *differential-reflection* operators which are deformations of the usual partial derivatives ∂_ξ; the commutativity $[L_P, L_Q] = 0$ is then an easy consequence of the commutativity of these differential-reflection operators. The commuting differential-reflection operators were discovered earlier by Ch. Dunkl in his work on multivariable orthogonal polynomials (see [**Dun89**]). We begin by briefly recalling the definition of Dunkl operators.

2.2.3. *Heckman's argument.* For a positive root $\alpha \in R_+$, denote by $\hat{s}_\alpha$ the reflection operator acting on functions in the natural way:

$$\hat{s}_\alpha : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \to \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] , \quad f(x) \mapsto f(s_\alpha(x)) .$$

For a nonzero $\xi \in \mathfrak{h}$, the Dunkl operator $\nabla_\xi(c) : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \to \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ is then defined using the formula[2]

$$\nabla_\xi(c) := \partial_\xi + \sum_{\alpha \in R_+} c_\alpha \frac{(\alpha, \xi)}{(\alpha, x)} \hat{s}_\alpha , \tag{2.1}$$

where $\alpha \mapsto c_\alpha$ is the multiplicity function introduced in Section 2.2.2.

THEOREM 2.2 ([**Dun89**]). *The operators ∇_ξ and ∇_η commute for all $\xi, \eta \in \mathfrak{h}$.*

We will sketch the proof of Theorem 2.2 (in the more general setting of complex reflection groups) in the end of this lecture. Now, we show how the Dunkl operators can be used to construct the commuting differential operators L_P.

First, observe that the operators (2.1) transform under the action of W in the same way as the partial derivatives, i. e. $\hat{w}\, \nabla_\xi = \nabla_{w(\xi)}\, \hat{w}$ for all $w \in W$ and $\xi \in \mathfrak{h}$. Hence, for any W-invariant polynomial $P \in \mathbb{C}[\mathfrak{h}^*]^W$, the differential-reflection operator $P(\nabla_\xi) : \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}] \to \mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]$ preserves the subspace $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^W$ of W-invariant functions. Moreover, it is easy to see that $P(\nabla_\xi)$ acts on this subspace as a *purely differential* operator. Indeed, using the obvious relations $\hat{s}_\alpha\, \partial_\xi = \partial_{s_\alpha(\xi)}\, \hat{s}_\alpha$ and $\hat{s}_\alpha\, \hat{f} = \hat{s}_\alpha(f)\, \hat{s}_\alpha$, one can move all nonlocal operators (reflections) in $P(\nabla_\xi)$ 'to the right'; then these nonlocal operators will act on invariant functions as the identity. Thus, for $P \in \mathbb{C}[\mathfrak{h}^*]^W$, the restriction of $P(\nabla_\xi)$ to $\mathbb{C}[\mathfrak{h}^{\mathrm{reg}}]^W$ is a W-invariant differential operator which we denote by $\mathrm{Res}^W[P(\nabla_\xi)] \in \mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W$. If $P(x) = |x|^2$ is the quadratic invariant in $\mathbb{C}[\mathfrak{h}^*]^W$, an easy calculation shows that

$$\mathrm{Res}^W |\nabla_\xi|^2 = \Delta_{\mathfrak{h}} - \sum_{\alpha \in R_+} \frac{c_\alpha(c_\alpha + 1)(\alpha, \alpha)}{(\alpha, x)^2} ,$$

which is exactly the generalized Calogero-Moser operator $H_{\mathfrak{g}}$ introduced in [**OP83**]. In general, if we set

$$L_P := \mathrm{Res}^W[P(\nabla_\xi)] , \quad \forall\, P \in \mathbb{C}[\mathfrak{h}^*]^W,$$

then all properties of Theorem 2.1 are easily seen to be satisfied. In particular, the key commutativity property $[L_P, L_Q] = 0$ follows from the equations

$$\begin{aligned} \mathrm{Res}^W[P(\nabla_\xi)]\ \mathrm{Res}^W[Q(\nabla_\xi)] &= \mathrm{Res}^W[P(\nabla_\xi)\, Q(\nabla_\xi)] \\ &= \mathrm{Res}^W[Q(\nabla_\xi)\, P(\nabla_\xi)] \\ &= \mathrm{Res}^W[Q(\nabla_\xi)]\ \mathrm{Res}^W[P(\nabla_\xi)] , \end{aligned}$$

where $P(\nabla_\xi)\, Q(\nabla_\xi) = Q(\nabla_\xi)\, P(\nabla_\xi)$ is a direct consequence of Theorem 2.2. Thus, the algebra homomorphism of Theorem 2.1 can be defined by the rule

$$\mathbb{C}[\mathfrak{h}^*]^W \to \mathcal{D}(\mathfrak{h}^{\mathrm{reg}})^W , \quad P \mapsto \mathrm{Res}^W[P(\nabla_\xi)] .$$

Despite its apparent simplicity this argument looks puzzling. What kind of an algebraic structure is hidden behind this calculation? This was one of the questions that led to the theory of rational Cherednik algebras developed in [**EG02b**]. In

[2]Strictly speaking, Dunkl introduced his operators in a slightly different form that is equivalent to (2.1) up to conjugation.

the next lecture, we will introduce these algebras in greater generality: for an arbitrary complex (i.e., not necessarily Coxeter) reflection group. The representation theory of rational Cherednik algebras is remarkably analogous to the classical representation theory of semisimple Lie algebras, and we will discuss this analogy in the subsequent lectures. We will also emphasize another less evident analogy that occurs only for *integral* multiplicity values. It turns out that for such values, the rational Cherednik algebras are closely related to the rings of (twisted) differential operators on certain singular algebraic varieties with 'good' singularities (*cf.* Proposition 5.10).

2.3. Finite reflection groups. The family of rational Cherednik algebras is associated to a finite reflection group. We begin by recalling the definition of such groups and basic facts from their invariant theory. The standard reference for this material is N. Bourbaki's book [**Bou68**].

Let V be a finite-dimensional vector space over a field k. We define two distinguished classes of linear automorphisms of V.

(1) $s \in \mathrm{GL}_k(V)$ is called a *pseudoreflection* if it has finite order > 1 and there is a hyperplane $H_s \subset V$ that is pointwise fixed by s. (If s is diagonalizable, this is the same as saying that all but one of the eigenvalues are equal to 1.)
(2) A pseudoreflection is a *reflection* if it is diagonalizable and has order 2. (In this case, the remaining eigenvalue is equal to -1.)

A *finite reflection group* on V is a finite subgroup $W \subset \mathrm{GL}_k(V)$ generated by pseudoreflections. Note that if W_1 is a finite reflection group on V_1 and W_2 is a finite reflection group on V_2, then $W = W_1 \times W_2$ is a finite reflection group on $V_1 \oplus V_2$. We say that W is indecomposable if it does not admit such a direct decomposition.

If $k = \mathbb{R}$, then all pseudoreflections are reflections. The classification of finite groups generated by reflections was obtained in this case by H. S. M. Coxeter [**Cox34**]. There are 4 infinite families of indecomposable Coxeter groups (A_n, B_n, D_n and $I_2(m)$) and 6 exceptional groups (E_6, E_7, E_8, F_4, H_3, H_4). Over the complex numbers $k = \mathbb{C}$, the situation is much more complicated. A complete list of indecomposable complex reflection groups was given by G. C. Shephard and J. A. Todd in 1954: it includes 1 infinite family $G(m, p, n)$ depending on 3 positive integer parameters (with p dividing m), and 34 exceptional groups (see [**ST54**]). A. Clark and J. Ewing used the Shephard-Todd results to classify the groups generated by pseudoreflections over an arbitrary field of characteristic coprime to the group order (see [**CE74**]).

There is a nice invariant-theoretic characterization of finite reflection groups. Namely, the invariants of a finite group $W \subset \mathrm{GL}_k(V)$ form a polynomial algebra if and only if W is generated by pseudoreflections. More precisely, we have

THEOREM 2.3. *Let V be a finite-dimensional faithful representation of a finite group W over a field k. Assume that either* $\mathrm{char}(k) = 0$ *or* $\mathrm{char}(k)$ *is coprime to* $|W|$*. Then the following properties are equivalent:*

(1) *W is generated by pseudoreflections,*
(2) *$k[V]$ is a free module over $k[V]^W$,*

(3) $k[V]^W = k[f_1, \dots, f_n]$ *is a free polynomial algebra*[3].

Over the complex numbers, this theorem was originally proved by Shephard and Todd [**ST54**] using their classification of pseudoreflection groups. Later Chevalley [**Che55**] gave a proof in the real case without using the classification, and Serre extended Chevalley's argument to the complex case. Below, we will give a homological proof of the (most interesting) implication (1) $\Rightarrow$ (2), which is due to L. Smith [**Smi85**]; it works in general in coprime characteristic.

First, we recall the following well-known result from commutative algebra.

LEMMA 2.4. *Let $A = \bigoplus_{i\geq 0} A_i$ be a non-negatively graded commutative k-algebra with $A_0 = k$, and let M be a non-negatively graded A-module. Then the following are equivalent:*

(1) *M is free,*
(2) *M is projective,*
(3) *M is flat,*
(4) $\mathrm{Tor}_1^A(k, M) = 0$, *where* $k = A/A_+$.

Note that the only nontrivial implication in the above lemma is: (4) $\Rightarrow$ (1). Its proof is standard homological algebra (see, e.g., [**Ser00**], Lemma 3, p. 92).

PROOF OF THEOREM 2.3, (1) $\Rightarrow$ (2). In view of Lemma 2.4, it suffices to prove that

$$\mathrm{Tor}_1^{k[V]^W}(k,\, k[V]) = 0\ .$$

Note that the non-negative grading and W-action on $k[V]$ induce a non-negative grading and W-action on $\mathrm{Tor}_1^{k[V]^W}(k, k[V])$. There are two steps in the proof. First, we show that the nonzero elements in $\mathrm{Tor}_1^{k[V]^W}(k, k[V])$ of minimal degree must be W-invariant, and then we show that W-invariant elements must be 0.

Given a pseudoreflection $s \in W$, choose a linear form $\alpha_s \in V^* \subset k[V]$ such that $\mathrm{Ker}(\alpha_s) = H_s$, and define the following operator

$$\Delta_s := \frac{1}{\alpha_s}(1-s) \in \mathrm{End}_{k[V]^W}(k[V])\ .$$

To see that this operator is well defined on $k[V]$, expand a polynomial $f \in k[V]$ as a Taylor series in α_s and note that the constant term of $(1-s)\cdot f$ is 0. To see that Δ_s is a $k[V]^W$-linear endomorphism, check the identity

$$\Delta_s(fg) = \Delta_s(f)g + s(f)\Delta_s(g)\ , \quad \forall\, f,\, g \in k[V]\ ,$$

and note that, for $f \in k[V]^W$, this identity becomes $\Delta_s(fg) = f\Delta_s(g)$. It is clear that Δ_s has degree -1. Now, since $\mathrm{Tor}_1^{k[V]^W}(k, -)$ is a functor on the category of $k[V]^W$-modules, we can apply it to the endomorphism Δ_s:

$$(\Delta_s)_* :\ \mathrm{Tor}_1^{k[V]^W}(k, k[V]) \to \mathrm{Tor}_1^{k[V]^W}(k, k[V])\ .$$

If $\xi \in \mathrm{Tor}_1^{k[V]^W}(k, k[V])$ has minimal degree, then $(\Delta_s)_*\,\xi = 0$ (since $(\Delta_s)_*$ must decrease degree by one). This implies that $s(\xi) = \xi$, which proves the claim that the nonzero elements of $\mathrm{Tor}_1^{k[V]^W}(k, k[V])$ of minimal degree are W-invariant.

[3]The polynomials $f_i \in k[V]$ generating $k[V]^W$ are not unique; however, their degrees $d_i = \deg(f_i)$ depend only on W: they are called the fundamental degrees of W.

Now, consider the averaging (Reynolds) operator[4]

$$\pi = \frac{1}{|W|}\sum_{g\in W} g : k[V] \to k[V]^W .$$

If we write $\iota : k[V]^W \hookrightarrow k[V]$ for the natural inclusion, then $\iota\circ\pi(f) = f$ for all $f \in k[V]^W$. The identity $\pi(\pi(f)g) = \pi(f)\pi(g)$ shows that π is a map of $k[V]^W$-modules. Applying then the functor $\mathrm{Tor}_1^{k[V]^W}(k, -)$ to the composition $\iota\circ\pi$, we get

$$\mathrm{Tor}_1^{k[V]^W}(k, k[V]) \xrightarrow{\pi_*} \mathrm{Tor}_1^{k[V]^W}(k, k[V]^W) \xrightarrow{\iota_*} \mathrm{Tor}_1^{k[V]^W}(k, k[V])$$

which is the *zero* map since $\mathrm{Tor}_1^{k[V]^W}(k, k[V]^W) = 0$. On the other hand, if $\xi \in \mathrm{Tor}_1^{k[V]^W}(k, k[V])^W$ is W-invariant, we have $\xi = (\iota\circ\pi)_*\,\xi = (\iota_*\circ\pi_*)\,\xi = 0$. Hence $\mathrm{Tor}_1^{k[V]^W}(k, k[V])^W = 0$ finishing the proof. □

2.4. Dunkl operators. From now on, we assume that $k = \mathbb{C}$. We fix a finite-dimensional complex vector space V and a finite reflection group W acting on V and introduce notation for the following objects:

(1) $\mathcal{A}$ is the set of hyperplanes fixed by the generating reflections of W,
(2) W_H is the (pointwise) stabilizer of a reflection hyperplane $H \in \mathcal{A}$; it is a cyclic subgroup of W of order $n_H \geq 2$,
(3) $(\,\cdot\,,\,\cdot\,) : V \times V \to \mathbb{C}$ is a positive definite Hermitian form on V, linear in the second factor and antilinear in the first,
(4) $x^* := (x, \,\cdot\,) \in V^*$ which gives an antilinear isomorphism $V \to V^*$,
(5) for $H \in \mathcal{A}$, fix $v_H \in V$ with $(v_H, x) = 0$ for all $x \in H$, and $\alpha_H \in V^*$ such that $H = \mathrm{Ker}(\alpha_H)$,
(6) $\delta := \prod_{H\in\mathcal{A}} \alpha_H \in \mathbb{C}[V]$ and $\delta^* := \prod_{H\in\mathcal{A}} v_H \in \mathbb{C}[V^*]$,
(7) $\boldsymbol{e}_{H,i} := \frac{1}{n_H}\sum_{w\in W_H} \det_V(w)^{-i}\, w \in \mathbb{C}W_H \subset \mathbb{C}W,\ i = 0, 1, \dots, n_H - 1\,.$

Some explanations are in order. The elements $e_{H,i}$ are idempotents which are generalizations of the primitive idempotents $(1-s)/2$ and $(1+s)/2$ for a real reflection s. The orders $n_H = |W_H|$ depend only on the orbit of H in $\mathcal{A}$. Under the action of W, δ^* transforms as the determinant character $\det_V : \mathrm{GL}(V) \to \mathbb{C}$ and δ transforms as the inverse determinant. Finally, to simplify notation for sums, we introduce the convention of conflating W-invariant functions on $\mathcal{A}$ with functions on $\mathcal{A}/W$, and we write indices cyclicly identifying the index sets $\{0, 1, \dots, n_H - 1\}$ with $\mathbb{Z}/n_H\mathbb{Z}$.

The rational Cherednik algebras associated to W will depend on complex parameters $k_C := \{k_{C,i}\}_{i=0}^{n_C-1}$, where C runs over the set of orbits $\mathcal{A}/W$; these parameters play the role of multiplicities c_α in the complex case. We will assume that $k_{C,0} = 0$ for all $C \in \mathcal{A}/W$. We will need to extend the ring of differential operators $\mathcal{D} = \mathcal{D}(V_{\mathrm{reg}})$ on $V_{\mathrm{reg}} := V \setminus (\cup_{H\in\mathcal{A}} H)$ by allowing group-valued coefficients. The group W acts naturally on $\mathcal{D}$, so we simply let $\mathcal{D}W := \mathcal{D} \rtimes W$ be the crossed

[4]Note that this operator is well defined since the characteristic of k is coprime to $|W|$.

product[5]. Then, for each $\xi \in V$, we define the operator $T_\xi(k) \in \mathcal{D}W$ by

$$T_\xi(k) := \partial_\xi - \sum_{H\in A} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H-1} n_H k_{H,i} e_{H,i}$$

It is an exercise in decoding the notation to check that in the case when W is a Coxeter group, the operator $T_\xi(k)$ agrees with the operator defined in (2.1), up to conjugation by the function $\prod_{H\in\mathcal{A}} \alpha_H^{k_H}$.

The following result was originally proved by Dunkl [**Dun89**] in the Coxeter case and in [**DO03**] in general. We end this lecture by outlining the proof given in [**DO03**].

LEMMA 2.5. *The operators $T_\xi(k)$ satisfy the following properties:*

(1) $[T_\xi(k), T_\eta(k)] = 0$*, i.e. the operators commute*
(2) $wT_\xi(k) = T_{w(\xi)}(k)w$

Note that one consequence of this lemma is that the linear map $\xi \mapsto T_\xi(k)$ yields an algebra embedding $\mathbb{C}[V^*] \hookrightarrow \mathcal{D}W$.

We think of the operators $T_\xi(k)$ as deformations of the directional derivatives ∂_ξ. One strategy for proving commutativity is to find the equations $\partial_\xi\partial_\eta - \partial_\eta\partial_\xi = 0$ "in nature," and then see if the situation in which they appear can be deformed. A natural place these equations appear is in the de Rham differential: $d^2 f = \sum_{i<j}(\partial_i\partial_j - \partial_j\partial_i) f\, dx_i \wedge dx_j$. In this case, commutativity of the partial derivatives is equivalent to the fact that the de Rham differential squares to zero.

We now proceed to deform the de Rham differential, with the eventual goal of showing that the deformed map *is* actually a differential. It turns out that in addition to deforming the de Rham differential, it is also convenient to deform the Euler derivation. First, for each $H \in \mathcal{A}$, define

$$a_H(k) := \sum_{i=0}^{n_H-1} n_H k_{H,i} e_{H,i} \in \mathbb{C}W_H$$

Let $K^\bullet = \mathbb{C}[V] \otimes \Lambda^\bullet V^*$ be the polynomial de Rham complex on V, and define $\Omega(k) \in \mathrm{End}(\mathbb{C}[V]) \otimes K^1$ by

$$\Omega(k) := \sum_{H\in A} a_H(k) \alpha_H^{-1} d\alpha_{\mathcal{H}}$$

The term $\alpha_H^{-1} d\alpha_H$ is the logarithmic differential of α_H, and $\Omega(k)$ is W-equivariant with respect to the diagonal action on K^1. Next we define $d(k) : \mathbb{C}[V] \to K^1$ via

$$d(k)(p) := dp + \Omega(k)(p)$$

(where $d(0)$ is the standard de Rham differential). We extend it to $d(k) : K^\bullet \to K^{\bullet+1}$ in the usual way - if $p \otimes \omega \in \mathbb{C}[V] \otimes \Lambda^\bullet V^*$, then $d(k)(p \otimes \omega) := d(k)(p) \wedge \omega$. Next, we define the standard Koszul differential $\partial : K^l \to K^{l-1}$ using

$$p \otimes dx_1 \wedge \cdots \wedge dx_l \mapsto \sum_{r=1}^{l} (-1)^{r+1} x_r p \otimes dx_1 \wedge \cdots \wedge d\hat{x}_r \wedge \cdots \wedge dx_l$$

[5]As a reminder, if W is a finite group acting on an algebra A by algebra automorphisms, then the crossed product $A \rtimes W$ is defined to be the vector space $A \otimes \mathbb{C}W$ with multiplication $(a \otimes w) \cdot (b \otimes v) = a\, w(b) \otimes wv$.

Finally, we define the (deformed) Euler vector field

$$E(k) := E(0) + \sum_{H \in A} a_H(k)$$

where $E(0)$ is the usual Euler derivation (i.e. the infinitesimal generator of the diagonal $\mathbb{C}^\times$ action on $K^\bullet$). The space $K^\bullet$ has a bigrading given by $K^\bullet = \oplus_m \oplus_{l=0}^{\dim V}$ $\mathbb{C}[V]_m \otimes \Lambda^l V^*$, and if we let K_m^l be one of the graded pieces and let $\omega \in K_m^l$, then

$$E(0)\omega = (l+m)\omega, \quad d(k)\omega \in K_{m-1}^{l+1}, \quad \partial\omega \in K_{m+1}^{l-1}$$

The deformations $E(k)$ and $d(k)$ are compatible in the following sense:

$$E(k) = \partial d(k) + d(k)\partial$$

The following important lemma is used to determine conditions on k which are sufficient to show $E(k)$ has a small kernel.

LEMMA 2.6 ([**DO03**], Lemma 2.5). *Let $z(k) = \sum_{H \in A} a_H(k) \in \mathbb{C}W$. Then*

(1) *The element $z(k)$ is central in $\mathbb{C}W$.*
(2) *For an irreducible representation τ of $\mathbb{C}W$, let $c_\tau(k)$ denote the (unique) eigenvalue for $z(k)$ on τ. Then $c_\tau(k)$ is a linear function of k with non-negative integer coefficients.*

The following theorem is the main step in the proof of commutativity of the Dunkl operators.

THEOREM 2.7 ([**DO03**], Theorem 2.9). *Assume that k is a parameter such that $-c_\tau(k) \notin \mathbb{N}$ for all irreducible τ. Then there exists a unique W-invariant linear isomorphism $S(k) : K^\bullet \to K^\bullet$ satisfying the properties*

(1) *$S(k)(K_m^l) \subset K_m^l$,*
(2) *The restriction of $S(k)$ to K_0^0 is the identity,*
(3) *$S(k)(p \otimes \omega) = (S(k)(p)) \otimes \omega$,*
(4) *$d(k)S(k) = S(k)d(0)$.*

The *intertwining operator* $S(k)$ is constructed by induction on m and l. It is not too difficult to show that the enumerated conditions imply that $S(k)$ is unique and invertible. The main point is to show the existence of $S(k)$. The proof of existence is essentially linear algebra using the fact that the assumption on the parameter k implies (via the previous lemma) that the kernel of $E(k)$ is exactly K_0^0.

COROLLARY 2.8. *The map $d(k)$ is a differential on $K^\bullet$.*

PROOF. If we assume $-c_\tau(k) \notin \mathbb{N}$ for all irreducible τ, then this corollary follows immediately from the existence of the intertwining operator (and the fact that $d(0)$ is a differential). However, the condition $d^2(k) = 0$ is a closed condition (either in Zariski or classical topology), so it must hold on a closed subset of the space of all parameters k. Since the theorem holds on an open dense set in this space, it implies the corollary for all parameter values. □

As mentioned above, the commutativity of Dunkl operators (Lemma 2.5) follows directly from this corollary. Indeed, let $e_i \in V$ and $x_i \in V^*$ be dual bases, and write $T_i = T_{e_i}(k)$. A straightforward computation shows that for all $f \in \mathbb{C}[V]$,

$$d(k)^2 f = \sum_{i<j} (T_i T_j - T_j T_i) f \otimes dx_i \wedge dx_j \ .$$

A different proof of commutativity can be found in [**EG02b**], Section 4.

REMARK 2.9. At first glance, the definition of the Dunkl operators may seem to be quite general. Indeed, at least in the real case, the formulas defining $T_\xi(k)$ make sense for an *arbitrary* finite hyperplane arrangement $\mathcal{A}$ (with prescribed multiplicities). However, the operators T_ξ thus defined will commute if and only if the hyperplane arrangement $\mathcal{A}$ comes from a finite reflection group (*cf.* [**Ves94**]).

3. Lecture 2

In this lecture, we introduce the rational Cherednik algebras and discuss their basic properties. We also define category $\mathcal{O}$, which is a category of modules over the Cherednik algebra subject to certain finiteness restrictions. This is a close analogue of the eponymous category of $\mathfrak{g}$-modules defined by J. Bernstein, I. Gelfand, and S. Gelfand for a semi-simple Lie algebra $\mathfrak{g}$ (see [**Hum08**]).

3.1. Rational Cherednik algebras. In this section, we will use the notation introduced in Section 2.4. We begin with the main definition.

DEFINITION 3.1 (*cf.* [**DO03**]). The *rational Cherednik algebra* $H_k(W)$ is the subalgebra of $\mathcal{D}W$ generated by $\mathbb{C}[V]$, $\mathbb{C}[V^*]$ and $\mathbb{C}W$. (The subalgebras $\mathbb{C}[V]$ and $\mathbb{C}W$ are embedded in $\mathcal{D}W$ in the natural way and are independent of k. On the other hand, the embedding of $\mathbb{C}[V^*]$ in $\mathcal{D}W$ is defined via the Dunkl operators $T_\xi(k)$ which certainly depend on k.)

It is also possible to give an 'abstract' definition of Cherednik algebras in terms of generators and relations. From this point of view, the previous definition is called the *Dunkl representation.* The key point is that the Dunkl representation is faithful. The algebra $H_k(W)$ is generated by the elements of V, V^* and W subject to the following relations

$$[x,\, x'] = 0\ , \quad [\xi,\, \xi'] = 0\ , \quad w\, x\, w^{-1} = w(x)\ , \quad w\, \xi\, w^{-1} = w(\xi)\ ,$$

$$[\xi,\, x] = \langle \xi,\, x\rangle + \sum_{H\in\mathcal{A}} \frac{\langle \alpha_H, \xi\rangle\, \langle x, v_H\rangle}{\langle \alpha_H, v_H\rangle} \sum_{i=0}^{n_H-1} n_H(k_{H,i} - k_{H,i+1})\, e_{H,i}\ .$$

where $x, x' \in V^*$ and $\xi, \xi' \in V$ and $w \in W$.

EXAMPLE 3.2. In the case $W = \mathbb{Z}_2$, the above relations are actually very simple. Specifically, $H_k(\mathbb{Z}_2)$ is generated by x, ξ and s satisfying

$$s^2 = 1\ , \quad sx = -xs\ , \quad s\xi = -\xi s\ , \quad [\xi, x] = 1 - 2ks\ .$$

The Dunkl operator corresponding to ξ is given by $\frac{d}{dx} - \frac{k}{x}(1-s)$ which acts naturally on $V_{\mathrm{reg}} = \mathbb{C}[x, x^{-1}]$, preserving $\mathbb{C}[x] \subset \mathbb{C}[x, x^{-1}]$.

3.2. Basic properties of $H_k(W)$. First, note that if $k = 0$, then $H_0 = \mathcal{D}(V) \rtimes W \subseteq \mathcal{D}(V_{\mathrm{reg}}) \rtimes W$, so we can view the Cherednik algebra as a deformation of the crossed product of a Weyl algebra $\mathcal{D}(V)$ with the group W. The next theorem collects several key properties of $H_k(W)$.

THEOREM 3.3 (see [**EG02b**]). *Let $H_k = H_k(W)$ be the family of Cherednik algebras associated to a complex reflection group W.*

(1) Universality: *$\{H_k\}$ is the universal deformation of H_0.*
(2) PBW property: *the linear map $\mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*] \to H_k$ induced by multiplication in H_k is a $\mathbb{C}[V]$-module isomorphism.*

(3) *Let $H_{\mathrm{reg}} = H_k[\delta^{-1}]$ denote the localization of H_k at the Ore subset $\{\delta^k\}_{k\in\mathbb{N}}$. Then the induced map $H_{\mathrm{reg}} \to \mathcal{D}W$ is an isomorphism of algebras.*

REMARK 3.4. Each statement deserves a comment:

(1) In general, universal deformations can rarely be realized algebraically, so the family of Cherednik algebras is somewhat exceptional.
(2) The name 'PBW property' comes from the Poincare-Birkhoff-Witt Theorem in Lie theory, which has a very similar statement. It is a fundamental property for many reasons. In particular, it is not obvious *a priori* that the generators and relations listed above give a nonzero algebra, so in some sense the important part of this statement is the "lower bound" on the size of H_k. (The "upper bound" is easy to see from the relations.)
(3) This justifies the notation $H_{\mathrm{reg}} := H_k[\delta^{-1}]$, since it shows that the localization is independent of the parameters k.

There are two filtrations on $\mathcal{D}W$ which are commonly used. The first is the *standard filtration*, where $\deg x = 1 = \deg \xi$ and $\deg w = 0$. The second is the *differential filtration*, where $\deg x = 0 = \deg w$, and $\deg \xi = 1$. Through the Dunkl embedding $H_k \hookrightarrow \mathcal{D}W$, these filtrations induce filtrations on H_k (with the same names), and for both filtrations we have $\mathrm{gr}(H_k) \cong \mathbb{C}[V \oplus V^*] \rtimes W$.

3.3. The spherical subalgebra. Each H_k contains a distinguished (nonunital) subalgebra $U_k = U_k(W)$ called the *spherical subalgebra*. We set $\boldsymbol{e} := \frac{1}{|W|}\sum_{w\in W} w \in \mathcal{D}W$ and define

$$U_k := \boldsymbol{e}H_k\boldsymbol{e}$$

(The additive and multiplicative structure of $\boldsymbol{e}H_k\boldsymbol{e}$ are induced from H_k, but the unit of $\boldsymbol{e}H_k\boldsymbol{e}$ is $\boldsymbol{e}$, not $1 \in H_k$.)

The spherical subalgebra U_k is closely related to H_k, however the exact relationship depends crucially on values of the parameter k. For each k, there is an algebra isomorphism $U_k \cong \mathrm{End}_{H_k}(\boldsymbol{e}H_k)$, and although $\boldsymbol{e}H_k$ is a f. g. projective module over H_k, this does not imply that U_k and H_k are Morita equivalent[6]. The problem is that for certain special values of k, the module $\boldsymbol{e}H_k$ is not a generator in the category of right H_k-modules. Such special values are called *singular*, and it is an interesting (and still open) question to precisely determine these values of k for a given W. For *generic* k, one can prove that H_k is a simple algebra (see Theorem 4.5 below), so in that case, the algebras H_k and U_k are Morita equivalent.

For $k = 0$, we know that $H_0 = \mathcal{D}(V) \rtimes W$. Since $\mathcal{D}(V)$ is isomorphic to a Weyl algebra, it is simple, which implies that $\mathcal{D}(V) \rtimes W$ is simple. This implies, in turn, that H_0 is Morita equivalent to U_0. We also remark that $U_0 = \boldsymbol{e}(\mathcal{D}(V) \rtimes W)\boldsymbol{e} \cong \mathcal{D}(V)^W$ via the identification $\boldsymbol{e}d\boldsymbol{e} \leftrightarrow d$. This allows a theorem analogous to Theorem 3.3(1):

THEOREM 3.5. *The family $\{U_k\}$ is a universal deformation of $\mathcal{D}(V)^W$. Also, we have $\mathrm{gr}(U_k) \cong \mathbb{C}[V \oplus V^*]^W$.*

In general, the algebra map $H_k \hookrightarrow \mathcal{D}W$ restricts to a map $U_k = \boldsymbol{e}H_k\boldsymbol{e} \to \boldsymbol{e}(\mathcal{D}W)\boldsymbol{e}$ which is an injective homomorphism of unital algebras. Heckman's restriction operation now becomes

$$\mathrm{Res}_k^W : U_k = \boldsymbol{e}H_k\boldsymbol{e} \hookrightarrow \boldsymbol{e}(\mathcal{D}W)\boldsymbol{e} \xrightarrow{\sim} \mathcal{D}(V_{\mathrm{reg}})^W \tag{3.1}$$

[6]We recall that two rings are *Morita equivalent* if their categories of left (or right) modules are equivalent. We refer to [**MR01**], Section 3.5, for basic Morita theory.

$\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}_-) \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}(\mathfrak{n}_+)$	$H_k(W) = \mathbb{C}[V] \otimes \mathbb{C}W \otimes \mathbb{C}[V^*]$
Weights for $\mathcal{U}(\mathfrak{g})$ are irreducible $\mathcal{U}(\mathfrak{h})$-modules, i.e. $\mu \in \mathfrak{h}^*$	'Weights' for H_k are irreducible W-modules
central character of M	the parameter k
the BGG category $\mathcal{O}$	category $\mathcal{O}'_k$
blocks $\mathcal{O}_\chi$ of $\mathcal{O}$	blocks $\mathcal{O}_k(\lambda)$

TABLE 1. Analogies between $\mathcal{U}(\mathfrak{g})$ and H_k

where the second map is an isomorphism given by $\boldsymbol{eDe} \mapsto D$. We call $\mathrm{Res}_k^W : U_k \hookrightarrow \mathcal{D}(V_{\mathrm{reg}})^W$ the *spherical Dunkl representation* of U_k. Note that Res_k^W is a deformation of the canonical embedding $\mathcal{D}(V)^W \hookrightarrow \mathcal{D}(V_{\mathrm{reg}})^W$ in the same way as the Dunkl embedding $H_k \hookrightarrow \mathcal{D}W$ is a deformation of the canonical map $\mathcal{D}(V) \rtimes W \hookrightarrow \mathcal{D}W$.

3.4. Category $\mathcal{O}$. In this section we discuss a subcategory of H_k-modules that shares many properties with the Berstein-Gelfand-Gelfand category $\mathcal{O}$ in Lie theory. (The BGG category $\mathcal{O}$ is a subcategory of representations of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ which are subject to certain finiteness conditions. A good exposition of this theory can be found in [**Hum08**].) Some analogies between $\mathcal{U}(\mathfrak{g})$ and $H_k(W)$ are listed in Table 1.

DEFINITION 3.6. We introduce the following subcategories of the category $\mathrm{mod}(H_k)$ of finitely generated H_k-modules:

$$\begin{aligned} \mathcal{O}'_k &:= \{M \in \mathrm{mod}(H_k) \mid \dim_{\mathbb{C}}(\mathbb{C}[V^*] \cdot m) < \infty\} \\ \mathcal{O}_k(\bar\lambda) &:= \{M \in \mathcal{O}'_k \mid (P - \bar\lambda(P))^N \cdot m = 0,\ N \gg 0\} \\ \mathcal{O}_k &:= \{M \in \mathcal{O}'_k \mid \xi^N \cdot m = 0,\ N \gg 0\} = \mathcal{O}_k(0) \end{aligned}$$

In these definitions, $\bar\lambda \in V^*/W$, (i.e. $\bar\lambda : \mathbb{C}[V]^W \to \mathbb{C}$), and the conditions are to hold for arbitrary elements m, P, and ξ (where $m \in M$ and $P \in \mathbb{C}[V^*]^W$ and $\xi \in V$):

From now on, we mainly discuss $\mathcal{O}_k$, which is called the *principal block* of category $\mathcal{O}'_k$. The following lemma is standard and closely mirrors the Lie situation.

LEMMA 3.7. *The objects of $\mathcal{O}'_k$ and $\mathcal{O}_k$ have the following properties:*
(1) *Each $M \in \mathcal{O}'_k$ (resp. $M \in \mathcal{O}_k$) is finitely generated over $\mathbb{C}[V] \subset H_k$.*
(2) *$\mathcal{O}'_k$ (resp. $\mathcal{O}_k$) is a stable Serre subcategory of* $\mathrm{mod}(H_k)$ *(i.e. is it an abelian subcategory closed under taking subobjects, quotients, and extensions).*
(3) *$\mathcal{O}'_k$ (resp. $\mathcal{O}_k$) is Artinian.*

Just as in the Lie case, the most important objects in $\mathcal{O}_k$ are obtained by inducing modules from subalgebras of H_k. In particular, fix an irreducible representation τ of W, and give it a $\mathbb{C}[V^*]$ module structure using $P \cdot x = P(0)x$ for $P \in \mathbb{C}[V^*]$ and $x \in \tau$. Since this action is W-invariant, it gives τ the structure of a $\mathbb{C}[V^*] \rtimes W$-module. We then define the *standard module* associated to τ by

$$M(\tau) := \mathrm{Ind}_{\mathbb{C}[V^*]\rtimes W}^{H_k}(\tau) = H_k \bigotimes_{\mathbb{C}[V^*]\rtimes W} \tau$$

These standard modules are analogues of Verma modules in Lie theory. (Of course, we can also induce other modules of $\mathbb{C}[V^*] \rtimes W$, but since we are working with the principal block $\mathcal{O}_k$, this definition is sufficient for our purposes.) The PBW property for H_k shows that $M(\tau) \cong \mathbb{C}[V] \otimes \tau$ as a $\mathbb{C}[V]$-module. Also, the relations in H_k make it clear that $M(\tau) \in \mathcal{O}_k$.

THEOREM 3.8. *As in the Lie case, the following properties hold.*

(1) *The set $\{M(\tau)\}_{\tau \in \mathrm{Irr}(W)}$ is a complete list of pairwise non-isomorphic indecomposable objects in $\mathcal{O}_k$.*
(2) *Each $M(\tau)$ has a* unique *simple quotient $L(\tau)$, and the $L(\tau)$ are a complete list of simple objects in $\mathcal{O}_k$*
(3) *The Jordan-Hölder property: Each $M \in \mathcal{O}_k$ has a finite filtration whose associated graded module is isomorphic to a sum of the $L(\tau)$ and is independent of the filtration.*
(4) *$\mathcal{O}_k$ is a highest weight category (in the sense of* [**CPS88**]).

For the proof of this theorem, we refer to [**DO03**], Proposition 2.27, and [**GGOR03**], Proposition 2.11 and Corollary 2.16.

4. Lecture 3

The category $\mathcal{O}$ defined at the end of the last lecture is related to the category of finite-dimensional representations of the *Iwahori-Hecke algebra* associated to W. The relation is given by a certain additive functor (called the *KZ functor*) which plays a fundamental role in representation theory of H_k. In this lecture, we will introduce this functor and discuss some of its applications. In particular, we define certain operations (*KZ twists*) on the set of isomorphism classes of irreducible W-modules depending on the (integral) parameter k and discuss interesting relations between these operations (conjectured by E. Opdam [**Opd95, Opd00**] and proved in [**BC11**]).

If A is an algebra, we write $\mathrm{Mod}(A)$ (respectively, $\mathrm{mod}(A)$) for the category of left (respectively, finitely generated left) A-modules.

4.1. The Knizhnik-Zamolodchikov (KZ) functor.

The KZ functor is defined as a composition of several functors, and we describe each one in turn. The key property that allows this construction is the well-known fact that a $\mathcal{D}_X$-module on a smooth algebraic variety X which is coherent as an $\mathcal{O}_X$-module is the same thing as a vector bundle with a flat connection on X.

The Dunkl embedding provides a natural functor $\mathrm{Mod}(H_k) \to \mathrm{Mod}(\mathcal{D}W)$ given by $M \mapsto \mathcal{D}W \otimes_{H_k} M$. We denote the output of this functor by M_{reg}. Next, we note that $\mathrm{Mod}(\mathcal{D}W)$ is naturally equivalent to the category of W-equivariant D-modules on V_{reg}. Since W acts freely on V_{reg}, this gives an equivalence $\mathrm{Mod}(\mathcal{D}W) \cong \mathrm{Mod}(\mathcal{D}(V_{\mathrm{reg}}/W))$. The category $\mathrm{Mod}(\mathcal{D}(V_{\mathrm{reg}}/W))$ contains a full subcategory of $\mathcal{O}$-coherent $\mathcal{D}$-modules (which are automatically $\mathcal{O}$-locally-free). There is an interpretation functor $\mathrm{Mod}_{\mathcal{O}}(\mathcal{D}(V_{\mathrm{reg}}/W)) \cong \mathrm{Vect}^f(V_{\mathrm{reg}}/W)$ which interprets an $\mathcal{O}$-locally-free $\mathcal{D}$-module as a vector bundle with a flat connection. Finally, the Riemann-Hilbert correspondence gives an equivalence of categories $\mathrm{Vect}^f(V_{\mathrm{reg}}/W) \cong \mathrm{mod}(B_W)$, where $B_W := \pi_1(V_{\mathrm{reg}}/W, *)$ is the Artin braid group.

DEFINITION 4.1. The $\mathtt{KZ}$ *functor* is defined by the composition of functors

$$\mathtt{KZ}_k : \mathcal{O}_k \to \mathrm{mod}(\mathcal{D}W) \cong \mathrm{mod}(\mathcal{D}(V_{\mathrm{reg}}/W)) \cong \mathrm{Vect}^f(V_{\mathrm{reg}}/W) \cong \mathrm{mod}(\mathbb{C}B_W)$$

For the standard modules $M(\tau)$ this composition of functors can be made quite explicit. Since $M(\tau) \cong \mathbb{C}[V] \otimes \tau$ as a $\mathbb{C}[V]$-module, it is free of rank $\dim_{\mathbb{C}}(\tau)$, and its localization $M(\tau)_{\mathrm{reg}}$ is isomorphic to $\mathbb{C}[V_{\mathrm{reg}}] \otimes \tau$ as a $\mathbb{C}[V_{\mathrm{reg}}]$-module. This identification allows one to interpret $M(\tau)_{\mathrm{reg}}$ as (sections of) a trivial vector bundle of rank $\dim_{\mathbb{C}} \tau$. The $\mathcal{D}$-module structure on $M(\tau)_{\mathrm{reg}}$ is given by

$$\partial_\xi(f \otimes v) = (\partial_\xi f) \otimes v + f \otimes (\partial_\xi v)$$

By definition, in M we have $\xi \cdot v = 0$, and since $\xi \mapsto T_\xi$ under the Dunkl embedding, we know $T_\xi(v) = 0$. Rewriting this, we see

$$\partial_\xi v - \sum_{H \in A} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H - 1} n_H k_{H,i} \boldsymbol{e}_{H,i}(v) = 0 \tag{4.1}$$

Combining these formulas, we obtain

$$\partial_\xi(f \otimes v) = \partial_\xi(f) \otimes v + \sum_{H \in A} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H - 1} n_H k_{H,i} f \otimes \boldsymbol{e}_{H,i}(v) \tag{4.2}$$

The right hand side is an explicit formula for the *KZ connection*, which is the regular flat connection on $M_{\mathrm{reg}} = \mathbb{C}[V] \otimes \tau$ given by the KZ functor. (Note the change in sign between the formula for the connection and for the Dunkl operator.) Horizontal sections $y : V_{\mathrm{reg}} \to \tau$ satisfy the *KZ equations*

$$\partial_\xi(y) + \sum_{H \in A} \frac{\alpha_H(\xi)}{\alpha_H} \sum_{i=0}^{n_H - 1} n_H k_{H,i} \boldsymbol{e}_{H,i}(y) = 0 \tag{4.3}$$

REMARK 4.2. The formulas (4.1) and (4.3) look very similar (other than the sign), but there is an important distinction. In the KZ connection (4.1) group elements act on the arguments of the functions involved, while in the KZ equation (4.3) they act on their values.

4.2. The Hecke algebra of W. It is natural to ask what the image of the `KZ` functor is. The answer is that for generic k, its image is the subcategory of $\mathbb{C}B_W$-modules that factor through a natural quotient of B_W called the *Hecke algebra* of W. We first record two facts about complex reflection groups the proofs of which can be found in [**BMR98**]:

(1) For all $H \in \mathcal{A}$, there is a unique $s_H \in W_H$ such that $\det(s_H) = e^{2\pi i/n_H}$.
(2) The braid group B_W is generated by the elements σ_H which are monodromy operators (around the $H \in \mathcal{A}$) corresponding to the s_H.

Following [**BMR98**], we define the Hecke algebra of W by

$$\mathcal{H}_k(W) := \mathbb{C}B_W \Big/ \left(\prod_{j=0}^{n_H - 1} (\sigma_H - (\det s_H)^{-j} e^{2\pi i k_{H,j}}) = 0 \right)_{H \in \mathcal{A}}$$

If $k_{H,i} \in \mathbb{Z}$, then the relations simplify to $\sigma_H^{n_H} = 1$, which shows that there is a canonical isomorphism $\mathcal{H}_k(W) \cong \mathbb{C}W$. In particular, $\dim_{\mathbb{C}} \mathcal{H}_k(W) = |W|$. Furthermore, if k is generic, then $\mathcal{H}_k(W)$ is semi-simple, and by rigidity of semisimple algebras , $\mathcal{H}_k(W)$ is generically isomorphic to $\mathbb{C}W$. However, it seems to be the case that the equality $\dim_{\mathbb{C}} \mathcal{H}_k(W) = |W|$ is still conjectural for a few exceptional W. (The equality is known for all Coxeter groups and for all but finitely many W.)

THEOREM 4.3 (see [**BEG03a**], [**GGOR03**], [**DO03**]). *For each k, the functor $\mathtt{KZ}_k : \mathcal{O}_k \to \mathrm{mod}(B_W)$ is an exact functor with image contained in $\mathrm{mod}(\mathcal{H}_k) \hookrightarrow \mathrm{mod}(B_W)$ (where the inclusion is induced by restriction of scalars).*

It is natural to ask whether this can be an equivalance. One obvious obstruction to a positive answer is the fact that the localization $H_k \to \mathcal{D}W$ can kill some modules. It turns out that this is the only obstruction. More precisely, if we let $\mathcal{O}_k^{\mathrm{tor}} := \{M \in \mathcal{O}_k \mid M_{\mathrm{reg}} = 0\}$, then we have the following theorem.

THEOREM 4.4 (see [**GGOR03**]). *Assume that* $\dim \mathcal{H}_k = |W|$. *Then*

(1) *The $\mathtt{KZ}_k$ functor induces an equivalence $\mathtt{KZ}_k : \mathcal{O}_k/\mathcal{O}_k^{\mathrm{tor}} \to \mathrm{mod}(\mathcal{H}_k)$.*
(2) *There is a 'big projective' $P \in \mathrm{Ob}(\mathcal{O}_k)$ and $Q \in \mathrm{Ob}(\mathrm{mod}(\mathcal{H}_k))$ such that $\mathcal{H}_k \cong \mathrm{End}_{\mathcal{O}_k}(P)$ and $\mathcal{O}_k \cong \mathrm{mod}(\mathrm{End}_{\mathcal{H}_k}(Q))$.*

One can interpret this theorem as showing that the structure of the category $\mathcal{O}_k$ is controlled by $\mathcal{H}_k$ for all k. The following result shows that the *algebra* structure of H_k also depends crucially on $\mathcal{H}_k$.

THEOREM 4.5. *Assume that* $\dim_{\mathbb{C}} \mathcal{H}_k = |W|$. *Then the following are equivalent:*

(1) *$\mathcal{H}_k$ is a semisimple algebra,*
(2) *$\mathcal{O}_k$ is a semisimple category,*
(3) *H_k is a simple algebra.*

If one of these conditions hold, then $\mathcal{O}_k^{\mathrm{tor}} = 0$ and $\mathcal{O}_k \cong \mathrm{mod}(\mathcal{H}_k)$. Furthermore, in this case the $M(\tau)$ are simple, i.e. $M(\tau) = L(\tau)$ for all $\tau \in \mathrm{Irr}(W)$.

For a detailed proof of Theorem 4.5, we refer to [**BC11**], Theorem 6.6, which combines the earlier results of [**BEG03a**], [**GGOR03**], [**DO03**] and is based on R. Vale's Ph.D. thesis (2006).

COROLLARY 4.6. *If the conditions of Theorem* 4.5 *hold, then H_k is Morita equivalent to its spherical subalgebra U_k.*

PROOF. Since H_k is simple, the two-sided ideal $H_k\boldsymbol{e}H_k$ must be H_k, which implies that $\boldsymbol{e}H_k$ is a generator in $\mathrm{mod}(H_k)$. Since $U_k \cong \mathrm{End}_{H_k}(\boldsymbol{e}H_k)$ and $\boldsymbol{e}H_k$ is projective, U_k and H_k are Morita equivalent. □

When the corollary applies the mutual equivalences can be written explicitly. The functor $\mathrm{mod}(H_k) \to \mathrm{mod}(U_k)$ is given by $M \mapsto \boldsymbol{e}M := \boldsymbol{e}H_k \otimes_{H_k} M$, and the functor $\mathrm{mod}(U_k) \to \mathrm{mod}(H_k)$ is given by $N \mapsto H_k\boldsymbol{e} \otimes_{U_k} N$.

4.3. Shift functors and KZ twists. The goal of this section is to relate the categories $\mathcal{O}_k$ for different values of k. There are several constructions of 'shift functors,' i.e. functors between different $\mathcal{O}_k$, the first of which was introduced in [**BEG03a**]. In this lecture we will focus on the functor introduced in [**BC11**]. The main idea is that the Dunkl embeddings all have the same target, and we can 'push forward' a module along one embedding and 'pull back' along another. This is analogous to the so-called Enright completion in Lie theory (see [**Jos82**]).

The first step is to enlarge $\mathcal{O}_k$ by defining $\mathcal{O}_k^{ln} \subset \mathrm{Mod}(H_k)$ to be all H_k-modules on which the $\xi \in V$ act locally nilpotently. Then $\iota_k : \mathcal{O}_k \hookrightarrow \mathcal{O}_k^{ln}$ is the natural inclusion whose image is the finitely generated modules. The inclusion $\mathcal{O}^{ln} \hookrightarrow \mathrm{Mod}(H_k)$ has a *right* adjoint $\mathfrak{r}_k : \mathrm{Mod}(H_k) \to \mathcal{O}_k^{ln}$ which outputs the largest submodule in $\mathcal{O}_k^{ln}$. In other words,

$$\mathfrak{r}_k(M) := \{m \in M \mid \xi^d m = 0, \forall \xi \in V, d \gg 0\}$$

Recall $\mathcal{D}W = \mathcal{D}(V_{\text{reg}}) \rtimes W$, and that the Dunkl embedding gives us an identification $H_k[\delta^{-1}] \cong \mathcal{D}W$. Write $\theta_k : H_k \to \mathcal{D}W$ for the localization map (which is just the Dunkl embedding). Also, write $\theta_k^* : \operatorname{Mod}(H_k) \to \operatorname{Mod}(\mathcal{D}W)$ for extension of scalars and $(\theta_k)_* : \operatorname{Mod}(\mathcal{D}W) \to \operatorname{Mod}(H_k)$ for the restriction of scalars.

Definition 4.7. For k, k' two parameter values, define $\mathcal{T}_{k\to k'} : \operatorname{Mod}(H_k) \to \operatorname{Mod}(H_{k'})$

$$\mathcal{T}_{k\to k'} = \mathfrak{r}_{k'}(\theta_{k'})_*\theta_k^*$$

Proposition 4.8. *$\mathcal{T}_{k\to k'}$ restricts to a functor $\mathcal{O}_k \to \mathcal{O}_{k'}$.*

Proof. Given $M \in \mathcal{O}_k$, let $N := (\theta_{k'})_*(\theta_k)^*M \in \operatorname{Mod}(H_{k'})$. To prove the claim we need only to show that $\mathfrak{r}_{k'}(N)$ is a finitely generated module over $H_{k'}$. Assuming the contrary, we may construct an infinite *strictly* increasing chain of submodules $N_0 \subset N_1 \subset N_2 \subset \ldots \subset \mathfrak{r}_{k'}(N) \subset M_{\text{reg}}$, with $N_i \in \mathcal{O}_{k'}$. Localizing this chain, we get an infinite chain of H_{reg}-submodules of M_{reg}. Since M_{reg} is finite over $\mathbb{C}[V_{\text{reg}}]$ and $\mathbb{C}[V_{\text{reg}}]$ is Noetherian, this localized chain stabilizes at some i. Thus, omitting finitely many terms, we may assume that $(N_i)_{\text{reg}} = (N_0)_{\text{reg}}$ for all i. In that case all the inclusions $N_i \subset N_{i+1}$ are essential extensions, and since each $N_i \in \mathcal{O}_{k'}$, the above chain of submodules can be embedded into an injective hull of N_0 in $\mathcal{O}_{k'}$ and hence stabilizes for $i \gg 0$. (The injective hulls in $\mathcal{O}_{k'}$ exist and have finite length, since $\mathcal{O}_{k'}$ is a highest weight category, see [**GGOR03**], Theorem 2.19.) This contradicts the assumption that the inclusions are strict. Thus, we conclude that $\mathfrak{r}_{k'}(N)$ is finitely generated. □

As one may expect, the properties of the shift functor depend on the parameters k and k'. We call a parameter k *regular* if $\mathcal{H}_k$ is semisimple, and we write $\operatorname{Reg}(W)$ for the set of regular parameters. It is proved in [**BC11**], Lemma 6.9, that $\operatorname{Reg}(W)$ is a connected set. We list some basic properties of the shift functor:

Lemma 4.9. *Let k, k', k'' be arbitrary complex multiplicites, and let $M \in \mathcal{O}_k$.*

(1) *If $k \in \operatorname{Reg}(W)$, then $\mathcal{T}_{k\to k}(M) = M$.*
(2) *If $k, k' \in \operatorname{Reg}(W)$ and M is simple, then $\mathcal{T}_{k\to k'}(M)$ is either 0 or simple.*
(3) *If $k, k', k'' \in \operatorname{Reg}(W)$, then $(\mathcal{T}_{k'\to k''}) \circ (\mathcal{T}_{k\to k'}) \cong \mathcal{T}_{k\to k''}$.*

Corollary 4.10. *If $k,' k \in \operatorname{Reg}(W)$, then the following are equivalent:*

(1) *$\mathcal{T}_{k\to k'}[M_k(\tau)] \cong M_{k'}(\tau')$;*
(2) *$M_k(\tau)_{\text{reg}} \cong M_{k'}(\tau')$ as H_{reg}-modules.*

4.4. KZ twists. In this section we assume that the parameter k is integral. Summarizing what we have said before, in this case we know that there is a canonical isomorphism $\mathcal{H}_k \cong \mathbb{C}W$, that the functor $\mathtt{KZ}_k : \mathcal{O}_k \to \operatorname{mod}(W)$ is an equivalence, and that the standard modules $M_k(\tau)$ are all simple.

Definition 4.11. The *KZ twist* $\operatorname{kz}_k : \operatorname{Irr}(W) \to \operatorname{Irr}(W)$ is the map induced by the KZ functor, i.e. $[\tau] \mapsto \mathtt{KZ}_k[M_k(\tau)]$.

This map was defined by Opdam in [**Opd95**] without the use of the Cherednik algebras (which hadn't been defined at the time). He showed that the maps satisfied the additivity property $\operatorname{kz}_k \circ \operatorname{kz}_{k'} = \operatorname{kz}_{k+k'}$ and conjectured that each kz_k was a permutation. This conjecture was proved as a corollary of the following theorem.

Theorem 4.12 (see [**BC11**], Theorem 7.11). *Let k and k' be complex multiplicities such that $k'_{H,i} - k_{H,i} \in \mathbb{Z}$ for all H and i. Then*

(1) $\mathcal{T}_{k\to k'}(M) \neq 0$ *for each standard module* $M = M(\tau) \in \mathcal{O}_k$*;*
(2) *if* $k, k' \in \mathrm{Reg}(W)$*, then* $\mathcal{T}_{k\to k'}[M(\tau)] = M_{k'}(\tau')$ *with* $\tau' = \mathrm{kz}_{k-k'}(\tau)$*.*

The theorem is first proved for integral k, k' and is then extended to all parameters using a deformation argument (and the fact that the integers are Zariski-dense in $\mathbb{C}$). The following is essentially a direct corollary of the second statement of the theorem and Lemma 4.9:

Corollary 4.13 ([**BC11**]). *The map* $k \mapsto \mathrm{kz}_k$ *is a homomorphism from the additive group of integral multiplicities to the permutation group of* $\{\mathrm{Irr}(W)\}$*.*

This result was conjectured by E. Opdam (see [**Opd95**] and [**Opd00**]).

5. Lecture 4

The notion of a quasi-invariant polynomial for a finite Coxeter group was introduced by O. Chalykh and A. Veselov in [**CV90**]. Although quasi-invariants are a natural generalization of invariants, they first appeared in a slightly disguised form (as symbols of commuting differential operators). More recently, the algebras of quasi-invariants and associated varieties have been studied in [**FV02, EG02a, BEG03a**] by means of representation theory and have found applications in other areas. In this lecture, we define quasi-invariants for an arbitrary complex reflection group and give new applications. This material is borrowed from [**BC11**].

5.1. Quasi-invariants for complex reflection groups. We will introduce a family of submodules of $\mathbb{C}[V]$ depending on the parameter k that interpolate between $\mathbb{C}[V]^W$ and $\mathbb{C}[V]$. These submodules are defined for *integral* values of k and can be interpreted as torsion-free coherent sheaves on certain (singular) algebraic varieties. The ring of invariant differential operators on such a variety turns out to be isomorphic to a spherical subalgebra U_k, and the modules of quasi-invariants become (via this isomorphism) objects of (the spherical analogue of) category $\mathcal{O}_k$. This explains the comment at the end of Section 2.2.

We first recall the definition of quasi-invariants for Coxeter groups from [**CV90**]. Let W be a Coxeter group with H a reflection hyperplane, and recall that s_H is the unique element of W_H (the pointwise stabilizer of H in W) with determinant -1. Then $f \in \mathbb{C}[V]^W$ if and only if $s_H(f) = f$ for all $H \in \mathcal{A}$. Let $k : \mathcal{A}/W \to \mathbb{Z}^{\geq 0}$.

Definition 5.1. The *quasi-invariants* (for W a Coxeter group) of parameter k are defined as $Q_k(W) := \{f \in \mathbb{C}[V] \mid s_H(f) \equiv f \bmod \langle \alpha_H \rangle^{2k_H}\}$.

Note that for extreme parameter values, $Q_0(W) = \mathbb{C}[V]$ and $Q_\infty[W] = \mathbb{C}[V]^W$.

We now return to the generality of complex reflection groups. Recall the idempotents $\boldsymbol{e}_{H,i} = \frac{1}{n_H}\sum_{w\in W_H}(\det w)^{-i} w$ for $i = 0, 1, \ldots, n_H - 1$.

Definition 5.2. For an arbitrary complex reflection group W, the *quasi-invariants* $Q_k \subset \mathbb{C}[V]$ are defined by

$$Q_k(W) := \{f \in \mathbb{C}[V] \mid \boldsymbol{e}_{H,-i}(f) \equiv 0 \bmod \langle \alpha_H \rangle^{n_H k_{H,i}},\ \forall\, 0 \leq i \leq n_H - 1,\, H \in \mathcal{A}\}$$

Note that in the Coxeter case, $s_H(f) \equiv f$ is equivalent to $(1 + \det(s_H)s_H)(f) \equiv 0$, so the two definitions agree in this case. Also, the condition holds automatically for $i = 0$, as we assumed that $k_{H,0} = 0$ for all $H \in \mathcal{A}$.

EXAMPLE 5.3. If $W = \mathbb{Z}/n\mathbb{Z}$ and $V = \mathbb{C}$, then $s(x) = e^{2i\pi/n}x$, the parameter k is $k = \{k_0 = 0, k_1, \ldots, k_{n-1}\}$, and

$$Q_k(\mathbb{Z}/n\mathbb{Z}) = \bigoplus_{i=0}^{n-1} x^{nk_i+1}\mathbb{C}[x^n] \tag{5.1}$$

In the Coxeter case, $Q_k(W)$ is a subring of $\mathbb{C}[V]$, but as the above example shows, this is not always true in the general case. However, there is a natural subring of $\mathbb{C}[V]$ associated to $Q_k(W)$, namely:

$$A_k(W) := \{P \in \mathbb{C}[V] \mid pQ_k(W) \subset Q_k(W)\} \tag{5.2}$$

We write $X_k = \mathrm{Spec}[A_k(W)]$.

LEMMA 5.4. *We list some properties of A_k and Q_k:*

(1) *$A_k(W) = Q_{k'}(W)$ for some parameter k'. In particular, both A_k and Q_k contain $\mathbb{C}[V]^W$ and are stable under the action of W.*
(2) *A_k is a finitely generated graded subalgebra of $\mathbb{C}[V]$, and Q_k is a finitely generated graded module over A_k of rank one.*
(3) *The field of fractions of A_k is $\mathbb{C}(V)$, and the integral closure of A_k in $\mathbb{C}(V)$ is $\mathbb{C}[V]$.*
(4) *X_k is an irreducible affine variety, and the normalization of X_k is $\mathbb{C}^n$.*
(5) *The normalization map $\pi_k : \mathbb{C}^n \to X_k$ is bijective*[7].
(6) *The preimage of the singular locus of X_k under π_k is the divisor $(\mathcal{A}, k)$.*

5.1.1. *$\mathbb{C}W$-valued quasi-invariants.* Our goal now is to show that $Q_k \subset \mathbb{C}[V]$ is preserved by the action of the spherical subalgebra U_k. However, this would be impossible to show by direct calculation, since for some complex reflection groups the minimial order of a non-constant W-invariant polynomial is 60. We therefore give a definition of quasi-invariants at the level of the Cherednik algebra itself, and then show that symmetrizing the $\mathbb{C}W$-valued quasi-invariants produces the quasi-invariants defined in the previous section.

The algebra $\mathcal{D}W$ can be viewed as a ring of W-equivariant differential operators on V_{reg}, and as such it acts naturally on the space of $\mathbb{C}W$-valued functions. More precisely, using the canonical inclusion $\mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W \hookrightarrow \mathcal{D}W$, we can identify $\mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W$ with the cyclic $\mathcal{D}W$-module $\mathcal{D}W/J$, where J is the left ideal of $\mathcal{D}W$ generated by $\partial_\xi \in \mathcal{D}W$, $\xi \in V$. Explicitly, in terms of generators, $\mathcal{D}W$ acts on $\mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W$ by

$$\begin{aligned} g(f \otimes u) &= gf \otimes u\,, \quad g \in \mathbb{C}[V_{\mathrm{reg}}]\ , \\ \partial_\xi(f \otimes u) &= \partial_\xi f \otimes u\,, \quad \xi \in V\ , \\ w(f \otimes u) &= f^w \otimes wu\,, \quad w \in W\ . \end{aligned} \tag{5.3}$$

Now, the restriction of scalars via the Dunkl representation $H_k(W) \hookrightarrow \mathcal{D}W$ makes $\mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W$ an $H_k(W)$-module. We will call the corresponding action of H_k the *differential action*. It turns out that, in the case of integral k's, the differential action of H_k is intimately related to quasi-invariants $Q_k = Q_k(W)$.

Besides the diagonal action (5.3), we will use another action of W on $\mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W$, which is trivial on the first factor: i. e., $f \otimes s \mapsto f \otimes ws$, where $w \in W$ and $f \otimes s \in \mathbb{C}[V_{\mathrm{reg}}] \otimes \mathbb{C}W$. We denote this action by $1 \otimes w$.

[7]The normalization map is bijective as a map of *sets*, but it is not an isomorphism of schemes.

Now, we define $\mathbf{Q}_k$ to be the subspace of $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ spanned by the elements φ satisfying

$$(1\otimes \boldsymbol{e}_{H,i})\,\varphi \equiv 0 \ \mathrm{mod}\ \langle\alpha_H\rangle^{n_H k_{H,i}}\otimes\mathbb{C}W\ , \tag{5.4}$$

for all $H\in\mathcal{A}$ and $i=0,1,\ldots,n_H-1$. Here, as in Definition 5.2, $\langle\alpha_H\rangle$ stands for the ideal of $\mathbb{C}[V]$ generated by α_H.

THEOREM 5.5. *If k is integral, then $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ contains a unique H_k-submodule $\mathbf{Q}'_k=\mathbf{Q}'_k(W)$, such that $\mathbf{Q}'_k$ is finite over $\mathbb{C}[V]\subset H_k$ and*

$$\boldsymbol{e}\,\mathbf{Q}'_k=\boldsymbol{e}(\,Q_k\otimes 1)\quad in\quad \mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W\ . \tag{5.5}$$

In fact, we have the equality $\mathbf{Q}_k=\mathbf{Q}'_k$.

As a simple consequence of the theorem, we get the following corollary.

COROLLARY 5.6. *Q_k is stable under the action of U_k on $\mathbb{C}[V_{\rm reg}]$ via the Dunkl representation* (3.1). *Thus Q_k is a U_k-module, with U_k acting on Q_k by invariant differential operators.*

PROOF. Theorem 5.5 implies that $\boldsymbol{e}H_k\boldsymbol{e}(\boldsymbol{e}\mathbf{Q}_k)\subseteq\boldsymbol{e}\mathbf{Q}_k$. Recall that for every element $\boldsymbol{e}L\boldsymbol{e}\in\boldsymbol{e}H_k\boldsymbol{e}$ we have $\boldsymbol{e}L\boldsymbol{e}=\boldsymbol{e}\,\mathrm{Res}\,L$, by the definition of the map (3.1). As a result,

$$\boldsymbol{e}\,(\mathrm{Res}\,L[Q_k]\otimes 1)=\boldsymbol{e}\,\mathrm{Res}\,L\,[Q_k\otimes 1]=(\boldsymbol{e}\,L\,\boldsymbol{e})[\mathbf{Q}_k]\subseteq\boldsymbol{e}\mathbf{Q}_k=\boldsymbol{e}(Q_k\otimes 1)\,.$$

It follows that $(\mathrm{Res}\,L)[Q_k]\subseteq Q_k$, since $\boldsymbol{e}\,(f\otimes 1)=0$ in $\mathbb{C}[V_{\rm reg}]\otimes\mathbb{C}W$ forces $f=0$. ☐

EXAMPLE 5.7. We illustrate Theorem 5.5 in the one-dimensional case. Let $W=\mathbb{Z}/n\mathbb{Z}$ and $k=(k_0,\ldots,k_{n-1})$ be as in Example 5.3. Then

$$\mathbf{Q}_k=\bigoplus_{i=0}^{n-1}x^{nk_i}\mathbb{C}[x]\otimes\boldsymbol{e}_i\ ,\qquad \boldsymbol{e}_i=\frac{1}{n}\sum_{w\in W}(\det w)^{-i}w\,. \tag{5.6}$$

Clearly, $\mathbf{Q}_k$ is stable under the action of W and $\mathbb{C}[x]$. On the other hand, if $k_i\in\mathbb{Z}$, a short calculation shows that the Dunkl operator $T:=\partial_x-x^{-1}\sum_{i=0}^{n-1}nk_i\boldsymbol{e}_i$ annihilates the elements $x^{nk_i}\otimes\boldsymbol{e}_i$, and hence preserves $\mathbf{Q}_k$ as well. Now, acting on $\mathbf{Q}_k$ by $\boldsymbol{e}=\boldsymbol{e}_0$ and using (5.1), we get

$$\boldsymbol{e}\mathbf{Q}_k=\bigoplus_{i=0}^{n-1}x^{nk_i+i}\mathbb{C}[x^n]\otimes\boldsymbol{e}_i=\bigoplus_{i=0}^{n-1}\boldsymbol{e}\,(x^{nk_i+i}\mathbb{C}[x^n]\otimes 1)=\boldsymbol{e}\,(Q_k\otimes 1)\ , \tag{5.7}$$

which agrees with Theorem 5.5.

5.1.2. *Differential operators on quasi-invariants.* We briefly recall the definition of differential operators in the algebro-geometric setting (see [**MR01**], Chap. 15).

Let A be a commutative algebra over $\mathbb{C}$, and let M be an A-module. The filtered ring of (linear) differential operators on M is defined by

$$\mathcal{D}_A(M):=\bigcup_{n\geq 0}\mathcal{D}^n_A(M)\ \subseteq\ \mathrm{End}_{\mathbb{C}}(M)\ ,$$

where $\mathcal{D}^0_A(M):=\mathrm{End}_A(M)$ and $\mathcal{D}^n_A(M)$, with $n\geq 1$, are given inductively:

$$\mathcal{D}^n_A(M):=\{D\in\mathrm{End}_{\mathbb{C}}(M)\mid[\,D,\,a\,]\in\mathcal{D}^{n-1}_A(M)\text{ for all }a\in A\}\ .$$

The elements of $\mathcal{D}^n_A(M) \setminus \mathcal{D}^{n-1}_A(M)$ are called *differential operators of order* n on M. Note that the commutator of two operators in $\mathcal{D}^n_A(M)$ of orders n and m has order at most $n+m-1$. Hence the associated graded ring $\operatorname{gr}\mathcal{D}_A(M) := \bigoplus_{n\geq 0}\mathcal{D}^n_A(M)/\mathcal{D}^{n-1}_A(M)$ is a commutative algebra.

If X is an affine variety with coordinate ring $A = \mathcal{O}(X)$, we denote $\mathcal{D}_A(A)$ by $\mathcal{D}(X)$ and call it the *ring of differential operators on* X. If X is irreducible, then each differential operator on X has a unique extension to a differential operator on $\mathbb{K} := \mathbb{C}(X)$, the field of rational functions of X, and thus we can identify (see [**MR01**], Theorem 15.5.5):

$$\mathcal{D}(X) = \{\, D \in \mathcal{D}(\mathbb{K}) \mid D(f) \in \mathcal{O}(X) \text{ for all } f \in \mathcal{O}(X)\,\} \ .$$

Slightly more generally, we have

LEMMA 5.8. *Suppose that* $M \subseteq \mathbb{K}$ *is a (nonzero)* A*-submodule of* $\mathbb{K}$*. Then*

$$\mathcal{D}_A(M) = \{\, D \in \mathcal{D}(\mathbb{K}) \mid D(f) \in M \text{ for all } f \in M\,\} \ .$$

We apply these concepts for $A = A_k$ and $M = Q_k$, writing $\mathcal{D}(Q_k)$ instead of $\mathcal{D}_A(M)$ in this case. By Lemma 5.4(3), $X_k = \operatorname{Spec}(A_k)$ is an irreducible variety with $\mathbb{K} = \mathbb{C}(V)$, so, by Lemma 5.8, we have

$$\mathcal{D}(Q_k) = \{D \in \mathcal{D}(\mathbb{K}) \,|\, D(f) \subseteq Q_k \text{ for all } f \in Q_k\,\} \ . \tag{5.8}$$

Note that the differential filtration on $\mathcal{D}(Q_k)$ is induced from the differential filtration on $\mathcal{D}(\mathbb{K})$. Thus (5.8) yields a canonical inclusion $\operatorname{gr}\mathcal{D}(Q_k) \subseteq \operatorname{gr}\mathcal{D}(\mathbb{K})$, with $\mathcal{D}^0(Q_k) = A_k$, see (5.2). In particular, if $k = \{0\}$, then $Q_k = \mathbb{C}[V]$ and (5.8) becomes the standard realization of $\mathcal{D}(V)$ as a subring of $\mathcal{D}(\mathbb{K})$.

Apart from Q_k, we may also apply Lemma 5.8 to $\mathbb{C}[V_{\rm reg}]$, which is naturally a subalgebra of $\mathbb{K} = \mathbb{C}(V)$. This gives the identification

$$\mathcal{D}(V_{\rm reg}) = \{D \in \mathcal{D}(\mathbb{K}) \,|\, D(f) \subseteq \mathbb{C}[V_{\rm reg}] \text{ for all } f \in \mathbb{C}[V_{\rm reg}]\,\} \ . \tag{5.9}$$

LEMMA 5.9. *With identifications* (5.8) *and* (5.9)*, we have*

$$\text{(i) } \mathcal{D}(Q_k) \subseteq \mathcal{D}(V_{\rm reg}) \quad \textit{and} \quad \text{(ii) } \operatorname{gr}\mathcal{D}(Q_k) \subseteq \operatorname{gr}\mathcal{D}(V) \ .$$

5.1.3. *Invariant differential operators.* Recall that, by Lemma 5.4, Q_k is stable under the action of W on $\mathbb{C}[V_{\rm reg}]$. Hence W acts naturally on $\mathcal{D}(Q_k)$, and this action is compatible with the inclusion of Lemma 5.9(i). It follows that $\mathcal{D}(Q_k)^W \subseteq \mathcal{D}(V_{\rm reg})^W$. Now, we recall the algebra embedding (3.1), which defines the Dunkl representation for the spherical subalgebra of H_k.

PROPOSITION 5.10. *The image of* $\operatorname{Res}: U_k \hookrightarrow \mathcal{D}(V_{\rm reg})^W$ *coincides with* $\mathcal{D}(Q_k)^W$*. Thus, the Dunkl representation of* U_k *yields an algebra isomorphism* $U_k \cong \mathcal{D}(Q_k)^W$*.*

Proposition 5.10 explains the remark at the end of Section 2.2.

COROLLARY 5.11. $\operatorname{gr}\mathcal{D}(V)$ *is a finite module over* $\operatorname{gr}\mathcal{D}(Q_k)$*. Consequently* $\operatorname{gr}\mathcal{D}(Q_k)$ *is a finitely generated (and hence, Noetherian) commutative* $\mathbb{C}$*-algebra.*

We are now in a position to state some of the main results of [**BC11**]. The first theorem can be viewed as a generalization of the Chevalley-Serre-Shephard-Todd Theorem (*cf.* Theorem 2.3, (1) $\Rightarrow$ (2)).

THEOREM 5.12 ([**BC11**], Theorem 1.1). Q_k *is a free module over* $\mathbb{C}[V]^W$ *of rank* $|W|$*.*

It would be nice to have a direct proof of this theorem generalizing the homological arguments presented in Section 2.3. Unfortunately, we are not aware of such a generalization. Instead, Theorem 5.12 is deduced as a consequence of the following much deeper result on the algebra of differential operators on quasi-invariants.

THEOREM 5.13 ([**BC11**], Theorem 1.2). *$\mathcal{D}(Q_k)$ is a simple ring, Morita equivalent to $\mathcal{D}(V)$.*

This result is surprising since, as explained above, $\mathcal{D}(Q_k)$ is isomorphic to the ring of (twisted) differential operators on a *singular* algebraic variety, and such rings usually do not have good properties. Combined with standard Morita theory Theorem 5.13 implies that $\mathcal{P} := \{D \in \mathcal{D}(\mathbb{K}) : D(f) \in Q_k \text{ for all } f \in \mathbb{C}[V]\}$ is a projective right ideal of $\mathcal{D}(V)$. This gives one of the only families of examples of non-free projective modules over higher Weyl algebras. In the Coxeter case, the Morita equivalence between the algebras $\mathcal{D}(Q_k)$ for integral k's was originally proved in [**BEG03a**].

Theorem 5.13 and Proposition 5.10 have another interesting consequence established in [**BC11**] using K-theoretic arguments.

THEOREM 5.14 ([**BC11**], Corollary 4.6). *$\mathcal{D}(Q_k)$ is a non-free projective module over $\mathcal{D}(Q_k)^W$.*

This result can be viewed as a noncommutative counterpart of Theorem 5.12.

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Notes on algebraic operads, graph complexes, and Willwacher's construction

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To Orit and Rosie

Abstract. We give a detailed proof of T. Willwacher's theorem (T. Willwacher, arXiv:1009.1654) which links the cohomology of the full graph complex fGC to the cohomology of the deformation complex of the operad Ger, governing Gerstenhaber algebras. We also present various prerequisites required for understanding the material of T. Willwacher (arXiv:1009.1654). In particular, we review operads, cooperads, and the cobar construction. We give a detailed exposition of the convolution Lie algebra and its properties. We prove a useful lifting property for maps from a dg operad obtained via the cobar construction. We describe in detail Willwacher's twisting construction, and then use it to work with various operads assembled from graphs, in particular, the full graph complex and its subcomplexes. These notes are loosely based on lectures given by the first author at the Graduate and Postdoc Summer School at the Center for Mathematics at Notre Dame (May 31 - June 4, 2011).

Contents

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1. Introduction

In his seminal paper [**22**], M. Kontsevich constructed an L_∞ quasi-isomorphism from the graded Lie algebra PV_d of polyvector fields on the affine space $\mathbb{R}^d$ to the differential graded (dg) Lie algebra of Hochschild cochains

$$C^\bullet(A) = \bigoplus_{m=0}^{\infty} \mathrm{Hom}(A^{\otimes\, m}, A) \tag{1.1}$$

for the polynomial algebra $A = \mathbb{R}[x^1, x^2, \dots, x^d]$. Among other things, this result implies that formal associative deformations of the algebra A can be described in terms of formal Poisson structures on $\mathbb{R}^d$.

According to [**23**], there exist many homotopy inequivalent L_∞ quasi-isomorphisms

$$\mathrm{PV}_d \rightsquigarrow C^\bullet(A) \tag{1.2}$$

from PV_d to $C^\bullet(A)$. More precisely, the full graph complex fGC (see Section 8) maps to the Chevalley-Eilenberg complex of PV_d and, using this map, one can define an action of the Lie algebra $H^0(\mathsf{fGC})$ on the homotopy classes of L_∞ quasi-isomorphisms (1.2).

In 1998, D. Tamarkin [**20**], [**37**] proposed a completely different approach to constructing L_∞ quasi-isomorphisms (1.2). His approach works for an arbitrary field $\mathbb{K}$ of characteristic zero and it is based on several deep results such as a proof of Deligne's conjecture on Hochschild complex [**6**], [**25**], [**33**], the formality for the operad of little discs [**38**], and the existence of a Drinfeld associator [**8**].

The main idea of Tamarkin's approach to Kontsevich's formality theorem is to use the existence of a Ger_∞-structure on the Hochschild complex $C^\bullet(A)$ (1.1), whose structure maps are expressed in terms of the cup product and insertions of cochains into a cochain. Showing the existence of such a Ger_∞-structure is the most difficult and the most interesting part of the proof. The construction of this Ger_∞-structure involves the choice of a Drinfeld associator. Furthermore, it is known [**39**] that different choices of Drinfeld associators result in homotopy inequivalent Ger_∞-structures.

According to [**8**], the set of Drinfeld associators forms a torsor (i.e. principle homogeneous space) for an infinite dimensional algebraic group GRT, which is called the Grothendieck-Teichmueller group[1]. This group is related to moduli of curves, to the absolute Galois group of the field of rationals, and to the theory of motives [**11**].

In preprint [**42**], T. Willwacher established remarkable links[2] between three objects: the group GRT, the full graph complex fGC and the deformation complex

[1]Following [**1**] we denote by GRT the unipotent radical of the group introduced by Drinfeld.
[2]We believe that the same link between the group GRT and the deformation complex of the operad Ger was established via different methods in paper [**10**] by B. Fresse.

of the operad Ger governing Gerstenhaber algebras. Using these links one can connect the above seemingly unrelated stories:

- Tamarkin's approach to Kontsevich's formality theorem based on the use of Drinfeld associators, and
- the action of the full graph complex fGC on L_∞ quasi-isomorphisms (1.2).

We refer the reader to [**4**], [**5**], and [**41**] for more details.

It is already clear that Willwacher's results have important consequences for deformation quantization, and they will certainly play a very influential role in future research. The details presented in [**42**], however, are technically subtle and difficult to access – even for experts. Many intermediate steps in the proofs are either left for the reader, or embedded in remarks and comments throughout the text. Moreover, several key statements are proved for a particular case, and then used in their full generality.

The goal of these notes is to give a detailed proof of T. Willwacher's theorem (See Theorem 13.2) which links the cohomology of the full graph complex fGC to the cohomology of the deformation complex of the operad Ger.

In addition, we also present here various prerequisites required for understanding the material of [**42**]. Thus, in Section 3, we review operads, cooperads, and the cobar construction. This construction assigns to a coaugmented cooperad $\mathcal{C}$ a free operad $\mathrm{Cobar}(\mathcal{C})$ with the differential defined in terms of the cooperad structure on $\mathcal{C}$. In Section 4, we give a detailed exposition of the convolution Lie algebra and its properties. In Section 5, we discuss homotopies of maps from $\mathrm{Cobar}(\mathcal{C})$ and prove a useful lifting property for such maps.

In Section 6 we describe in detail Willwacher's twisting construction Tw which assigns to a dg operad $\mathcal{O}$ and a map[3] (of dg operads)

$$\Lambda\mathsf{Lie}_\infty \to \mathcal{O} \tag{1.3}$$

another dg operad $\mathrm{Tw}\mathcal{O}$. We refer to $\mathrm{Tw}\mathcal{O}$ as *the twisted version* of the (dg) operad $\mathcal{O}$.

Algebras over $\mathrm{Tw}\mathcal{O}$ (satisfying minor technical conditions) can be identified with $\mathcal{O}$-algebras equipped with a chosen Maurer-Cartan element for the $\Lambda\mathsf{Lie}_\infty$-structure induced by the map (1.3). It is the twisting construction which gives us a convenient framework for working with various operads assembled from graphs, in particular, the full graph complex and its subcomplexes.

In Section 7, we introduce the operad Gra and define an embedding from the operad Ger to Gra.

In Section 8, we introduce the full graph complex fGC and its "connected part" $\mathsf{fGC}_{\mathrm{conn}} \subset \mathsf{fGC}$. We also present a link between fGC and its subcomplex $\mathsf{fGC}_{\mathrm{conn}}$. This link allows us to reduce the question of computing cohomology of fGC to the question of computing cohomology of $\mathsf{fGC}_{\mathrm{conn}}$.

Section 9 is devoted to a thorough analysis of the dg operad $\mathrm{Tw}\mathsf{Gra}$ and its various suboperads. Several useful statements about suboperads of $\mathrm{Tw}\mathsf{Gra}$ and the operad Ger are assembled in the commutative diagram (9.75) at the end of Section 9.

[3]The dg operad $\Lambda\mathsf{Lie}_\infty$ differs from the dg operad Lie_∞ governing L_∞-algebras by a degree shift. Namely, $\Lambda\mathsf{Lie}_\infty$-structures on a cochain complex V are in bijection with Lie_∞-structures on $\mathbf{s}^{-1}V$.

In Section 10, we use the results of the previous section to deduce deeper statements about the full graph complex fGC. In particular, we prove the decomposition theorem for the graph cohomology (see Theorem 10.4).

In Section 11, we introduce the deformation complex (11.2) of the operad Ger and prove a technical statement about this complex.

In Section 12, we consider the convolution Lie algebra $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ with the differential coming from a natural composition $\mathrm{Cobar}(\mathsf{Ger}^{\vee}) \to \mathsf{Ger} \to \mathsf{Gra}$. We prove that the cohomology of $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ is spanned by the class of a single given vector. In particular, $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ does not have non-zero cohomology classes "coming from arities ≥ 3". This statement is a version of Tamarkin's rigidity theorem for the Gerstenhaber algebra PV_d of polyvector fields on $\mathbb{K}^d$, which is one of the corner stones of Tamarkin's proof of Kontsevich's formality theorem.

Section 13 is the culmination of our notes. In this section, we give a proof of Theorem 13.2 which links the cohomology of the "connected part" of the full graph complex fGC to the cohomology of the "connected part" of the deformation complex of the operad Ger. The cohomology of the full graph complex and the cohomology of the deformation complex of the operad Ger can be easily expressed in terms of the cohomology of their "connected parts".

The proof of Theorem 13.2 is assembled from several building blocks. First, this proof relies on Corollary 9.25 which links the operad Ger to a suboperad of the dg operad $\mathrm{Tw}\mathsf{Gra}$. Second, it relies on technical Theorem 11.9 which is given in Subsection 11.2. This theorem states that the (extended) deformation complex of the operad Ger is quasi-isomorphic to a certain subcomplex. Finally, the proof of Theorem 13.2 relies on a version of Tamarkin's rigidity (see Corollary 12.2).

We should remark that the proof of Theorem 13.2 given here is not different from the one outlined in Willwacher's preprint [**42**]. We only make the logic "more linear" and fill in many omitted details.

Appendices A, B, C contain proofs of three useful statements: a lemma on a quasi-isomorphism between filtered complexes, the theorem on the Harrison homology of the cofree cocommutative coalgebra, and a version of the Goldman-Millson theorem [**19**]. Although all these statements are well known, it is hard to find in the literature proofs which are formulated in the desired generality.

Many minor steps in proofs are left as exercises, which are formulated in the body of the text. Appendix D at the end of the paper contains solutions to some of these exercises.

Theorem 13.2 accounts for only 30% of results of T. Willwacher's preprint [**42**]. So we hope to write a separate paper, in which we will give a detailed proof of Willwacher's theorem which links the full graph complex to the Lie algebra $\mathfrak{grt}$ of the Grothendieck-Teichmueller group GRT.

In our exposition, we tried to follow (or rather not to follow) Serre's suggestions from his famous lecture [**36**]. We hope that this text will be useful both for specialists working on operads and deformation quantization, and for graduate students interested in this subject.

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Summer School at the Center for Mathematics at Notre Dame (May 31 - June 4, 2011). We would like to thank Samuel Evens and Michael Gekhtman for organizing such a wonderful summer school. We are thankful to our previous institution, the UC Riverside, in which we started discussing topics related to this paper. V.A.D. would like to thank Ezra Getzler for his kind offer to use his office during V.A.D.'s visit of Northwestern University in May of 2011. V.A.D. is also thankful to Brian Paljug for his remarks about early versions of the draft.

1.1. Notation and Conventions. The base field $\mathbb{K}$ has characteristic zero. For a set X we denote by $\mathbb{K}\langle X\rangle$ the $\mathbb{K}$-vector space of finite linear combinations of elements in X .

The underlying symmetric monoidal category $\mathfrak{C}$ is often the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces or the category $\mathsf{grVect}_{\mathbb{K}}$ of $\mathbb{Z}$-graded $\mathbb{K}$-vector spaces. We will frequently use the ubiquitous combination "dg" (differential graded) to refer to algebraic objects in $\mathsf{Ch}_{\mathbb{K}}$. For a homogeneous vector v in a cochain complex (or a graded vector space), $|v|$ denotes the degree of v . We denote by $\mathbf{s}$ (resp. $\mathbf{s}^{-1}$) the operation of suspension (resp. desuspension). Namely, for a cochain complex (or a graded vector space) $\mathcal{V}$, we have

$$\big(\mathbf{s}\,\mathcal{V}\big)^{\bullet} = \mathcal{V}^{\bullet-1}\,, \qquad \big(\mathbf{s}^{-1}\,\mathcal{V}\big)^{\bullet} = \mathcal{V}^{\bullet+1}\,.$$

The notation $\mathbf{1}$ is reserved for the unit of the underlying symmetric monoidal category $\mathfrak{C}$

By a commutative algebra we always mean commutative and associative algebra. The notation Lie (resp. Com, Ger) is reserved for the operad governing Lie algebras (resp. commutative algebras without unit, Gerstenhaber algebras without unit). Dually, the notation coLie (resp. coCom) is reserved for the cooperad governing Lie coalgebras (resp. cocommutative coalgebras without counit). The notation $\mathrm{CH}(x,y)$ is reserved for the Campbell-Hausdorff series in x and y.

The notation S_n is reserved for the symmetric group on n letters and $\mathrm{Sh}_{p_1,\dots,p_k}$ denotes the subset of $(p_1,\dots,p_k)$-shuffles in S_n, i.e. $\mathrm{Sh}_{p_1,\dots,p_k}$ consists of elements $\sigma \in S_n$, $n = p_1 + p_2 + \dots + p_k$ such that

$$\sigma(1) < \sigma(2) < \dots < \sigma(p_1),$$

$$\sigma(p_1+1) < \sigma(p_1+2) < \dots < \sigma(p_1+p_2),$$

$$\dots$$

$$\sigma(n-p_k+1) < \sigma(n-p_k+2) < \dots < \sigma(n)\,.$$

For $i \le j \le n$ we denote by $\varsigma_{i,j}$ the following cycle in S_n

$$\varsigma_{i,j} = \begin{cases} (i, i+1, \dots, j-1, j) & \text{if } i<j\,, \\ \mathsf{id} & \text{if } i=j\,. \end{cases} \tag{1.4}$$

It is clear that

$$\varsigma_{i,j}^{-1} = \begin{cases} (j, j-1, \dots, i+1, i) & \text{if } i<j\,, \\ \mathsf{id} & \text{if } i=j\,. \end{cases} \tag{1.5}$$

For example, the set $\{\varsigma_{i,n}\}_{1\le i\le n}$ is exactly the set $\mathrm{Sh}_{n-1,1}$ of $(n-1,1)$-shuffles and $\{\varsigma_{1,i}^{-1}\}_{1\le i\le n}$ is the set $\mathrm{Sh}_{1,n-1}$ of $(1,n-1)$-shuffles.

For a group G and a G-module W, the notation W^G (resp. W_G) is reserved for the subspace of G-invariants (resp. the quotient space of G-coinvariants).

For an operad $\mathcal{O}$ (resp. a cooperad $\mathcal{C}$) and a cochain complex V, the notation $\mathcal{O}(V)$ (resp. $\mathcal{C}(V)$) is reserved for the free $\mathcal{O}$-algebra (resp. cofree $\mathcal{C}$-coalgebra). Namely,

$$\mathcal{O}(V) := \bigoplus_{n\geq 0} \Big(\mathcal{O}(n) \otimes V^{\otimes n}\Big)_{S_n}, \tag{1.6}$$

$$\mathcal{C}(V) := \bigoplus_{n\geq 0} \Big(\mathcal{C}(n) \otimes V^{\otimes n}\Big)^{S_n}. \tag{1.7}$$

For an augmented operad $\mathcal{O}$ (in $\mathsf{Ch}_{\mathbb{K}}$) we denote by $\mathcal{O}_\circ$ the kernel of the augmentation. Dually, for a coaugmented cooperad $\mathcal{C}$ (in $\mathsf{Ch}_{\mathbb{K}}$) we denote by $\mathcal{C}_\circ$ the cokernel of the coaugmentation. (We refer the reader to Subsections 3.3.1 and 3.5.1 for more details.)

For a groupoid $\mathcal{G}$ the notation $\pi_0(\mathcal{G})$ is reserved for the set of its isomorphism classes.

A directed graph (resp. graph) Γ is a pair $(V(\Gamma), E(\Gamma))$, where $V(\Gamma)$ is a finite non-empty set and $E(\Gamma)$ is a set of ordered (resp. unordered) pairs of elements of $V(\Gamma)$. Elements of $V(\Gamma)$ are called vertices and elements of $E(\Gamma)$ are called edges. We say that a directed graph (resp. graph) Γ is labeled if it is equipped with a bijection between the set $V(\Gamma)$ and the set of numbers $\{1, 2, \dots, |V(\Gamma)|\}$. We allow a graph with the empty set of edges.

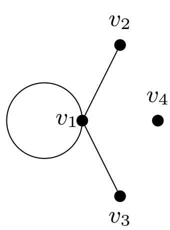


FIG. 1. A graph Γ

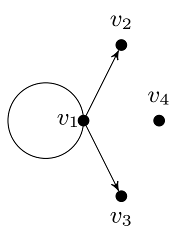


FIG. 2. A directed graph Γ'

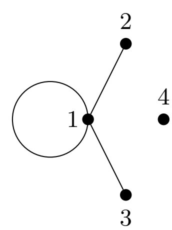


FIG. 3. A labeled graph Γ''

For example, the graph Γ on figure 1 has

$$V(\Gamma) = \{v_1, v_2, v_3, v_4\} \qquad \text{and} \qquad E(\Gamma) = \{\{v_1, v_1\}, \{v_2, v_1\}, \{v_1, v_3\}\}.$$

For the directed graph Γ' on figure 2 we have

$$V(\Gamma') = \{v_1, v_2, v_3, v_4\} \qquad \text{and} \qquad E(\Gamma') = \{(v_1, v_1), (v_1, v_2), (v_1, v_3)\}.$$

Finally, figure 3 gives us an example of a labeled graph.

A valency of a vertex v in a (directed) graph Γ is the total number of its appearances in the pairs $E(\Gamma)$. For example, vertex v_1 in the graph on figure 2 has valency 4.

2. Trees

A connected graph without cycles is called a tree. In this paper all trees are *planted*, i.e. each tree has a marked vertex (called the *root*) and this marked vertex has valency 1. (In particular, each tree has at least one edge.) The edge adjacent to the root vertex is called the *root edge*. Non-root vertices of valency 1 are called *leaves*. A vertex is called *internal* if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge adjacent to a leaf is called *external*. We allow a degenerate tree, that is a tree with exactly two vertices (the root vertex and a leaf) connected by a single edge. A tree $\mathbf{t}$ is called *planar* if, for every internal vertex v of $\mathbf{t}$, the set of edges terminating at v carries a total order.

Let us recall that for every planar tree $\mathbf{t}$ the set $V(\mathbf{t})$ of all its vertices is equipped with a natural total order. To define this total order on $V(\mathbf{t})$ we introduce the function

$$\mathcal{N} : V(\mathbf{t}) \to V(\mathbf{t})\,. \tag{2.1}$$

To a non-root vertex v the function $\mathcal{N}$ assigns the next vertex along the (unique) path connecting v to the root vertex. Furthermore $\mathcal{N}$ sends the root vertex to the root vertex.

Let v_1, v_2 be two distinct vertices of $\mathbf{t}$. If v_1 lies on the path which connects v_2 to the root vertex then we declare that

$$v_1 < v_2\,.$$

Similarly, if v_2 lies on the path which connects v_1 to the root vertex then we declare that

$$v_2 < v_1\,.$$

If neither of the above options realize then there exist numbers k_1 and k_2 such that

$$\mathcal{N}^{k_1}(v_1) = \mathcal{N}^{k_2}(v_2) \tag{2.2}$$

but

$$\mathcal{N}^{k_1-1}(v_1) \neq \mathcal{N}^{k_2-1}(v_2)\,.$$

Since the tree $\mathbf{t}$ is planar the set of $\mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1))$ is equipped with a total order. Furthermore, since both vertices $\mathcal{N}^{k_1-1}(v_1)$ and $\mathcal{N}^{k_2-1}(v_2)$ belong to the set $\mathcal{N}^{-1}(\mathcal{N}^{k_1}(v_1))$, we may compare them with respect to this order.

We declare that, if $\mathcal{N}^{k_1-1}(v_1) < \mathcal{N}^{k_2-1}(v_2)$, then

$$v_1 < v_2\,.$$

Otherwise we set $v_2 < v_1$.

It is not hard to see that the resulting relation $<$ on $V(\mathbf{t})$ is indeed a total order.

The total order on $V(\mathbf{t})$ can be defined graphically. Indeed, draw a planar tree $\mathbf{t}$ on the plane. Then choose a small tubular neighborhood of $\mathbf{t}$ on the plane and walk along its boundary starting from a vicinity of the root vertex in the clockwise direction. On our way, we will meet each vertex of $\mathbf{t}$ at least once. So we declare that $v_1 < v_2$ if the first occurrence of v_1 precedes the first occurrence of v_2.

For example, consider the planar tree depicted on figure 4. Following the path drawn around this tree we get

$$r < v_1 < v_2 < v_3 < v_4 < v_5 < v_6\,.$$

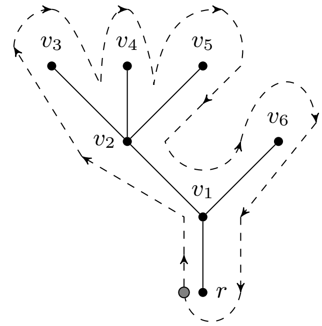


FIG. 4. We start walking around the planar tree from the gray circle

Keeping this order in mind, we can say things like "the first vertex", "the second vertex", and "the i-th vertex" of a planar tree $\mathbf{t}$. In fact, the first vertex of a tree is always its root vertex.

We have an obvious bijection between the set of edges $E(\mathbf{t})$ of a tree $\mathbf{t}$ and the subset of vertices:

$$V(\mathbf{t}) \setminus \{\text{root vertex}\}\,. \tag{2.3}$$

This bijection assigns to a vertex v in (2.3) its outgoing edge.

Thus the canonical total order on the set (2.3) gives us a natural total order on the set of edges $E(\mathbf{t})$.

For our purposes we also extend the total orders on the sets $V(\mathbf{t})\setminus\{\text{root vertex}\}$ and $E(\mathbf{t})$ to the disjoint union

$$V(\mathbf{t}) \setminus \{\text{root vertex}\} \sqcup E(\mathbf{t}) \tag{2.4}$$

by declaring that a vertex is bigger than its outgoing edge. For example, the root edge is the minimal element in the set (2.4).

2.1. Groupoid of labeled planar trees. Let n be a non-negative integer. An n-labeled planar tree $\mathbf{t}$ is a planar tree equipped with an injective map

$$\mathfrak{l} : \{1, 2, \dots, n\} \to L(\mathbf{t}) \tag{2.5}$$

from the set $\{1, 2, \dots, n\}$ to the set $L(\mathbf{t})$ of leaves of $\mathbf{t}$. Although the set $L(\mathbf{t})$ has a natural total order we do not require that the map (2.5) is monotonous.

The set $L(\mathbf{t})$ of leaves of an n-labeled planar tree $\mathbf{t}$ splits into the disjoint union of the image $\mathfrak{l}(\{1, 2, \dots, n\})$ and its complement. We call leaves in the image of $\mathfrak{l}$ *labeled.*

A vertex x of an n-labeled planar tree $\mathbf{t}$ is called *nodal* if it is neither a root vertex, nor a labeled leaf. We denote by $V_{\text{nod}}(\mathbf{t})$ the set of all nodal vertices of $\mathbf{t}$. Keeping in mind the canonical total order on the set of all vertices of $\mathbf{t}$ we can say things like "the first nodal vertex", "the second nodal vertex", and "the i-th nodal vertex".

EXAMPLE 2.1. An example of a 4-labeled planar tree is depicted on figure 5. On figures we use small white circles for nodal vertices and small black circles for labeled leaves and the root vertex.

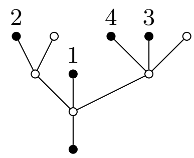


FIG. 5. A 4-labeled planar tree

For our purposes we need to upgrade the set of n-labeled planar trees to the groupoid $\mathsf{Tree}(n)$. Objects of $\mathsf{Tree}(n)$ are n-labeled planar trees and morphisms are non-planar isomorphisms of the corresponding (non-planar) trees compatible with labeling. The groupoid $\mathsf{Tree}(n)$ is equipped with an obvious left action of the symmetric group S_n.

As far as we know the groupoid $\mathsf{Tree}(n)$ was originally introduced by E. Getzler and M. Kapranov in [**17**]. However, here we do not exactly follow the notation from [**17**].

The notation $\mathsf{Tree}_2(n)$ is reserved for the full sub-category of $\mathsf{Tree}(n)$ whose objects are n-labeled planar trees with exactly 2 nodal vertices. It is not hard to see that every object in $\mathsf{Tree}_2(n)$ has at most $n+1$ leaves. Due to Exercise 2.2, isomorphism classes of $\mathsf{Tree}_2(n)$ are in bijection with the union

$$\bigsqcup_{0 \le p \le n} \mathrm{Sh}_{p,n-p}\,. \tag{2.6}$$

EXERCISE 2.2. Let us assign to a shuffle $\tau \in \mathrm{Sh}_{p,n-p}$ the n-labeled planar tree depicted on figure 6. Prove that this assignment gives us a bijection between the

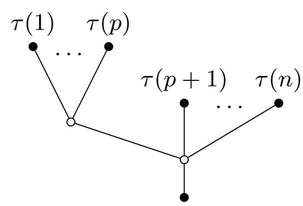


FIG. 6. Here τ is a $(p, n-p)$-shuffle

set (2.6) and the set of isomorphism classes in $\mathsf{Tree}_2(n)$.

REMARK 2.3. The groupoid $\mathsf{Tree}_2(0)$ has exactly one object (see figure 7) and hence exactly one isomorphism class. The groupoid $\mathsf{Tree}_2(1)$ has three objects and two isomorphisms classes. Representatives of isomorphism classes in $\mathsf{Tree}_2(1)$ are depicted on figures 8 and 9.

2.2. Insertions of trees. Let $\widetilde{\mathbf{t}}$ be an n-labeled planar tree with a non-empty set of nodal vertices. If the i-th nodal vertex of $\widetilde{\mathbf{t}}$ has m_i incoming edges then for every m_i-labeled planar tree $\mathbf{t}$ we can define the insertion $\bullet_i$ of the tree $\mathbf{t}$ into the i-th nodal vertex of $\widetilde{\mathbf{t}}$. The resulting planar tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is also n-labeled.

If $m_i = 0$ then $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is obtained via identifying the root edge of $\mathbf{t}$ with edge originating at the i-th nodal vertex.

If $m_i > 0$ then the tree $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is built following these steps:

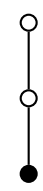


FIG. 7. The unique object in $\mathsf{Tree}_2(0)$

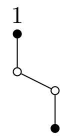


FIG. 8. A tree in $\mathsf{Tree}_2(1)$

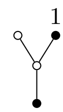


FIG. 9. A tree in $\mathsf{Tree}_2(1)$

- First, we denote by $E_i(\widetilde{\mathbf{t}})$ the set of edges terminating at the i-th nodal vertex of $\widetilde{\mathbf{t}}$. Since $\widetilde{\mathbf{t}}$ is planar, the set $E_i(\widetilde{\mathbf{t}})$ comes with a total order;
- second, we erase the i-th nodal vertex of $\widetilde{\mathbf{t}}$;
- third, we identify the root edge of $\mathbf{t}$ with the edge of $\widetilde{\mathbf{t}}$ which originates at the i-th nodal vertex;
- finally, we identify the edges of $\mathbf{t}$ adjacent to labeled leaves with the edges in the set $E_i(\widetilde{\mathbf{t}})$ following this rule: the external edge with label j gets identified with the j-th edge in the set $E_i(\widetilde{\mathbf{t}})$. In doing this, we keep the same planar structure on $\mathbf{t}$, so, in general, branches of $\widetilde{\mathbf{t}}$ move around.

EXAMPLE 2.4. Let $\widetilde{\mathbf{t}}$ be the 4-labeled planar tree depicted on figure 5 and $\mathbf{t}$ be the 3-labeled planar tree depicted on figure 10. Then the insertion $\widetilde{\mathbf{t}} \bullet_1 \mathbf{t}$ of $\mathbf{t}$ into the first nodal vertex of $\widetilde{\mathbf{t}}$ is shown on figure 11. Figure 12 illustrates the construction algorithm of $\widetilde{\mathbf{t}} \bullet_1 \mathbf{t}$ step by step.

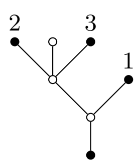


FIG. 10. A 3-labeled planar tree $\mathbf{t}$

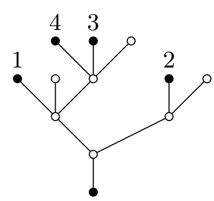


FIG. 11. The 4-labeled planar tree $\widetilde{\mathbf{t}} \bullet_1 \mathbf{t}$

3. Operads, pseudo-operads, and their dual versions

3.1. Collections. By a *collection* we mean the sequence $\{P(n)\}_{n\ge 0}$ of objects of the underlying symmetric monoidal category $\mathfrak{C}$ such that for each n, the object $P(n)$ is equipped with a left action of the symmetric group S_n.

Given a collection P we form covariant functors for $n \ge 0$

$$\underline{P}_n : \mathsf{Tree}(n) \to \mathfrak{C}\,.$$

To an n-labelled planar tree $\mathbf{t}$ the functor $\underline{P}_n$ assigns the object

$$\underline{P}_n(\mathbf{t}) = \bigotimes_{x \in V_{\mathrm{nod}}(\mathbf{t})} P(m(x))\,, \tag{3.1}$$

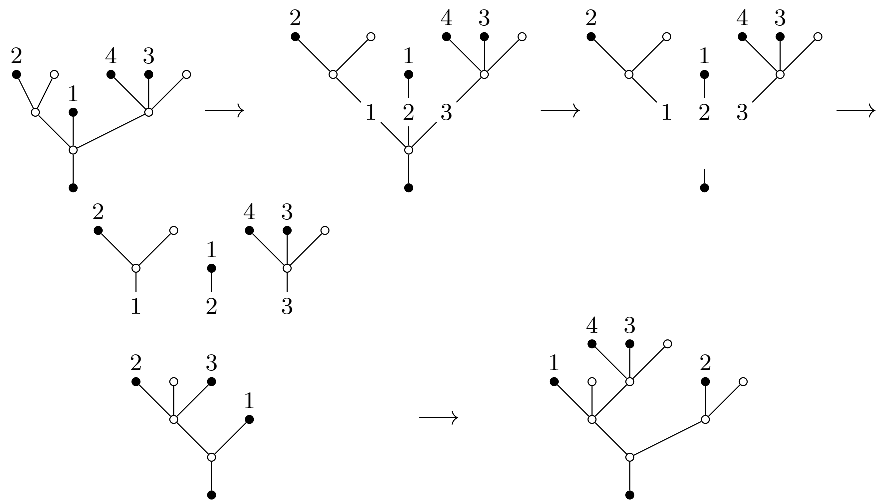


FIG. 12. Algorithm for constructing $\widetilde{\mathbf{t}} \bullet_1 \mathbf{t}$

where $V_{\mathrm{nod}}(\mathbf{t})$ is the set of all nodal vertices of $\mathbf{t}$, the notation $m(x)$ is reserved for the number of edges terminating at the vertex x, and the order of the factors in the right hand side of the equation agrees with the natural order on the set $V_{\mathrm{nod}}(\mathbf{t})$.

To define the functor $\underline{P}_n$ on the level of morphisms we use the actions of the symmetric groups and the structure of the symmetric monoidal category $\mathfrak{C}$ in the obvious way.

EXAMPLE 3.1. Let $\mathbf{t}_1$ (resp. $\mathbf{t}_2$) be a 2-labeled planar tree depicted on figure 13 (resp. figure 14). There is a unique morphism λ from $\mathbf{t}_1$ to $\mathbf{t}_2$ in $\mathsf{Tree}(2)$. For

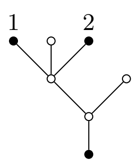


FIG. 13. A 2-labeled planar tree $\mathbf{t}_1$

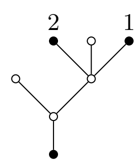


FIG. 14. A 2-labeled planar tree $\mathbf{t}_2$

these trees we have

$$\underline{P}_2(\mathbf{t}_1) = P(2) \otimes P(3) \otimes P(0) \otimes P(0)\,,$$

$$\underline{P}_2(\mathbf{t}_2) = P(2) \otimes P(0) \otimes P(3) \otimes P(0)\,,$$

and the morphism

$$\underline{P}_2(\lambda) : P(2) \otimes P(3) \otimes P(0) \otimes P(0) \to P(2) \otimes P(0) \otimes P(3) \otimes P(0)$$

is the composition

$$\underline{P}_2(\lambda) = (1 \otimes \beta) \circ (\sigma_{12} \otimes \sigma_{13} \otimes 1 \otimes 1),$$

where σ_{12} (resp. σ_{13}) is the corresponding transposition in S_2 (resp. in S_3) and β is the braiding

$$\beta : \big(P(3) \otimes P(0)\big) \otimes P(0) \to P(0) \otimes \big(P(3) \otimes P(0)\big)\,.$$

3.2. Pseudo-operads. We now recall that a *pseudo-operad* is a collection $\{P(n)\}_{n \ge 0}$ equipped with multiplication maps

$$\mu_{\mathbf{t}} : \underline{P}_n(\mathbf{t}) \to P(n) \tag{3.2}$$

for all n-labeled trees $\mathbf{t}$ and for all $n \ge 0$. These multiplications should satisfy the axioms which we list below.

First, we require that for the standard corolla $\mathbf{q}_n$ (see figures 15, 16) the mul-

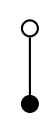


FIG. 15. The corolla $\mathbf{q}_0$

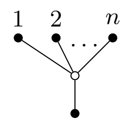


FIG. 16. The corolla $\mathbf{q}_n$ for $n \ge 1$

tiplication map $\mu_{\mathbf{q}_n}$ is the identity morphism on $P(n)$

$$\mu_{\mathbf{q}_n} = \mathsf{id}_{P(n)}\,. \tag{3.3}$$

Second, we require that the operations (3.2) are S_n-equivariant

$$\mu_{\sigma(\mathbf{t})} = \sigma \circ \mu_{\mathbf{t}}\,, \qquad \forall\ \sigma \in S_n,\ \mathbf{t} \in \mathsf{Tree}(n)\,. \tag{3.4}$$

Third, for every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in $\mathsf{Tree}(n)$ we have

$$\mu_{\mathbf{t}'} \circ \underline{P}_n(\lambda) = \mu_{\mathbf{t}}\,. \tag{3.5}$$

Finally, we need to formulate the associativity axiom for multiplications (3.2). For this purpose we consider the following quadruple $(\widetilde{\mathbf{t}}, i, m_i, \mathbf{t})$ where $\widetilde{\mathbf{t}}$ is an n-labeled planar tree with k nodal vertices, $1 \le i \le k$, m_i is the number of edges terminating at the i-th nodal vertex of $\widetilde{\mathbf{t}}$, and $\mathbf{t}$ is an m_i-labeled planar tree.

The associativity axioms states that for each such quadruple $(\widetilde{\mathbf{t}}, i, m_i, \mathbf{t})$ we have

$$\mu_{\widetilde{\mathbf{t}}} \circ (\mathsf{id} \otimes \cdots \otimes \mathsf{id} \otimes \underbrace{\mu_{\mathbf{t}}}_{i\text{-th spot}} \otimes \mathsf{id} \otimes \cdots \otimes \mathsf{id}) \circ \beta_{\widetilde{\mathbf{t}}, i, m_i, \mathbf{t}} = \mu_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}} \tag{3.6}$$

where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is the n-labeled planar tree obtained by inserting $\mathbf{t}$ into the i-th nodal vertex of $\widetilde{\mathbf{t}}$ and $\beta_{\widetilde{\mathbf{t}}, i, m_i, \mathbf{t}}$ is the isomorphism in $\mathfrak{C}$ which is responsible for "putting tensor factors in the correct order".

To define the isomorphism $\beta_{\widetilde{\mathbf{t}}, i, m_i, \mathbf{t}}$ we observe that the source of the map $\mu_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}}$ is

$$\bigotimes_{x \in V_{\mathrm{nod}}(\widetilde{\mathbf{t}} \bullet_i \mathbf{t})} P(m(x)) \tag{3.7}$$

where $m(x)$ denotes the number of edges of $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ terminating at the nodal vertex x and the order of factors agrees with the total order on the set $V_{\mathrm{nod}}(\widetilde{\mathbf{t}} \bullet_i \mathbf{t})$. The source of the map

$$\mu_{\tilde{\mathbf{t}}} \circ (\mathsf{id} \otimes \cdots \otimes \mathsf{id} \otimes \underbrace{\mu_{\mathbf{t}}}_{i-th\ spot} \otimes \mathsf{id} \otimes \cdots \otimes \mathsf{id}) \tag{3.8}$$

is also the product (3.7) with a possibly different order of tensor factors. The map $\beta_{\tilde{\mathbf{t}},i,m_i,\mathbf{t}}$ in (3.6) is the isomorphism in $\mathfrak{C}$ which connects the source of $\mu_{\tilde{\mathbf{t}}\bullet_i \mathbf{t}}$ to the source of (3.8).

Given integers $n \geq 1$, $k \geq 0$, $1 \leq i \leq n$ and a permutation $\sigma \in S_{n+k-1}$ we can form the $(n+k-1)$-labeled planar tree $\mathbf{t}^{n,k,i}_{\sigma}$ shown on figure 17.

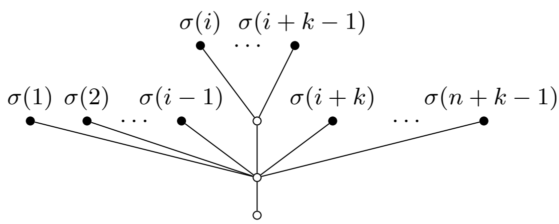


FIG. 17. The $(n+k-1)$-labeled planar tree $\mathbf{t}^{n,k,i}_{\sigma}$

In the case when $\mathfrak{C} = \mathsf{Ch}_{\mathbb{K}}$ or $\mathfrak{C} = \mathsf{grVect}_{\mathbb{K}}$, it is convenient to use a slightly different notation for values of the multiplication map $\mu_{\mathbf{t}^{n,k,i}_{\sigma}}$ corresponding to the tree $\mathbf{t}^{n,k,i}_{\sigma}$. More precisely, for a vector $v \in P(n)$ and $w \in P(k)$ of a pseudo-operad P we set

$$v\big(\sigma(1), \dots \sigma(i-1), w\big(\sigma(i), \dots, \sigma(i+k-1)\big), \sigma(i+k), \dots, \sigma(n+k-1)\big) := \mu_{\mathbf{t}^{n,k,i}_{\sigma}}(v, w)\,. \tag{3.9}$$

Recall that, for $\sigma = \mathsf{id} \in S_{n+k-1}$, the multiplication

$$\mu_{\mathbf{t}^{n,k,i}_{\mathsf{id}}} : P(n) \otimes P(k) \to P(n+k-1)$$

is called *the elementary insertion* and often denoted by $\circ_i$. Namely, for $v \in P(n)$ and $w \in P(k)$ we have[4]

$$v \circ_i w := \mu_{\mathbf{t}^{n,k,i}_{\mathsf{id}}}(v, w)\,. \tag{3.10}$$

It is not hard to see that a pseudo-operad structure on a collection P (in $\mathsf{Ch}_{\mathbb{K}}$ or $\mathsf{grVect}_{\mathbb{K}}$) is uniquely determined by elementary insertions (3.10). All the remaining multiplications (3.2) can be expressed in terms of (3.10) using the axioms of a pseudo-operad.

Thus, it is not hard to see that, the following definition of a pseudo-operad is equivalent to ours.

DEFINITION 3.2 (Definition 17, [**32**]). A pseudo-operad in $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$) is a collection P in $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$) equipped with maps

$$\circ_i : P(n) \otimes P(k) \to P(n+k-1)\,, \qquad 1 \leq i \leq n \tag{3.11}$$

[4]Numbers n and k are suppressed from the notation.

satisfying the associativity axiom and equivariance axioms. The associativity axiom states that for all homogeneous vectors a, b, c in $P(n_a), P(n_b)$, and $P(n_c)$, respectively and for all $1 \le i \le n_a$ and $1 \le j \le n_a + n_b - 1$

$$(a \circ_i b) \circ_j c = \begin{cases} (-1)^{|b||c|} (a \circ_j c) \circ_{i+n_c-1} b & \text{if } j < i\,, \\ a \circ_i (b \circ_{j-i+1} c) & \text{if } i \le j \le i + n_b - 1\,, \\ (-1)^{|b||c|} (a \circ_{j-n_b+1} c) \circ_i b & \text{if } j \ge i + n_b\,. \end{cases} \tag{3.12}$$

The equivariance axioms state that for all $1 \le p \le n_b - 1$ and $1 \le k \le n_a - 1$ we have

$$a \circ_i (\sigma_{p\,(p+1)} b) = \sigma_{(p+i-1)\,(p+i)} (a \circ_i b)\,, \tag{3.13}$$

$$(\sigma_{k\,(k+1)} a) \circ_i b = \begin{cases} \sigma_{k\,(k+1)} (a \circ_i b) & \text{if } k+1 < i \\ \varsigma_{i-1,\, i+n_b-1} (a \circ_{i-1} b) & \text{if } k+1 = i \\ \varsigma^{-1}_{i,\, i+n_b} (a \circ_{i+1} b) & \text{if } k+1 = i \\ \sigma_{(k+n_b-1)\,(k+n_b)} (a \circ_i b) & \text{if } k > i\,. \end{cases} \tag{3.14}$$

Here $\sigma_{i_1\,i_2}$ denotes the transposition (i_1, i_2) and ς_{i_1, i_2} is the cycle defined in (1.4).

In [**32**] a pseudo-operad is called non-unital Markl's operad.

3.3. Operads. An operad is a pseudo-operad P with unit, that is a map

$$\mathbf{u} : \mathbf{1} \to P(1) \tag{3.15}$$

for which the compositions

$$\begin{aligned} P(n) \cong P(n) \otimes \mathbf{1} \stackrel{\mathsf{id}\otimes \mathbf{u}}{\longrightarrow} P(n) \otimes P(1) \stackrel{\circ_i}{\longrightarrow} P(n) \\ P(n) \cong \mathbf{1} \otimes P(n) \stackrel{\mathbf{u}\otimes \mathsf{id}}{\longrightarrow} P(1) \otimes P(n) \stackrel{\circ_1}{\longrightarrow} P(n) \end{aligned} \tag{3.16}$$

coincide with the identity map on $P(n)$.

Morphisms of pseudo-operads and operads are defined in the obvious way.

EXAMPLE 3.3. For an object $\mathcal{V}$ of $\mathfrak{C}$ we denote by $\mathsf{End}_{\mathcal{V}}$ the following collection[5]

$$\mathsf{End}_{\mathcal{V}}(n) = \mathrm{Hom}(\mathcal{V}^{\otimes\, n}, \mathcal{V})\,. \tag{3.17}$$

This collection is equipped with the obvious structure of an operad. Namely, the elementary insertions

$$\circ_i \colon \mathsf{End}_{\mathcal{V}}(n) \otimes \mathsf{End}_{\mathcal{V}}(m) \to \mathsf{End}_{\mathcal{V}}(n+m-1)$$

are defined by the equation

$$f \circ_i g := f \circ (\mathsf{id}^{\otimes\,(i-1)} \otimes g \otimes \mathsf{id}^{\otimes\,(n-i)})$$

and the unit

$$\mathbf{u} : \mathbf{1} \to \mathrm{Hom}(\mathcal{V}, \mathcal{V})$$

corresponds to the isomorphism $\mathbf{1} \otimes \mathcal{V} \cong \mathcal{V}$. We call $\mathsf{End}_{\mathcal{V}}$ the *endomorphism operad* of $\mathcal{V}$.

[5]We tacitly assume that the symmetric monoidal category $\mathfrak{C}$ has inner Hom.

This example plays an important role because it is used in the definition of an algebra over an operad. Namely, an *algebra over an operad* P (in $\mathfrak{C}$) is an object $\mathcal{V}$ of $\mathfrak{C}$ together with an operad map

$$P \to \mathsf{End}_{\mathcal{V}}\,.$$

It is not hard to see that an object $\mathcal{V}$ in $\mathfrak{C}$ is an algebra over an operad P if and only if $\mathcal{V}$ is equipped with a collection of multiplication maps

$$\mu_{\mathcal{V}} : P(n) \otimes \mathcal{V}^{\otimes\, n} \to \mathcal{V} \qquad n \ge 0\,, \tag{3.18}$$

satisfying the associativity axiom, the equivariance axiom and the unitality axiom formulated, for instance, in [**32**, Proposition 24] .

EXERCISE 3.4. Let $\mathfrak{C} = \mathsf{grVect}_{\mathbb{K}}$. Consider the collections

$$\mathsf{Com}_u(n) = \mathbb{K}\,, \tag{3.19}$$

and

$$\mathsf{Com}(n) = \begin{cases} \mathbb{K} & \text{if } n \ge 1\,, \\ \mathbf{0} & \text{if } n = 0\,. \end{cases} \tag{3.20}$$

with the trivial S_n-action on $\mathsf{Com}(n)$ (resp. $\mathsf{Com}_u(n)$) . The collections Com and Com_u are equipped with the obvious operad structures. For Com_u we have

$$\circ_i = \mathsf{id} \ : \ \mathsf{Com}_u(n) \otimes \mathsf{Com}_u(k) \cong \mathbb{K} \otimes \mathbb{K} \to \mathsf{Com}_u(n+k-1) \cong \mathbb{K}\,,$$

$$\mathbf{u} = \mathsf{id} \ : \ \mathbb{K} \ \to \ \mathsf{Com}_u(1) \cong \mathbb{K}\,,$$

and for Com we have

$$\circ_i = \mathsf{id} \ : \ \mathsf{Com}(n) \otimes \mathsf{Com}(k) \cong \mathbb{K} \otimes \mathbb{K} \to \mathsf{Com}(n+k-1) \cong \mathbb{K}\,,$$

if $k \neq 0$, and

$$\mathbf{u} = \mathsf{id} \ : \ \mathbb{K} \ \to \ \mathsf{Com}(1) \cong \mathbb{K}\,.$$

Show that Com_u-algebras (resp. Com-algebras) are exactly unital (resp. non-unital) commutative algebras.

EXERCISE 3.5. Let P and $\mathcal{O}$ be operads (resp. pseudo-operad) in $\mathfrak{C}$. Show that the collection $P \otimes \mathcal{O}$ with

$$P \otimes \mathcal{O}(n) = P(n) \otimes \mathcal{O}(n)$$

is naturally an operad (resp. pseudo-operad) . *For this exercise it may be more convenient to use Markl's definition* [**32**, Definition 17].

EXERCISE 3.6 (The operad Λ). Let $\mathfrak{C} = \mathsf{grVect}_{\mathbb{K}}$ or $\mathfrak{C} = \mathsf{Ch}_{\mathbb{K}}$. Consider the collection Λ

$$\Lambda(n) = \begin{cases} \mathbf{s}^{1-n}\mathrm{sgn}_n & \text{if } n \ge 1 \\ \mathbf{0} & \text{if } n = 0 \end{cases} \tag{3.21}$$

with sgn_n being the sign representation of S_n . Let

$$\circ_i \ : \ \Lambda(n) \otimes \Lambda(k) \to \Lambda(n+k-1)$$

be the operations defined by

$$1_n \circ_i 1_k = (-1)^{(1-k)(n-i)} 1_{n+k-1}\,, \tag{3.22}$$

where 1_m denotes the generator $\mathbf{s}^{1-m}1 \in \mathbf{s}^{1-m}\mathrm{sgn}_m$. Prove that (3.22) together with the obvious unit map $\mathbf{u} = \mathsf{id} : \mathbb{K} \to \Lambda(1) \cong \mathbb{K}$ equip the collection Λ with a structure of an operad. Show that Λ-algebra structures on $\mathcal{V}$ are in bijection with Com-algebra structure on $\mathbf{s}^{-1}\mathcal{V}$.

EXERCISE 3.7. For an operad $\mathcal{O}$ in the category $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$) we denote by $\Lambda\mathcal{O}$ the operad

$$\Lambda\mathcal{O} := \Lambda \otimes \mathcal{O}\,. \tag{3.23}$$

Show that $\Lambda\mathcal{O}$-algebra structures on a cochain complex (resp. graded vector space) $\mathcal{V}$ are in bijection with $\mathcal{O}$-algebra structures on $\mathbf{s}^{-1}\,\mathcal{V}$.

EXAMPLE 3.8. Let Lie be the operad which governs Lie algebras. An algebra over $\Lambda\mathsf{Lie}$ in $\mathsf{grVect}_{\mathbb{K}}$ is a graded vector space $\mathcal{V}$ equipped with the binary operation:

$$\{\,,\,\} : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V}$$

of degree -1 satisfying the identities:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\}\,,$$

$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0\,,$$

where v_1, v_2, v_3 are homogeneous vectors in $\mathcal{V}$.

EXERCISE 3.9 (Free algebra over an operad $\mathcal{O}$). Let $\mathcal{O}$ be an operad in the category $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$). Show that for every cochain complex (resp. graded vector space) $\mathcal{V}$ the direct sum

$$\mathcal{O}(\mathcal{V}) := \bigoplus_{n=0}^{\infty} \Big(\mathcal{O}(n) \otimes \mathcal{V}^{\otimes\, n}\Big)_{S_n} \tag{3.24}$$

carries a natural structure of an algebra over $\mathcal{O}$. Prove that the $\mathcal{O}$-algebra $\mathcal{O}(\mathcal{V})$ is free. In other words, the assignment

$$\mathcal{V} \to \mathcal{O}(\mathcal{V})$$

upgrades to a functor which is left adjoint to the forgetful functor from the category of $\mathcal{O}$-algebras to the category $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$).

3.3.1. *Augmented operads.* In this subsection $\mathfrak{C}$ is either $\mathsf{Ch}_{\mathbb{K}}$ or $\mathsf{grVect}_{\mathbb{K}}$.

Let us observe that the collection

$$*(n) = \begin{cases} \mathbb{K} & \text{if } n = 1 \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{3.25}$$

is equipped with the unique structure of an operad. In fact, $*$ is the initial object in the category of operads (in $\mathfrak{C}$).

An augmentation of an operad $\mathcal{O}$ is an operad morphism

$$\varepsilon : \mathcal{O} \to *\,.$$

Given a pseudo-operad P in $\mathfrak{C}$ we can always form an operad by formally adjoining a unit. The resulting operad is naturally augmented.

Furthermore, the kernel of the augmentation for any augmented operad is naturally a pseudo-operad. According to [**32**, Proposition 21], these two constructions give us an equivalence between the category of augmented operads and the category pseudo-operads.

For an augmented operad $\mathcal{O}$ we denote by $\mathcal{O}_\circ$ the kernel of its augmentation.

EXERCISE 3.10. Show that the operads Com and Lie have natural augmentations. Prove that the operad Com_u (from Exercise 3.4) does not admit an augmentation.

3.3.2. *Example: the operad* Ger. Let us recall that a *Gerstenhaber algebra* is a graded vector space V equipped with a commutative (and associative) product (without identity) and a degree -1 binary operation $\{\ ,\ \}$ which satisfies the following relations:

$$\{v_1, v_2\} = (-1)^{|v_1||v_2|}\{v_2, v_1\}\,, \tag{3.26}$$

$$\{v, v_1 v_2\} = \{v, v_1\}v_2 + (-1)^{|v_1||v|+|v_1|}v_1\{v, v_2\}\,, \tag{3.27}$$

(3.28)
$$\{\{v_1, v_2\}, v_3\} + (-1)^{|v_1|(|v_2|+|v_3|)}\{\{v_2, v_3\}, v_1\} + (-1)^{|v_3|(|v_1|+|v_2|)}\{\{v_3, v_1\}, v_2\} = 0\,.$$

In particular, $(V, \{\ ,\ \})$ is a $\Lambda\mathsf{Lie}$-algebra.

To define spaces of the operad Ger governing Gerstenhaber algebras we introduce the free Gerstenhaber algebra Ger_n in n dummy variables $a_1, a_2, \dots, a_n$ of degree 0. Next we set $\mathsf{Ger}(0) = \mathbf{0}$ and $\mathsf{Ger}(1) = \mathbb{K}$. Then we declare that, for $n \geq 2$, $\mathsf{Ger}(n)$ is spanned by monomials of Ger_n in which each dummy variable a_i appears exactly once.

The symmetric group S_n acts on $\mathsf{Ger}(n)$ by permuting the dummy variables and the elementary insertions are defined in the obvious way.

EXAMPLE 3.11. Let us consider the monomials $u = \{a_2, a_3\}a_1\{a_4, a_5\} \in \mathsf{Ger}(5)$ and $w = \{a_1, a_2\} \in \mathsf{Ger}(2)$ and compute the insertions $u \circ_2 w$, $u \circ_4 w$ and $w \circ_1 u$. We get

$$u \circ_2 w = -\{\{a_2, a_3\}, a_4\}a_1\{a_5, a_6\}\,, \qquad u \circ_4 w = \{a_2, a_3\}a_1\{\{a_4, a_5\}, a_6\}\,,$$

$$w \circ_1 u = \{\{a_2, a_3\}a_1\{a_4, a_5\}, a_6\} =$$
$$= \{a_6, \{a_2, a_3\}a_1\{a_4, a_5\}\} = \{a_6, \{a_2, a_3\}\}a_1\{a_4, a_5\}$$
$$-\{a_2, a_3\}\{a_6, a_1\}\{a_4, a_5\} - \{a_2, a_3\}a_1\{a_6, \{a_4, a_5\}\}\,.$$

(Note that the insertions obey the usual Koszul rule for signs.)

It is easy to see that the operad Ger is generated by the monomials $a_1 a_2, \{a_1, a_2\} \in \mathsf{Ger}(2)$ and algebras over the operad Ger are Gerstenhaber algebras. It is also easy to see that the monomial $\{a_1, a_2\}$ generates a suboperad of Ger isomorphic to $\Lambda\mathsf{Lie}$. The operad Ger carries the obvious augmentation.

We would like to remark that the space $\mathsf{Ger}(n)$ is spanned by monomials $v \in \mathsf{Ger}(n)$ of the form

(3.29)
$$v = v_1(a_{i_{11}}, a_{i_{12}}, \dots, a_{i_{1p_1}})\, v_2(a_{i_{21}}, a_{i_{22}}, \dots, a_{i_{2p_2}})\, \dots\, v_t(a_{i_{t1}}, a_{i_{t2}}, \dots, a_{i_{tp_t}})\,,$$

where $v_1, v_2, \dots, v_t$ are $\Lambda\mathsf{Lie}$-words in p_1, p_2, $\dots$, p_t variables, respectively, without repetitions and

$$\{i_{11}, i_{12}, \dots, i_{1p_1}\} \sqcup \{i_{21}, i_{22}, \dots, i_{2p_2}\} \sqcup \dots \sqcup \{i_{t1}, i_{t2}, \dots, i_{tp_t}\}$$

is a partition of the set of indices $\{1, 2, \dots, n\}$. So, from now on, by a monomial in $\mathsf{Ger}(n)$ we mean a monomial of the form (3.29).

EXERCISE 3.12. Consider the ordered partitions of the set $\{1, 2, \dots, n\}$

$$\{i_{11}, i_{12}, \dots, i_{1p_1}\} \sqcup \{i_{21}, i_{22}, \dots, i_{2p_2}\} \sqcup \dots \sqcup \{i_{t1}, i_{t2}, \dots, i_{tp_t}\} \tag{3.30}$$

satisfying the following properties:

- for each $1 \le \beta \le t$ the index $i_{\beta p_\beta}$ is the biggest among $i_{\beta 1}, \dots, i_{\beta(p_\beta - 1)}$
- $i_{1p_1} < i_{2p_2} < \dots < i_{tp_t}$ (in particular, $i_{tp_t} = n$).

Prove that the monomials

$$\{a_{i_{11}}, \dots, \{a_{i_{1(p_1-1)}}, a_{i_{1p_1}}\}..\} \dots \{a_{i_{t1}}, \dots, \{a_{i_{t(p_t-1)}}, a_{i_{tp_t}}\}..\} \tag{3.31}$$

corresponding to all ordered partitions (3.30) satisfying the above properties form a basis of $\mathsf{Ger}(n)$. Use this fact to show that

$$\dim(\mathsf{Ger}(n)) = n!\,.$$

3.4. Pseudo-cooperads. Reversing the arrows in the definition of a pseudo-operad we get the definition of a pseudo-cooperad. More precisely, a pseudo-cooperad is a collection Q equipped with comultiplication maps

$$\Delta_{\mathbf{t}} : Q(n) \to \underline{Q}_n(\mathbf{t})\,, \tag{3.32}$$

which satisfy a similar list of axioms.

Just as for pseudo-operads, we have

$$\Delta_{\mathbf{q}_n} = \mathsf{id}_{Q(n)}\,, \tag{3.33}$$

where $\mathbf{q}_n$ is the standard corolla (see figures 15, 16).

We also require that the operations (3.32) are S_n-equivariant

$$\Delta_{\sigma(\mathbf{t})} \circ \sigma = \Delta_{\mathbf{t}}\,, \qquad \forall\, \sigma \in S_n,\ \mathbf{t} \in \mathsf{Tree}(n)\,. \tag{3.34}$$

For every morphism $\lambda : \mathbf{t} \to \mathbf{t}'$ in $\mathsf{Tree}(n)$ we have

$$\Delta_{\mathbf{t}'} = \underline{Q}_n(\lambda) \circ \Delta_{\mathbf{t}}\,. \tag{3.35}$$

Finally, to formulate the coassociativity axiom for (3.32), we consider the following quadruple $(\widetilde{\mathbf{t}}, i, m_i, \mathbf{t})$ where $\widetilde{\mathbf{t}}$ is an n-labeled planar tree with k nodal vertices, $1 \le i \le k$, m_i is the number of edges terminating at the i-th nodal vertex of $\widetilde{\mathbf{t}}$, and $\mathbf{t}$ is an m_i-labeled planar tree.

The coassociativity axioms states that for each such quadruple $(\widetilde{\mathbf{t}}, i, m_i, \mathbf{t})$ we have

$$(\mathsf{id} \otimes \dots \otimes \mathsf{id} \otimes \underbrace{\Delta_{\mathbf{t}}}_{i\text{-th spot}} \otimes \mathsf{id} \otimes \dots \otimes \mathsf{id}) \circ \Delta_{\widetilde{\mathbf{t}}} = \beta_{\widetilde{\mathbf{t}}, i, m_i, \mathbf{t}} \circ \Delta_{\widetilde{\mathbf{t}} \bullet_i \mathbf{t}}\,, \tag{3.36}$$

where $\widetilde{\mathbf{t}} \bullet_i \mathbf{t}$ is the n-labeled planar tree obtained by inserting $\mathbf{t}$ into the i-th nodal vertex of $\widetilde{\mathbf{t}}$ and $\beta_{\widetilde{\mathbf{t}}, i, m_i, \mathbf{t}}$ is the isomorphism in $\mathfrak{C}$ which is responsible for "putting tensor factors in the correct order".

Just as for pseudo-operads, a pseudo-cooperad structure on a collection Q is uniquely determined by the comultiplications:

$$\Delta_i := D_{\mathbf{t}^{n,k,i}_{\mathsf{id}}} : Q(n + k - 1) \to Q(n) \otimes Q(k)\,, \tag{3.37}$$

where $\{\mathbf{t}^{n,k,i}_{\sigma}\}_{\sigma \in S_{n+k-1}}$ is the family of labeled planar trees depicted on figure 17.

The comultiplications (3.37) are called *elementary co-insertions.*

3.5. Cooperads. We recall that a cooperad is a pseudo-cooperad Q with counit, that is a map

$$\mathbf{u}^* : Q(1) \to \mathbf{1} \tag{3.38}$$

for which the compositions

$$\begin{array}{c} Q(n) \xrightarrow{\Delta_i} Q(n) \otimes Q(1) \xrightarrow{\mathsf{id}\otimes \mathbf{u}^*} Q(n) \otimes \mathbf{1} \cong Q(n) \\ Q(n) \xrightarrow{\Delta_1} Q(1) \otimes Q(n) \xrightarrow{\mathbf{u}^*\otimes \mathsf{id}} \mathbf{1} \otimes Q(n) \cong Q(n) \end{array} \tag{3.39}$$

coincide with the identity map on $Q(n)$.

Morphisms of pseudo-cooperads and cooperads are defined in the obvious way.

Unfortunately there is no natural notion of "endomorphism cooperad". So a coalgebra over a cooperad Q is defined as an object $\mathcal{V}$ in $\mathfrak{C}$ equipped with a collection of comultiplication maps

$$\Delta_{\mathcal{V}} : \mathcal{V} \to Q(n) \otimes \mathcal{V}^{\otimes n} \qquad n \geq 0\,, \tag{3.40}$$

satisfying axioms which are dual to the associativity axiom, the equivariance axiom and the unitality axiom from [**32**, Proposition 24].

3.5.1. *Coaugmented cooperads.* In this subsection $\mathfrak{C}$ is either $\mathsf{Ch}_{\mathbb{K}}$ or $\mathsf{grVect}_{\mathbb{K}}$.

It is not hard to see that the collection $*$ (3.25) is equipped with the unique cooperad structure. Furthermore, $*$ is the terminal object in the category of cooperads.

We say that a cooperad $\mathcal{C}$ is coaugmented if we have a cooperad morphism

$$\varepsilon' : * \to \mathcal{C}\,. \tag{3.41}$$

Given a pseudo-cooperad $\mathcal{C}$ we can always form a cooperad by formally adjoining a counit. The resulting cooperad is naturally coaugmented.

Furthermore, the cokernel of the coaugmentation for any coaugmented cooperad is naturally a pseudo-cooperad. Dualizing the line of arguments in [**32**, Proposition 21] we see that these two constructions give an equivalence between the category of coaugmented cooperads and the category of pseudo-cooperads.

For a coaugmented cooperad $\mathcal{C}$ we will denote by $\mathcal{C}_\circ$ the cokernel of the coaugmentation.

Just as for operads (see Exercise 3.5), the tensor product of two cooperads is naturally a cooperad. Furthermore, the collection Λ (3.21) introduced in Exercise 3.6 carries a cooperad structure with the following elementary co-insertions:

$$\Delta_i(1_{n+k-1}) = (-1)^{(1-k)(n-i)} 1_n \otimes 1_k\,, \tag{3.42}$$

where 1_m denotes the generator $\mathbf{s}^{1-m}1 \in \mathbf{s}^{1-m}\mathrm{sgn}_m$.

For a cooperad $\mathcal{C}$ in the category $\mathsf{Ch}_{\mathbb{K}}$ or $\mathsf{grVect}_{\mathbb{K}}$ we denote by $\Lambda\mathcal{C}$ the cooperad

$$\Lambda\mathcal{C} := \Lambda \otimes \mathcal{C}\,. \tag{3.43}$$

Just as for operads (see Exercise 3.7), it is easy to see that $\Lambda\mathcal{C}$-coalgebra structures on a cochain complex (or a graded vector space) $\mathcal{V}$ are in bijection with $\mathcal{C}$-coalgebra structures on $\mathbf{s}^{-1}\mathcal{V}$.

EXERCISE 3.13 (Cofree coalgebra over a cooperad $\mathcal{C}$). Let $\mathcal{C}$ be a cooperad in the category $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$). Show that for every cochain complex (resp.

graded vector space) $\mathcal{V}$ the direct sum

$$\mathcal{C}(\mathcal{V}) := \bigoplus_{n=0}^{\infty} \Big(\mathcal{C}(n) \otimes \mathcal{V}^{\otimes\, n} \Big)^{S_n} \tag{3.44}$$

carries a natural structure of a coalgebra over $\mathcal{C}$. Prove that the $\mathcal{C}$-coalgebra $\mathcal{C}(\mathcal{V})$ is cofree[6]. In other words, the assignment

$$\mathcal{V} \to \mathcal{C}(\mathcal{V})$$

upgrades to a functor which is right adjoint to the forgetful functor from the category of $\mathcal{C}$-coalgebras to the category $\mathsf{Ch}_{\mathbb{K}}$ (resp. $\mathsf{grVect}_{\mathbb{K}}$).

3.6. Free operad. In this section $\mathfrak{C} = \mathsf{Ch}_{\mathbb{K}}$ or $\mathsf{grVect}_{\mathbb{K}}$.

Let Q be a collection. Following [**2**, Section 5.8] the spaces $\big\{\Psi\mathbb{OP}(Q)(n)\big\}_{n\ge 0}$ of the free pseudo-operad generated by the collection Q are

$$\Psi\mathbb{OP}(Q)(n) = \mathsf{colim}\, \underline{Q}_n \tag{3.45}$$

where $\underline{Q}_n$ is the functor from the groupoid $\mathsf{Tree}(n)$ to $\mathfrak{C}$ defined in Subsection 3.1.

The pseudo-operad structure on $\Psi\mathbb{OP}(Q)$ is defined in the obvious way using grafting of trees and the free operad $\mathbb{OP}(Q)$ generated by Q is obtained from $\Psi\mathbb{OP}(Q)$ by formally adjoining the unit.

Unfolding (3.45) we see that $\Psi\mathbb{OP}(Q)(n)$ is the quotient of the direct sum

$$\bigoplus_{\mathbf{t}\in\mathsf{Tree}(n)} \underline{Q}_n(\mathbf{t}) \tag{3.46}$$

by the subspace spanned by vectors of the form

$$(\mathbf{t}, X) - (\mathbf{t}', \underline{Q}_n(\lambda)(X))$$

where $\lambda : \mathbf{t} \to \mathbf{t}'$ is a morphism in $\mathsf{Tree}(n)$ and $X \in \underline{Q}_n(\mathbf{t})$.

Thus it is convenient to represent vectors in $\Psi\mathbb{OP}(Q)$ and in $\mathbb{OP}(Q)$ by labeled planar trees with nodal vertices decorated by vectors in Q. The decoration is subject to this rule: if $m(x)$ is the number of edges which terminate at a nodal vertex x then x is decorated by a vector $v_x \in Q(m(x))$.

If a decorated tree $\mathbf{t}'$ is obtained from a decorated tree $\mathbf{t}$ by applying an element $\sigma \in S_{m(x)}$ to incoming edges of a vertex x and replacing the vector v_x by $\sigma^{-1}(v_x)$ then $\mathbf{t}'$ and $\mathbf{t}$ represent the same vectors in $\Psi\mathbb{OP}(Q)$ (and in $\mathbb{OP}(Q)$).

Example 3.14. Let Q be a collection. Figure 18 shows a 4-labeled planar tree $\mathbf{t}$ decorated by vectors $v_1 \in Q(3)$, $v_2 \in Q(2)$ and $v_3 \in Q(1)$. Figure 19 shows another decorated tree with $v_1' = \sigma_{23}(v_1)$ and $v_2' = \sigma_{12}(v_2)$, where σ_{23} and σ_{12} are the corresponding transpositions in S_3 and S_2, respectively. According to our discussion, these decorated trees represent the same vector in $\mathbb{OP}(Q)(4)$.

Remark 3.15. In view of the above description, generators $X \in Q(n)$ of the free operad $\mathbb{OP}(Q)$ can be also written in the form

$$(\mathbf{q}_n, X)\,,$$

where $\mathbf{q}_n$ is the standard n-corolla (see figures 15, 16).

[6] All $\mathcal{C}$-coalgebras are assumed to be nilpotent in the sense of [**20**, Section 2.4.1].

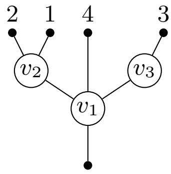


FIG. 18. A 4-labeled decorated tree $\mathbf{t}$

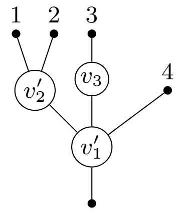


FIG. 19. A 4-labeled decorated tree $\widetilde{\mathbf{t}}$. Here $v'_1 = \sigma_{23}(v_1)$ and $v'_2 = \sigma_{12}(v_2)$

3.7. Cobar construction. The underlying symmetric monoidal category $\mathfrak{C}$ is the category $\mathsf{Ch}_{\mathbb{K}}$ of unbounded cochain complexes of $\mathbb{K}$-vector spaces.

The cobar construction Cobar [**9**], [**16**], [**18**], [**27**, Section 6.5] is a functor from the category of coaugmented cooperads in $\mathsf{Ch}_{\mathbb{K}}$ to the category of augmented operads in $\mathsf{Ch}_{\mathbb{K}}$. It is used to construct free resolutions for operads.

Let $\mathcal{C}$ be a coaugmented cooperad in $\mathsf{Ch}_{\mathbb{K}}$. As an operad in the category $\mathsf{grVect}_{\mathbb{K}}$, $\mathrm{Cobar}(\mathcal{C})$ is freely generated by the collection $\mathbf{s}\,\mathcal{C}_\circ$

$$\mathrm{Cobar}(\mathcal{C}) = \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)\,, \tag{3.47}$$

where $\mathcal{C}_\circ$ denotes the cokernel of the coaugmentation.

To define the differential $\partial^{\mathrm{Cobar}}$ on $\mathrm{Cobar}(\mathcal{C})$, we recall that $\mathsf{Tree}_2(n)$ is the full subcategory of $\mathsf{Tree}(n)$ which consists of n-labeled planar trees with exactly 2 nodal vertices and $\pi_0(\mathsf{Tree}_2(n))$ is the set of isomorphism classes in the groupoid $\mathsf{Tree}_2(n)$. Due to Exercise 2.2, the set $\pi_0(\mathsf{Tree}_2(n))$ is in bijection with $(p, n-p)$-shuffles for all $0 \le p \le n$.

Since the operad $\mathrm{Cobar}(\mathcal{C})$ is freely generated by the collection $\mathbf{s}\,\mathcal{C}_\circ$, it suffices to define the differential $\partial^{\mathrm{Cobar}}$ on generators.

We have

$$\partial^{\mathrm{Cobar}} = \partial' + \partial''\,,$$

with

$$\partial'(X) = -\mathbf{s}\,\partial_{\mathcal{C}}\mathbf{s}^{-1}\,X\,, \tag{3.48}$$

and

$$\partial''(X) = -\sum_{z \in \pi_0(\mathsf{Tree}_2(n))} (\mathbf{s}\otimes\mathbf{s})\big(\mathbf{t}_z; \Delta_{\mathbf{t}_z}(\mathbf{s}^{-1}\,X)\big)\,, \tag{3.49}$$

where $X \in \mathbf{s}\,\mathcal{C}_\circ(n)$, $\mathbf{t}_z$ is any representative of the isomorphism class $z \in \pi_0(\mathsf{Tree}_2(n))$, and $\partial_{\mathcal{C}}$ is the differential on $\mathcal{C}$. The axioms of a pseudo-cooperad imply that the right hand side of (3.49) does not depend on the choice of representatives $\mathbf{t}_z$.

EXERCISE 3.16. Identity

$$\partial' \circ \partial' = 0$$

readily follows from $(\partial_{\mathcal{C}})^2 = 0$. Use the compatibility of the differential $\partial_{\mathcal{C}}$ with the cooperad structure and the coassociativity axiom (3.36) to deduce the identities

$$\partial' \circ \partial'' + \partial'' \circ \partial' = 0 \tag{3.50}$$

and

$$\partial'' \circ \partial'' = 0\,. \tag{3.51}$$

4. Convolution Lie algebra

Let P (resp. Q) be a dg pseudo-operad (resp. a dg pseudo-cooperad).

We consider the following cochain complex

$$\mathrm{Conv}(Q,P) = \prod_{n\ge 0} \mathrm{Hom}_{S_n}(Q(n), P(n))\,. \tag{4.1}$$

with the binary operation $\bullet$ defined by the formula

$$f \bullet g(X) = \sum_{z\in\pi_0(\mathsf{Tree}_2(n))} \mu_{\mathbf{t}_z}(f\otimes g\ \Delta_{\mathbf{t}_z}(X))\,, \tag{4.2}$$

$$f,g\in \mathrm{Conv}(Q,P), \qquad X\in Q(n)\,,$$

where $\mathbf{t}_z$ is any representative of the isomorphism class $z\in\pi_0(\mathsf{Tree}_2(n))$. The axioms of pseudo-operad (resp. pseudo-cooperad) imply that the right hand side of (4.2) does not depend on the choice of representatives $\mathbf{t}_z$.

It follows directly from the definition that the operation $\bullet$ is compatible with the differential on $\mathrm{Conv}(Q,P)$ coming from Q and P. Furthermore, we claim that

Proposition 4.1. *The bracket*

$$[f,g] = \big(f\bullet g - (-1)^{|f|\,|g|} g\bullet f\big)$$

satisfies the Jacobi identity.

Proof. We will prove the proposition by showing that the operation (4.2) satisfies the axiom of the pre-Lie algebra

$$(f\bullet g)\bullet h - f\bullet(g\bullet h) = (-1)^{|g||h|}(f\bullet h)\bullet g - (-1)^{|g||h|} f\bullet(h\bullet g)\,, \tag{4.3}$$

where f,g,h are homogeneous vectors in $\mathrm{Conv}(Q,P)$.

The expression $\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)(X)$ can be rewritten as

$$\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)(X) =$$

$$\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\ \sum_{z'\in\pi_0(\mathsf{Tree}_2(m_1(z)))} \mu_{\mathbf{t}_z}\circ(\mu_{\mathbf{t}_{z'}}\otimes \mathsf{id})\circ(f\otimes g\otimes h)\circ(\Delta_{\mathbf{t}_{z'}}\otimes\mathsf{id})\circ\Delta_{\mathbf{t}_z}(X) -$$
$$\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\ \sum_{z'\in\pi_0(\mathsf{Tree}_2(m_2(z)))} \mu_{\mathbf{t}_z}\circ(\mathsf{id}\otimes\mu_{\mathbf{t}_{z'}})\circ(f\otimes g\otimes h)\circ(\mathsf{id}\otimes\Delta_{\mathbf{t}_{z'}})\circ\Delta_{\mathbf{t}_z}(X)\,,$$

where $m_1(z)$ (resp. $m_2(z)$) is the number of edges terminating at the first (resp. the second) nodal vertex of the planar tree $\mathbf{t}_z$.

Due to the axioms for the maps $\mu_{\mathbf{t}}$ and $\Delta_{\mathbf{t}}$, we get

$$\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)(X)$$
$$\sum_{p,q\ge 0}\ \sum_{\tau\in \mathrm{Sh}_{p,q,n-p-q}} \mu_{\mathbf{t}_\tau}\big((f\otimes g\otimes h)\,\Delta_{\mathbf{t}_\tau}(X)\big)$$

where $\mathbf{t}_\tau$ is the n-labeled planar tree depicted on figure 20.

The set $\{\mathbf{t}_\tau \mid \tau\in\mathrm{Sh}_{p,q,n-p-q}\}$ is stable under the obvious isomorphism λ which switches the second nodal vertex with the third one. Hence, we have

$$\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)(X) =$$

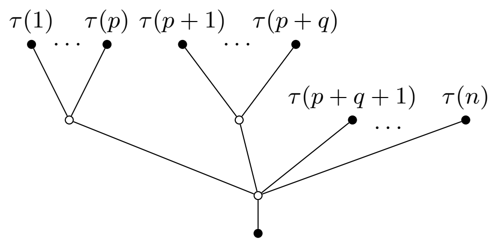


FIG. 20. τ is a $(p,q,n-p-q)$-shuffle

$$\sum_{p,q\geq 1}\sum_{\tau\in \mathrm{Sh}_{p,q,n-p-q}} \mu_{\lambda(\mathbf{t}_\tau)}\big((f\otimes g\otimes h)\,\Delta_{\lambda(\mathbf{t}_\tau)}(X)\big)\,.$$

Using axioms (3.5) and (3.35) and the fact that f is equivariant with respect to the action of the symmetric group, we can rewrite the latter expression as follows

$$\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)\,(X) =$$

$$\sum_{p,q\geq 1}\sum_{\tau\in \mathrm{Sh}_{p,q,n-p-q}} \mu_{\mathbf{t}_\tau}\circ \underline{P}_n(\lambda)\circ(f\otimes g\otimes h)\circ \underline{Q}_n(\lambda)\,\Delta_{\mathbf{t}_\tau}(X) =$$

$$\sum_{p,q\geq 1}\sum_{\tau,\alpha}(-1)^{|X_2^{\tau,\alpha}|\,|X_3^{\tau,\alpha}|}\,\mu_{\mathbf{t}_\tau}\circ \underline{P}_n(\lambda)\circ(f\otimes g\otimes h)\,(\sigma_{12}(X_1^{\tau,\alpha}),X_3^{\tau,\alpha},X_2^{\tau,\alpha}) =$$

$$\sum_{p,q\geq 1}\sum_{\tau,\alpha}(-1)^{\varepsilon(\tau,\alpha,g,h)}\,\mu_{\mathbf{t}_\tau}\circ \underline{P}_n(\lambda)\,(\sigma_{12}f(X_1^{\tau,\alpha}),g(X_3^{\tau,\alpha}),h(X_2^{\tau,\alpha})) \tag{4.4}$$

where σ_{12} is the transposition $(1,2)$,

$$\Delta_{\mathbf{t}_\tau}(X) = \sum_\alpha (X_1^{\tau,\alpha},X_2^{\tau,\alpha},X_3^{\tau,\alpha}), \tag{4.5}$$

and

$$\varepsilon(\tau,\alpha,g,h) = |X_2^{\tau,\alpha}|\,|X_3^{\tau,\alpha}| + |h|(|X_1^{\tau,\alpha}|+|X_3^{\tau,\alpha}|) + |g||X_1^{\tau,\alpha}|\,. \tag{4.6}$$

Applying $\underline{P}_n(\lambda)$ to $(\sigma_{12}f(X_1^{\tau,\alpha}),g(X_3^{\tau,\alpha}),h(X_2^{\tau,\alpha}))$ in (4.4) we get

$$\big((f\bullet g)\bullet h - f\bullet(g\bullet h)\big)\,(X) =$$

$$\sum_{p,q\geq 1}\sum_{\tau,\alpha}(-1)^{\varepsilon(\tau,\alpha,g,h)}(-1)^{(|g|+|X_3^{\tau,\alpha}|)(|h|+|X_2^{\tau,\alpha}|)}\,\mu_{\mathbf{t}_\tau}\,(f(X_1^{\tau,\alpha}),h(X_2^{\tau,\alpha}),g(X_3^{\tau,\alpha})) =$$

$$\sum_{p,q\geq 1}\sum_{\tau,\alpha}(-1)^{\widetilde{\varepsilon}(\tau,\alpha,g,h)}\,\mu_{\mathbf{t}_\tau}\,\big((f\otimes h\otimes g)\,\Delta_{\mathbf{t}_\tau}(X)\big)\,,$$

where

$$\widetilde{\varepsilon}(\tau,\alpha,g,h) = \varepsilon(\tau,\alpha,g,h)+(|g|+|X_3^{\tau,\alpha}|)(|h|+|X_2^{\tau,\alpha}|)+|h||X_1^{\tau,\alpha}|+|g|(|X_1^{\tau,\alpha}|+|X_2^{\tau,\alpha}|)\,. \tag{4.7}$$

The direct computation shows that

$$\widetilde{\varepsilon}(\tau,\alpha,g,h) = |g||h| \mod 2$$

and the proposition follows. $\square$

4.1. A useful modification $\mathrm{Conv}^{\oplus}(Q,P)$**.** Let us observe that for every dg pseudo-operad P and every dg pseudo-cooperad Q the subcomplex

$$\mathrm{Conv}^{\oplus}(Q,P) := \bigoplus_{n\geq 0} \mathrm{Hom}_{S_n}(Q(n), P(n)) \subset \mathrm{Conv}(Q,P) \tag{4.8}$$

is closed with respect to the pre-Lie operation (4.2). Thus $\mathrm{Conv}^{\oplus}(Q,P)$ is a dg Lie subalgebra of $\mathrm{Conv}(Q,P)$.

We often use this subalgebra in our notes to prove facts about its completion $\mathrm{Conv}(Q,P)$.

4.2. Example: the dg Lie algebra $\mathrm{Conv}(\mathcal{C}_\circ, \mathsf{End}_V)$**.** Let V be a cochain complex, $\mathcal{C}$ be a coaugmented dg cooperad, and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation. We denote by $\mathcal{C}(V)$ the cofree $\mathcal{C}$-coalgebra cogenerated by V. Furthermore, we denote by p_V the natural projection

$$p_V : \mathcal{C}(V) \to V\,. \tag{4.9}$$

In this subsection we interpret $\mathrm{Conv}(\mathcal{C}_\circ, \mathsf{End}_V)$ as a subalgebra in the dg Lie algebra $\mathrm{coDer}(\mathcal{C}(V))$ of coderivations of $\mathcal{C}(V)$.

Let us recall [**16**, Proposition 2.14] that the map

$$\mathcal{D} \mapsto p_V \circ \mathcal{D} \tag{4.10}$$

defines an isomorphism of cochain complexes

$$\mathrm{coDer}(\mathcal{C}(V)) \cong \mathrm{Hom}(\mathcal{C}(V), V)\,.$$

Then we observe that coderivations $\mathcal{D} \in \mathrm{coDer}(\mathcal{C}(V))$ satisfying the property

$$\mathcal{D}\Big|_V = 0 \tag{4.11}$$

form a dg Lie subalgebra of $\mathrm{coDer}(\mathcal{C}(V))$. We denote this dg Lie subalgebra by $\mathrm{coDer}'(\mathcal{C}(V))$.

Next, we remark that the formula

$$p \circ \mathcal{D}_f(\gamma; v_1, v_2, \dots, v_n) = f(\gamma)(v_1, v_2, \dots, v_n) \tag{4.12}$$

$$f \in \mathrm{Conv}(\mathcal{C}_\circ, \mathsf{End}_V), \qquad \gamma \in \mathcal{C}_\circ(n)\,, \qquad v_1, v_2, \dots, v_n \in V$$

defines a map (of graded vector spaces)

$$f \mapsto \mathcal{D}_f : \mathrm{Conv}(\mathcal{C}_\circ, \mathsf{End}_V) \to \mathrm{coDer}'(\mathcal{C}(V))\,. \tag{4.13}$$

Finally, we claim that[7]

PROPOSITION 4.2. *For every cochain complex V and for every coaugmented dg cooperad $\mathcal{C}$ the map* (4.13) *is an isomorphism of dg Lie algebras*

$$\mathrm{Conv}(\mathcal{C}_\circ, \mathsf{End}_V) \cong \mathrm{coDer}'(\mathcal{C}(V))\,.$$

A proof of this proposition is straightforward so we leave it as an exercise.

EXERCISE 4.3. Prove Proposition 4.2.

[7]Proposition 4.2 is a version of [**16**, Proposition 2.15].

4.3. What if $Q(n)$ is finite dimensional for all n? Let us assume that the pseudo-cooperad Q satisfies the property

PROPERTY 4.4. For each n the graded vector space $Q(n)$ is finite dimensional.

Due to this property we have

$$\mathrm{Conv}(Q,P) \cong \prod_{n\ge 0} \big(P(n)\otimes Q^*(n)\big)^{S_n}\,. \tag{4.14}$$

where $Q^*(n)$ denotes the linear dual of the vector space $Q(n)$.

The collection $Q^* := \{Q^*(n)\}_{n\ge 0}$ is naturally a pseudo-operad and we can express the pre-Lie structure (4.2) in terms of elementary insertions on P and Q^*. Namely, given two vectors

$$X = \sum_{n\ge 0} v_n \otimes w_n\,, \qquad X' = \sum_{n\ge 0} v'_n \otimes w'_n$$

in

$$\prod_{n\ge 0} \big(P(n)\otimes Q^*(n)\big)^{S_n}\,,$$

we have

$$X \bullet X' = \sum_{n\ge 1, m\ge 0} (-1)^{|v'_m||w_n|} \sum_{\sigma\in \mathrm{Sh}_{m,n-1}} \sigma(v_n \circ_1 v'_m)\otimes \sigma(w_n\circ_1 w'_m)\,. \tag{4.15}$$

4.4. The functors $\mathrm{Conv}(Q,?)$ and $\mathrm{Conv}(?,P)$ preserve quasi-isomorphisms. It often happens that a pseudo-cooperad Q is equipped with a cocomplete ascending filtration

$$\mathbf{0} = \mathcal{F}^0 Q \subset \mathcal{F}^1 Q \subset \mathcal{F}^2 Q \subset \dots \tag{4.16}$$

$$\mathrm{colim}_m\, \mathcal{F}^m Q(n) = Q(n) \qquad \forall\ n$$

which is compatible with comultiplications $\Delta_{\mathbf{t}}$ in the following sense:

$$\Delta_{\mathbf{t}}\big(\mathcal{F}^m Q(n)\big) \subset \bigoplus_{q_1+q_2+\dots+q_k=m} \mathcal{F}^{q_1}Q(r_1)\otimes \mathcal{F}^{q_2}Q(r_2)\otimes\dots\otimes \mathcal{F}^{q_k}Q(r_k)\,, \tag{4.17}$$

where $\mathbf{t}$ is an n-labeled planar tree with k nodal vertices and r_i is the number of edges terminating at the i-th nodal vertex of $\mathbf{t}$.

DEFINITION 4.5. If a pseudo-cooperad Q is equipped with such a filtration then we say that Q is cofiltered.

EXERCISE 4.6. Let us recall that a coaugmented (dg) cooperad $\mathcal{C}$ is called reduced if

$$\mathcal{C}(0) = \mathbf{0}\,, \qquad \text{and} \qquad \mathcal{C}(1) = \mathbb{K}\,.$$

For every reduced coaugmented cooperad $\mathcal{C}$, the cokernel of the coaugmentation $\mathcal{C}_\circ$ carries the ascending filtration "by arity":

$$\mathcal{F}^m\mathcal{C}_\circ(n) = \begin{cases} \mathcal{C}_\circ(n) & \text{if} \quad n\le m+1 \\ \mathbf{0} & \text{otherwise}\,. \end{cases} \tag{4.18}$$

Show that this filtration is cocomplete and compatible with comultiplications $\Delta_{\mathbf{t}}$ in the sense of (4.17).

For any dg operad P and for any cofiltered dg pseudo-cooperad Q the dg Lie algebra $\mathrm{Conv}(Q,P)$ is equipped with the descending filtration

$$\mathrm{Conv}(Q,P) = \mathcal{F}_1\mathrm{Conv}(Q,P) \supset \mathcal{F}_2\mathrm{Conv}(Q,P) \supset \mathcal{F}_3\mathrm{Conv}(Q,P) \supset \dots, \tag{4.19}$$

$$\mathcal{F}_m\mathrm{Conv}(Q,P) = \{ f \in \mathrm{Conv}(Q,P) \mid f(X) = 0 \quad \forall\, X \in \mathcal{F}^{m-1}(Q) \}.$$

Inclusion (4.17) implies that the filtration on $\mathrm{Conv}(Q,P)$ is compatible with the Lie bracket. Furthermore, since the filtration on Q is cocomplete, the filtration (4.19) is complete

$$\lim_{m} \mathrm{Conv}(Q,P) \big/ \mathcal{F}_m\mathrm{Conv}(Q,P) = \mathrm{Conv}(Q,P). \tag{4.20}$$

Any morphism of dg pseudo-operads

$$f : P \to P'. \tag{4.21}$$

induces the obvious map of dg Lie algebras

$$f_* : \mathrm{Conv}(Q,P) \to \mathrm{Conv}(Q,P'). \tag{4.22}$$

We claim that

THEOREM 4.7. *If the map (4.21) is a quasi-isomorphism then so is the map (4.22). In addition, if Q is cofiltered, then the restriction of f_* onto $\mathcal{F}_m\mathrm{Conv}(Q,P)$*

$$f_* \Big|_{\mathcal{F}_m\mathrm{Conv}(Q,P)} \; : \; \mathcal{F}_m\mathrm{Conv}(Q,P) \to \mathcal{F}_m\mathrm{Conv}(Q,P')$$

is a quasi-isomorphism for all m.

PROOF. According to [**40**, Section 1.4], every cochain complex of $\mathbb{K}$-vector spaces is chain homotopy equivalent to its cohomology.

Therefore, there exist collections of maps

$$g_n : P'(n) \to P(n) \tag{4.23}$$

$$\chi_n : P(n) \to P(n) \tag{4.24}$$

$$\chi'_n : P'(n) \to P'(n) \tag{4.25}$$

such that

$$f_n \circ g_n - \mathsf{id}_{P'(n)} = \partial \chi'_n + \chi'_n \partial, \tag{4.26}$$

and

$$g_n \circ f_n - \mathsf{id}_{P(n)} = \partial \chi_n + \chi_n \partial. \tag{4.27}$$

In other words, $f_n \circ g_n$ (resp. $g_n \circ f_n$) is homotopic to $\mathsf{id}_{P'(n)}$ (resp. $\mathsf{id}_{P(n)}$).

In general, the set of maps $\{g_n\}_{n \ge 0}$ gives us neither a map of operads nor a map of the underlying collections. Similarly, the maps (4.24) and (4.25) may not be S_n-equivariant.

For this reason we switch from the set $\{g_n\}_{n\ge 0}$ to the set

$$\widetilde{g}_n = \frac{1}{(n!)^2} \sum_{\sigma,\tau \in S_n} \sigma \circ g_n \circ \tau. \tag{4.28}$$

It is easy to see these new maps $\widetilde{g}_n$ give us a morphism of the underlying collections. Moreover, equations (4.26) and (4.27) imply the identities

$$f_n \circ \widetilde{g}_n - \mathsf{id}_{P'(n)} = \partial \widetilde{\chi'}_n + \widetilde{\chi'}_n \partial, \tag{4.29}$$

and

$$\widetilde{g}_n \circ f_n - \mathsf{id}_{P(n)} = \partial \widetilde{\chi}_n + \widetilde{\chi}_n \partial \tag{4.30}$$

with S_n-equivariant homotopy operators

$$\widetilde{\chi}_n = \frac{1}{(n!)^2} \sum_{\sigma, \tau \in S_n} \sigma \circ \chi_n \circ \tau\,, \tag{4.31}$$

$$\widetilde{\chi'}_n = \frac{1}{(n!)^2} \sum_{\sigma, \tau \in S_n} \sigma \circ \chi'_n \circ \tau\,. \tag{4.32}$$

Let us now consider the map

$$\widetilde{g}_* : \mathrm{Conv}(Q, P') \to \mathrm{Conv}(Q, P) \tag{4.33}$$

In general $\widetilde{g}_*$ is not compatible with the Lie brackets. Regardless, using equations (4.29), (4.30) and S_n-equivariance of the homotopy operators (4.31), (4.32), it is not hard to see that the compositions $f_* \circ \widetilde{g}_*$ and $\widetilde{g}_* \circ f_*$ are homotopic to $\mathsf{id}_{\mathrm{Conv}(Q,P')}$ and $\mathsf{id}_{\mathrm{Conv}(Q,P)}$, respectively.

Thus f_* is indeed a quasi-isomorphism.

To prove the second statement we denote by f_*^m and $\widetilde{g}_*^m$ the restriction of f_* and $\widetilde{g}_*$ onto

$$\mathcal{F}_m \mathrm{Conv}(Q, P) \qquad \text{and} \qquad \mathcal{F}_m \mathrm{Conv}(Q, P')$$

respectively.

Using the same homotopy operators (4.31), (4.32), it is not hard to see that the compositions $f_*^m \circ \widetilde{g}_*^m$ and $\widetilde{g}_*^m \circ f_*^m$ are homotopic to $\mathsf{id}_{\mathcal{F}_m \mathrm{Conv}(Q,P')}$ and $\mathsf{id}_{\mathcal{F}_m \mathrm{Conv}(Q,P)}$, respectively.

Theorem 4.7 is proved. □

EXERCISE 4.8. Using the ideas of the above proof, show that the (contravariant) functor $\mathrm{Conv}(?, P)$ also preserves quasi-isomorphisms.

5. To invert, or not to invert: that is the question

Let $\mathcal{C}$ be a coaugmented dg cooperad and $\mathcal{C}_\circ$ be the cokernel of the coaugmentation. This section is devoted to the lifting property for maps from the dg operad $\mathrm{Cobar}(\mathcal{C})$. The material contained in this section is an adaptation of constructions from [**34**] to the setting of dg operads.

First, we observe that, since $\mathrm{Cobar}(\mathcal{C})$ is freely generated by $\mathbf{s}\,\mathcal{C}_\circ$, any map of dg operads

$$F : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O} \tag{5.1}$$

is uniquely determined by its restriction to generators:

$$F \Big|_{\mathbf{s}\,\mathcal{C}_\circ} \ : \ \mathbf{s}\,\mathcal{C}_\circ \to \mathcal{O}\,.$$

Hence, composing the latter map with the suspension operator $\mathbf{s}$, we get a degree one element

$$\alpha_F \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}) \tag{5.2}$$

in the dg Lie algebra $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$.

EXERCISE 5.1. Prove that the compatibility of F with the differentials on $\mathrm{Cobar}(\mathcal{C})$ and $\mathcal{O}$ is equivalent to the Maurer-Cartan equation for the element α_F (5.2) in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$.

Thus we arrive at the following proposition

PROPOSITION 5.2. *For an arbitrary coaugmented dg cooperad $\mathcal{C}$ and for an arbitrary dg operad $\mathcal{O}$, the correspondence*

$$F \mapsto \alpha_F$$

is a bijection between the set of maps (of dg operads) (5.1) *and the set of Maurer-Cartan elements in* $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$. $\square$

Combining Proposition 4.2 with 5.2 we deduce the following Corollary

COROLLARY 5.3. *For every coaugmented dg cooperad $\mathcal{C}$ and for every cochain complex V the set of* $\mathrm{Cobar}(\mathcal{C})$*-algebra structures on V is in bijection with the set of Maurer-Cartan elements in the dg Lie algebra*

$$\mathrm{coDer}'(\mathcal{C}(V)) := \Big\{ \mathcal{D} \in \mathrm{coDer}(\mathcal{C}(V)) \ \Big| \ \mathcal{D}\Big|_V = 0 \Big\}. \ \square$$

5.1. Homotopies of maps from $\mathrm{Cobar}(\mathcal{C})$**.** Let $\Omega^\bullet(\mathbb{K}) = \mathbb{K}[t] \oplus \mathbb{K}[t]dt$ be the polynomial de Rham algebra on the affine line with dt sitting in degree 1. We denote by

$$\mathcal{O}^I := \mathcal{O} \otimes \Omega^\bullet(\mathbb{K})$$

the dg operad with underlying collection

$$\{\mathcal{O}(n) \otimes \Omega^\bullet(\mathbb{K})\}_{n \ge 0}.$$

We also denote by p_0, p_1 the obvious maps of dg operads

$$\begin{aligned} & p_0 : \mathcal{O}^I \to \mathcal{O}, \quad p_0(X) = X\Big|_{t=0,\ dt=0}, \\ & p_1 : \mathcal{O}^I \to \mathcal{O}, \quad p_1(X) = X\Big|_{t=1,\ dt=0}. \end{aligned} \tag{5.3}$$

For our purposes we will use the following "pedestrian" definition of homotopy between maps $F, \widetilde{F} : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$.

DEFINITION 5.4. We say that maps of dg operads

$$F, \widetilde{F} : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$$

are <u>homotopic</u> if there exists a map of dg operads

$$H : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}^I$$

such that

$$F = p_0 \circ H \qquad \text{and} \qquad \widetilde{F} = p_1 \circ H.$$

REMARK 5.5. Definition 5.4 leaves out many questions and some of these questions may be answered by constructing a closed model structure on a subcategory of dg operads satisfying certain technical conditions. Unfortunately, many dg operads which show up in applications do not satisfy required technical conditions. We hope that all such issues will be resolved in yet another "infinity" treatise [**30**] of J. Lurie.

We now state a theorem which characterizes homotopic maps from $\mathrm{Cobar}(\mathcal{C})$ in terms of the corresponding Maurer-Cartan elements in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$.

THEOREM 5.6. *Let $\mathcal{O}$ be an arbitrary dg operad and $\mathcal{C}$ be a coaugmented dg cooperad for which the pseudo-operad $\mathcal{C}_\circ$ is cofiltered (see Definition 4.5) and the vector space*

$$\bigoplus_{n \ge 0} \mathcal{F}^m \mathcal{C}_\circ(n) \tag{5.4}$$

is finite dimensional for all m. Then two maps of dg operads

$$F, \widetilde{F} : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$$

are homotopic if and only if the corresponding Maurer-Cartan elements

$$\alpha_F, \alpha_{\widetilde{F}} \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$$

are isomorphic[8].

PROOF. Let

$$\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\} \tag{5.5}$$

be the dg Lie subalgebra of $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})[[t]]$ which consists of infinite series

$$f = \sum_{k=0}^{\infty} f_k t^k \,, \qquad f_k \in \mathcal{F}_{m_k} \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}) \tag{5.6}$$

satisfying the condition

$$m_1 \le m_2 \le m_3 \le \dots \qquad \lim_{k \to \infty} m_k = \infty \,. \tag{5.7}$$

Combining condition (5.7) together with the fact that the filtration on $\mathcal{C}_\circ$ is cocomplete, we conclude that, for every $X \in \mathcal{C}_\circ(n)$ and for every f in (5.5), the sum

$$\sum_{k=0}^{\infty} f_k(X) t^k$$

has only finitely many non-zero terms.

Therefore, the formula

$$\Psi(f)(X) = \sum_{k=0}^{\infty} f_k(X) t^k \,, \qquad X \in \mathcal{C}_\circ(n) \tag{5.8}$$

defines a map

$$\Psi : \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\} \to \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}[t]) \,.$$

Let us now consider a vector $g \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}[t])$.

Since the vector spaces

$$\bigoplus_{n \ge 0} \mathcal{F}^{m-1} \mathcal{C}_\circ(n)$$

are finite dimensional, for each m there exists a positive integer N_m such that the polynomials

$$g(X) = \sum_{k \ge 0} g_k(X) t^k$$

[8] We view Maurer-Cartan elements as objects of the Deligne groupoid. See Appendix C for details.

have degrees $\le N_m$ for all $X \in \mathcal{F}^{m-1}\mathcal{C}_\circ(n)$ and for all n. Moreover, the integers $\{N_m\}_{m\ge 1}$ can be chosen in such a way that

$$N_1 \le N_2 \le N_3 \le \dots .$$

Therefore, the formula:

$$\Psi'(g) = \sum_{k=0}^{\infty} \Psi'_k(g) t^k \tag{5.9}$$

$$\Psi'_k(g)(X) := g_k(X)\,, \qquad X \in \mathcal{C}_\circ$$

defines a map

$$\Psi' : \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}[t]) \to \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\}\,.$$

Furthermore, it is easy to see that Ψ' is the inverse of Ψ.

Thus the dg Lie algebras $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}[t])$ and $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\}$ are naturally isomorphic.

To prove the "only if" part we start with a map of dg operads

$$H : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}^I$$

which establishes a homotopy between F and $\widetilde{F}$ and let

$$\alpha_H = \alpha_H^{(1)} + \alpha_H^{(0)}\, dt \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}^I) \tag{5.10}$$

be the Maurer-Cartan element corresponding to H. Here $\alpha_H^{(1)}$ (resp. $\alpha_H^{(0)}$) is a degree 1 (resp. degree 0) vector in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}[t]) \cong \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\}$.

The Maurer-Cartan equation for α_H

$$dt \frac{d}{dt}\alpha_H + \partial \alpha_H + \frac{1}{2}[\alpha_H, \alpha_H] = 0$$

is equivalent to the pair of equations

$$\partial \alpha_H^{(1)} + \frac{1}{2}[\alpha_H^{(1)}, \alpha_H^{(1)}] = 0 \tag{5.11}$$

and

$$\frac{d}{dt}\alpha_H^{(1)} = \partial \alpha_H^{(0)} - [\alpha_H^{(0)}, \alpha_H^{(1)}]\,. \tag{5.12}$$

Using equations (5.11) and (5.12) we deduce from [**4**, Theorem C.1, App. C] that the Maurer-Cartan elements

$$\alpha_H^{(1)}\Big|_{t=0} \qquad \text{and} \qquad \alpha_H^{(1)}\Big|_{t=1}$$

in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$ are connected by the action of the group

$$\exp\Big(\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\Big)\,.$$

Since

$$\alpha_H^{(1)}\Big|_{t=0} = \alpha_F \qquad \text{and} \qquad \alpha_H^{(1)}\Big|_{t=1} = \alpha_{\widetilde{F}}$$

we conclude that the "only if" part is proved.

We leave the easier "if" part as an exercise. (See Exercise 5.7 below.) □

EXERCISE 5.7. Prove the "if" part of Theorem 5.6.

We now deduce the following corollary.

COROLLARY 5.8. *Let $\mathcal{C}$ be a coaugmented dg cooperad which satisfies the conditions of Theorem* 5.6. *If $U : \mathcal{O} \to \mathcal{O}'$ is a quasi-isomorphism of dg operads then for every operad morphism $F' : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}'$ there exists a morphism $F : \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$ such that the diagram*

$$\begin{tikzcd} & \mathrm{Cobar}(\mathcal{C}) \arrow[dl, dotted, "F"'] \arrow[d, "F'"] \\ \mathcal{O} \arrow[r, "U"] & \mathcal{O}' \end{tikzcd} \tag{5.13}$$

commutes up to homotopy. Moreover the morphism F is determined uniquely up to homotopy.

PROOF. The map U induces the homomorphism of dg Lie algebras

$$U_* : \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}) \to \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}')\,.$$

Due to Theorem 4.7 U_* is a quasi-isomorphism of dg Lie algebras. Moreover, the restriction of U_*

$$U_*\Big|_{\mathcal{F}_m \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})} : \mathcal{F}_m \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}) \to \mathcal{F}_m \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}')$$

is also a quasi-isomorphism of dg Lie algebras for all m.

Hence, Theorem C.2 from Appendix C implies that U_* induces a bijection between the isomorphism classes of Maurer-Cartan elements in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$ and in $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}')$.

Thus, the statements of the corollary follow immediately from Theorem 5.6. □

5.2. Models for homotopy algebras. Developing the machinery of algebraic operads is partially motivated by the desire to blend together concepts of abstract algebra and concepts of homotopy theory [**28**], [**30**], [**31**].

Thus, in homotopy theory, the notions of Lie algebra, commutative algebra, and Gerstenhaber algebra are replaced by their ∞-versions (a.k.a homotopy versions): L_∞-algebras, Com_∞-algebras and Ger_∞-algebras, respectively. These are examples of *homotopy algebras*.

In this paper we will go into a general philosophy for homotopy algebras and instead limit ourselves to conventional definitions.

DEFINITION 5.9. An <u>Lie_∞-algebra</u> (a.k.a. <u>L_∞-algebra</u>) is an algebra (in $\mathsf{Ch}_{\mathbb{K}}$) over the operad

$$\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda \mathsf{coCom})\,. \tag{5.14}$$

DEFINITION 5.10. A <u>Com_∞-algebra</u> is an algebra (in $\mathsf{Ch}_{\mathbb{K}}$) over the operad

$$\mathsf{Com}_\infty = \mathrm{Cobar}(\Lambda \mathsf{coLie})\,. \tag{5.15}$$

Finally,

DEFINITION 5.11. A <u>Ger_∞-algebra</u> is an algebra (in $\mathsf{Ch}_{\mathbb{K}}$) over the operad

$$\mathsf{Ger}_\infty = \mathrm{Cobar}(\mathsf{Ger}^\vee)\,, \tag{5.16}$$

where

$$\mathsf{Ger}^\vee = \big(\Lambda^{-2}\mathsf{Ger}\big)^*\,, \tag{5.17}$$

and * is the operation of taking linear dual.

The above definitions are partially motivated by the observation that the operads Lie_∞, Com_∞ and Ger_∞ are free resolutions of the operads Lie, Com and Ger, respectively.

Thus the canonical quasi-isomorphism of dg operads

$$U_{\mathsf{Lie}} : \mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda\mathsf{coCom}) \ \to \ \mathsf{Lie} \tag{5.18}$$

corresponds to the Maurer-Cartan element[9]

$$\alpha_{\mathsf{Lie}} = [a_1, a_2] \otimes b_1 b_2 \in \mathrm{Conv}(\Lambda\mathsf{coCom}_\circ, \mathsf{Lie}) \cong \prod_{n \ge 2} \Big(\mathsf{Lie}(n) \otimes \Lambda^{-1}\mathsf{Com}(n)\Big)^{S_n},$$

where $[a_1, a_2]$ (resp. $b_1 b_2$) denotes the canonical generator of $\mathsf{Lie}(2)$ (resp. $\Lambda^{-1}\mathsf{Com}(2)$).

Similarly, the canonical quasi-isomorphism of dg operads

$$U_{\mathsf{Com}} : \mathsf{Com}_\infty = \mathrm{Cobar}(\Lambda\mathsf{coLie}) \ \to \ \mathsf{Com} \tag{5.19}$$

corresponds to the Maurer-Cartan element

$$\alpha_{\mathsf{Com}} = a_1 a_2 \otimes \{b_1, b_2\} \in \mathrm{Conv}(\Lambda\mathsf{coLie}_\circ, \mathsf{Com}) \cong \prod_{n \ge 2} \Big(\mathsf{Com}(n) \otimes \Lambda^{-1}\mathsf{Lie}(n)\Big)^{S_n},$$

where $a_1 a_2$ (resp. $\{b_1, b_2\}$) denotes the canonical generator of $\mathsf{Com}(2)$ (resp. $\Lambda^{-1}\mathsf{Lie}(2)$).

Finally the canonical quasi-isomorphism of dg operads

$$U_{\mathsf{Ger}} : \mathsf{Ger}_\infty = \mathrm{Cobar}(\mathsf{Ger}^\vee) \ \to \ \mathsf{Ger} \tag{5.20}$$

corresponds to the Maurer-Cartan element

$$\alpha_{\mathsf{Ger}} = a_1 a_2 \otimes \{b_1, b_2\} + \{a_1, a_2\} \otimes b_1 b_2 \in \mathrm{Conv}(\mathsf{Ger}^\vee_\circ, \mathsf{Ger}) = \prod_{n \ge 2} \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n},$$

where $a_1 a_2$, $\{a_1, a_2\}$ are the canonical generators of $\mathsf{Ger}(2)$ and $b_1 b_2$, $\{b_1, b_2\}$ are the canonical generators of $\Lambda^{-2}\mathsf{Ger}(2)$.

We should remark that here, instead of Lie algebras and L_∞-algebras we often deal with $\Lambda\mathsf{Lie}_\infty$-algebras. It is not hard to see that $\Lambda\mathsf{Lie}_\infty$-algebras are algebras in $\mathsf{Ch}_{\mathbb{K}}$ over the operad

$$\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom}). \tag{5.21}$$

Furthermore, the canonical quasi-isomorphism

$$U_{\Lambda\mathsf{Lie}} : \Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \ \to \ \Lambda\mathsf{Lie} \tag{5.22}$$

corresponds to the Maurer-Cartan element

$$\alpha_{\Lambda\mathsf{Lie}} = \{a_1, a_2\} \otimes b_1 b_2 \in \mathrm{Conv}(\Lambda^2\mathsf{coCom}_\circ, \Lambda\mathsf{Lie}) \cong \prod_{n \ge 2} \Big(\Lambda\mathsf{Lie}(n) \otimes \Lambda^{-2}\mathsf{Com}(n)\Big)^{S_n},$$

where $\{a_1, a_2\}$ (resp. $b_1 b_2$) denotes the canonical generator of $\Lambda\mathsf{Lie}(2)$ (resp. $\Lambda^{-2}\mathsf{Com}(2)$).

[9]Recall that, due to Proposition 5.2, operads maps from $\mathrm{Cobar}(\mathcal{C})$ to $\mathcal{O}$ are identified with Maurer-Cartan element of $\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$.

5.2.1. $\Lambda\mathsf{Lie}_\infty$-*algebras.* Let V be a cochain complex.

Since $\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom})$, Corollary 5.3 implies that $\Lambda\mathsf{Lie}_\infty$-algebra structure on V is a choice of degree 1 coderivation

$$\mathcal{D} \in \mathrm{coDer}(\Lambda^2\mathsf{coCom}(V))$$

satisfying the Maurer-Cartan equation

$$\partial\mathcal{D} + \frac{1}{2}[\mathcal{D},\mathcal{D}] = 0 \tag{5.23}$$

together with the condition

$$\mathcal{D}\Big|_V = 0\,.$$

On the other hand, according to [**16**, Proposition 2.14], any coderivation of $\Lambda^2\mathsf{coCom}(V)$ is uniquely determined by its composition $p_V \circ \mathcal{D}$ with the projection

$$p_V : \Lambda^2\mathsf{coCom}(V) \to V\,.$$

Thus, since

$$\Lambda^2\mathsf{coCom}(V) = \mathbf{s}^2 S(\mathbf{s}^{-2}V),$$

an $\Lambda\mathsf{Lie}_\infty$-structure on V is determined by the infinite sequence of multi-ary operations

$$\{\,,\,,\dots,\,\}_n = p_V \circ \mathcal{D}\,\mathbf{s}^{2n-2} : S^n(V) \to V\,, \qquad n \ge 2 \tag{5.24}$$

where the n-th operation $\{\,,\,,\dots,\,\}_n$ carries degree $3-2n$.

The Maurer-Cartan equation (5.23) is equivalent to the sequence of the following quadratic relations on operations (5.24):

$$\partial\{v_1, v_2, \dots, v_n\}_n + \sum_{i=1}^{n} (-1)^{|v_1|+\dots+|v_{i-1}|}\{v_1,\dots,v_{i-1},\partial v_i, v_{i+1},\dots,v_n\}_n + \tag{5.25}$$

$$\sum_{p=2}^{n-1}\sum_{\sigma\in \mathrm{Sh}_{p,n-p}} (-1)^{\varepsilon(\sigma,v_1,\dots,v_n)}\{\{v_{\sigma(1)},\dots,v_{\sigma(p)}\}_p, v_{\sigma(p+1)},\dots,v_{\sigma(n)}\}_{n-p+1} = 0\,,$$

where ∂ is the differential on V and $(-1)^{\varepsilon(\sigma,v_1,\dots,v_n)}$ is the sign factor determined by the usual Koszul rule.

REMARK 5.12. Even though there is an obvious bijection between $\Lambda\mathsf{Lie}_\infty$-structures on V and L_∞-structures on $\mathbf{s}^{-1}V$, it is often easier to deal with signs in formulas for $\Lambda\mathsf{Lie}_\infty$-structures.

6. Twisting of operads

Let $\mathcal{O}$ be a dg operad equipped with a map

$$\widehat{\varphi} : \Lambda\mathsf{Lie}_\infty \to \mathcal{O}\,. \tag{6.1}$$

Let V be an algebra over $\mathcal{O}$. Using the map $\widehat{\varphi}$, we equip V with an $\Lambda\mathsf{Lie}_\infty$-structure.

If we assume, in addition, that V is equipped with a complete descending filtration

$$V \supset F_1V \supset F_2V \supset F_3V \supset \dots\,, \qquad V = \lim_k V \Big/ F_kV \tag{6.2}$$

and the $\mathcal{O}$-algebra structure on V is compatible with this filtration then we may define Maurer-Cartan elements of V as degree 2 elements $\alpha \in F_1 V$ satisfying the equation

$$\partial(\alpha) + \sum_{n \geq 2} \frac{1}{n!} \{\alpha, \alpha, \dots, \alpha\}_n = 0\,, \tag{6.3}$$

where ∂ is the differential on V and $\{\cdot, \cdot, \dots, \cdot\}_n$ is the n-th operation of the $\Lambda\mathsf{Lie}_\infty$-structure on V.

Given such a Maurer-Cartan element α we can twist the differential on V and insert α into various $\mathcal{O}$-operations on V. This way we get a new algebra structure on V.

It turns out that this new algebra structure is governed by an operad $\mathrm{Tw}\mathcal{O}$ which is built from the pair $(\mathcal{O}, \widehat{\varphi})$.

This section is devoted to the construction of $\mathrm{Tw}\mathcal{O}$.

6.1. Intrinsic derivations of an operad. Let $\mathcal{O}$ be an dg operad. We recall that a $\mathbb{K}$-linear map

$$\delta : \bigoplus_{n \geq 0} \mathcal{O}(n) \to \bigoplus_{n \geq 0} \mathcal{O}(n)$$

is an *operadic derivation* if for every $a \in \mathcal{O}(n)$, $\delta(a) \in \mathcal{O}(n)$ and for all homogeneous vectors $a_1 \in \mathcal{O}(n)$, $a_2 \in \mathcal{O}(k)$

$$\delta(a_1 \circ_i a_2) = \delta(a_1) \circ_i a_2 + (-1)^{|\delta||a_1|} a_1 \circ_i \delta(a_2)\,, \qquad \forall\ 1 \leq i \leq n\,.$$

Let us now observe that the operation $\circ_1$ equips $\mathcal{O}(1)$ with a structure of a dg associative algebra. We consider $\mathcal{O}(1)$ as a dg Lie algebra with the Lie bracket being the commutator.

We claim that

PROPOSITION 6.1. *The formula*

$$\delta_b(a) = b \circ_1 a - (-1)^{|a||b|} \sum_{i=1}^{n} a \circ_i b \tag{6.4}$$

with

$$b \in \mathcal{O}(1), \qquad \text{and} \qquad a \in \mathcal{O}(n)$$

defines an operadic derivation of $\mathcal{O}$ *for every* $b \in \mathcal{O}(1)$.

Operadic derivations of the form (6.4) are called *intrinsic.*

PROOF. Let $a_1 \in \mathcal{O}(n_1)$ and $a_2 \in \mathcal{O}(n_2)$. Then for every $b \in \mathcal{O}(1)$ and $1 \leq j \leq n_1$ we have

$$\delta_b(a_1 \circ_j a_2) = b \circ_1 (a_1 \circ_j a_2) - (-1)^{(|a_1|+|a_2|)|b|} \sum_{i=1}^{n_1+n_2-1} (a_1 \circ_j a_2) \circ_i b =$$

$$(b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1||b|} \sum_{\substack{1 \leq i \leq n_1 \\ i \neq j}} (a_1 \circ_i b) \circ_j a_2 - (-1)^{(|a_1|+|a_2|)|b|} \sum_{i=1}^{n_2} a_1 \circ_j (a_2 \circ_i b)$$

$$= (b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1||b|} \sum_{i=1}^{n_1} (a_1 \circ_i b) \circ_j a_2$$

$$+(-1)^{|a_1||b|}(a_1 \circ_j b) \circ_j a_2 - (-1)^{(|a_1|+|a_2|)|b|} \sum_{i=1}^{n_2} a_1 \circ_j (a_2 \circ_i b)$$

$$= (b \circ_1 a_1) \circ_j a_2 - (-1)^{|a_1||b|} \sum_{i=1}^{n_1} (a_1 \circ_i b) \circ_j a_2$$

$$+(-1)^{|a_1||b|} a_1 \circ_j (b \circ_1 a_2) - (-1)^{(|a_1|+|a_2|)|b|} \sum_{i=1}^{n_2} a_1 \circ_j (a_2 \circ_i b)$$

$$= \delta_b(a_1) \circ_j a_2 + (-1)^{|a_1||b|} a_1 \circ_j \delta_b(a_2)\,.$$

Hence δ_b is indeed an operadic derivation of $\mathcal{O}$.

It remains to verify the identity

$$[\delta_{b_1}, \delta_{b_2}] = \delta_{[b_1, b_2]} \tag{6.5}$$

and we leave this step as an exercise. □

EXERCISE 6.2. Verify identity (6.5).

6.2. Construction of the operad $\widetilde{\mathbf{Tw}}\mathcal{O}$. Let us recall that, since $\Lambda\mathsf{Lie}_\infty = \mathrm{Cobar}(\Lambda^2\mathsf{coCom})$, the morphism (6.1) is determined by a Maurer-Cartan element

$$\varphi \in \mathrm{Conv}(\Lambda^2\mathsf{coCom}_\circ, \mathcal{O})\,. \tag{6.6}$$

The n-th space of $\Lambda^2\mathsf{coCom}_\circ$ is the trivial S_n-module placed in degree $2-2n$:

$$\Lambda^2\mathsf{coCom}(n) = \mathbf{s}^{2-2n}\mathbb{K}\,.$$

So we have

$$\mathrm{Conv}(\Lambda^2\mathsf{coCom}_\circ, \mathcal{O}) = \prod_{n\geq 2} \mathrm{Hom}_{S_n}(\mathbf{s}^{2-2n}\mathbb{K}, \mathcal{O}(n)) = \prod_{n\geq 2} \mathbf{s}^{2n-2}\big(\mathcal{O}(n)\big)^{S_n}\,.$$

For our purposes we will need to extend the dg Lie algebra $\mathrm{Conv}(\Lambda^2\mathsf{coCom}_\circ, \mathcal{O})$ to

$$\mathcal{L}_{\mathcal{O}} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathcal{O}) = \prod_{n\geq 1} \mathrm{Hom}_{S_n}(\mathbf{s}^{2-2n}\mathbb{K}, \mathcal{O}(n))\,. \tag{6.7}$$

It is clear that

$$\mathcal{L}_{\mathcal{O}} = \prod_{n\geq 1} \mathbf{s}^{2n-2}\big(\mathcal{O}(n)\big)^{S_n}\,.$$

For $n, r \geq 1$ we realize the group S_r as the following subgroup of S_{r+n}

$$S_r \cong \big\{\sigma \in S_{r+n} \mid \sigma(i) = i\,, \quad \forall\, i > r\big\}\,. \tag{6.8}$$

In other words, for every $n \geq 1$, the group S_r may be viewed as subgroup of S_{r+n} permuting the first r letters. We set S_0 to be the trivial group.

Using this embedding of S_r into S_{n+r} we introduce the following collection ($n \geq 0$)

$$\widetilde{\mathrm{Tw}}\mathcal{O}(n) = \prod_{r\geq 0} \mathrm{Hom}_{S_r}(\mathbf{s}^{-2r}\mathbb{K}, \mathcal{O}(r+n))\,. \tag{6.9}$$

It is clear that

$$\widetilde{\mathrm{Tw}}\mathcal{O}(n) = \prod_{r\geq 0} \mathbf{s}^{2r}\big(\mathcal{O}(r+n)\big)^{S_r}\,.$$

To define an operad structure on (6.9) we denote by 1_r the generator $\mathbf{s}^{-2r}\,1 \in \mathbf{s}^{-2r}\mathbb{K}$. Then the identity element $\mathbf{u}$ in $\widetilde{\mathrm{Tw}}\mathcal{O}(1)$ is given by

$$\mathbf{u}(1_r) = \begin{cases} \mathbf{u}_{\mathcal{O}} & \text{if } r = 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{6.10}$$

where $\mathbf{u}_{\mathcal{O}} \in \mathcal{O}(1)$ is the identity element for the operad $\mathcal{O}$.

Next, for $f \in \widetilde{\mathrm{Tw}}\mathcal{O}(n)$ and $g \in \widetilde{\mathrm{Tw}}\mathcal{O}(m)$, we define the i-th elementary insertion $\circ_i$ $1 \le i \le n$ by the formula

$$f \circ_i g(1_r) = \sum_{p=0}^{r} \sum_{\sigma \in \mathrm{Sh}_{p,r-p}} \mu_{\mathbf{t}_{\sigma,i}}\big(f(1_p) \otimes g(1_{r-p})\big)\,. \tag{6.11}$$

where the tree $\mathbf{t}_{\sigma,i}$ is depicted on figure 21.

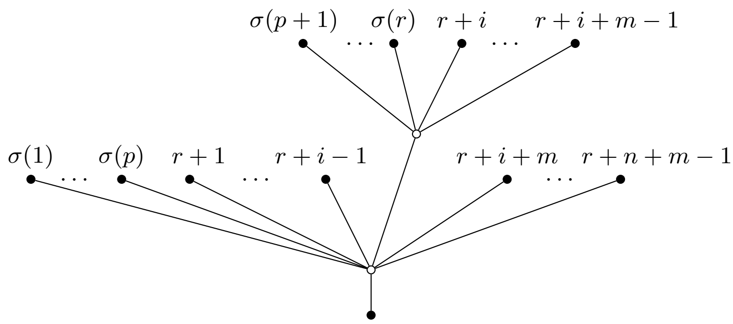


FIG. 21. Here σ is a $(p, r-p)$-shuffle

To see that the element $f \circ_i g(1_r) \in \mathcal{O}(r+n+m-1)$ is S_r-invariant one simply needs to use the fact that every element $\tau \in S_r$ can be uniquely presented as the composition $\tau_{sh} \circ \tau_{p,r-p}$, where τ_{sh} is a $(p, r-p)$-shuffle and $\tau_{p,r-p} \in S_p \times S_{r-p}$.

Let $f \in \widetilde{\mathrm{Tw}}\mathcal{O}(n)$, $g \in \widetilde{\mathrm{Tw}}\mathcal{O}(m)$, $h \in \widetilde{\mathrm{Tw}}\mathcal{O}(k)$, $1 \le i \le n$, and $1 \le j \le m$. To check the identity

$$f \circ_i (g \circ_j h) = (f \circ_i g) \circ_{j+i-1} h \tag{6.12}$$

we observe that

$$f \circ_i (g \circ_j h)(1_r) = \sum_{p=0}^{r} \sum_{\sigma \in \mathrm{Sh}_{p,r-p}} \mu_{\mathbf{t}_{\sigma,i}}\big(f(1_p) \otimes (g \circ_j h)(1_{r-p})\big)$$

$$= \sum_{p_1+p_2+p_3=r} \sum_{\sigma \in \mathrm{Sh}_{p_1,p_2+p_3}} \sum_{\sigma' \in \mathrm{Sh}_{p_2,p_3}} \mu_{\mathbf{t}_{\sigma,i}} \circ (1 \otimes \mu_{\mathbf{t}_{\sigma',j}})\big(f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3})\big)$$

$$= \sum_{p_1+p_2+p_3=r} \sum_{\tau \in \mathrm{Sh}_{p_1,p_2,p_3}} \mu_{\mathbf{t}_{\tau,i,j}}\big(f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3})\big)\,,$$

where the tree $\mathbf{t}_{\tau,i,j}$ is depicted on figure 22. Similar calculations show that

$$(f \circ_i g) \circ_{j+i-1} h = \sum_{p_1+p_2+p_3=r} \sum_{\tau \in \mathrm{Sh}_{p_1,p_2,p_3}} \mu_{\mathbf{t}_{\tau,i,j}}\big(f(1_{p_1}) \otimes g(1_{p_2}) \otimes h(1_{p_3})\big)\,,$$

with $\mathbf{t}_{\tau,i,j}$ being the tree depicted on figure 22.

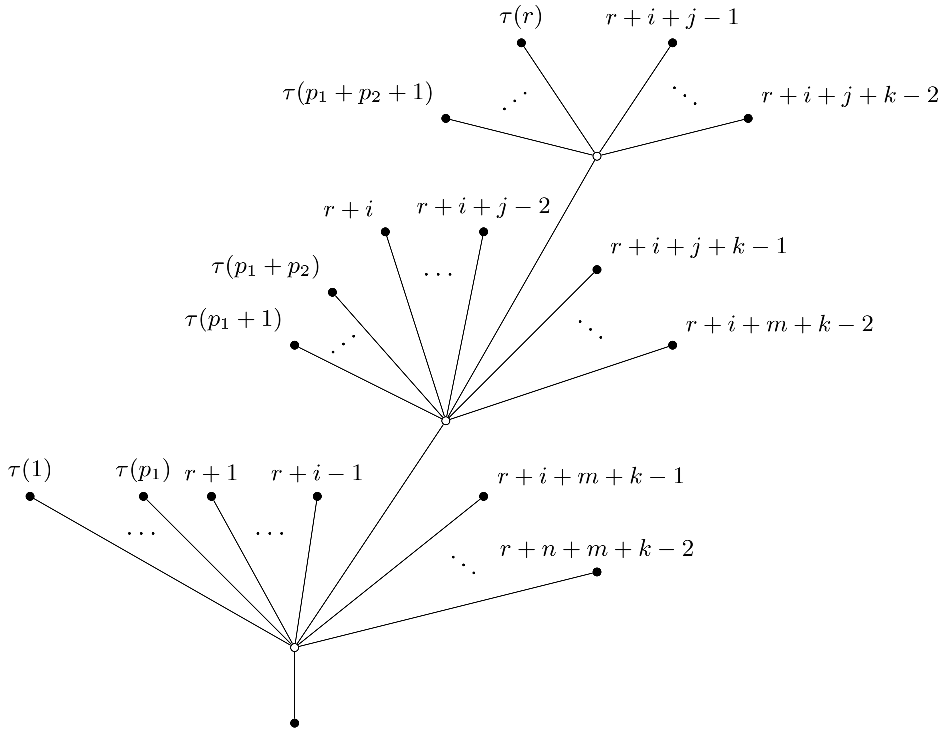


FIG. 22. Here τ is a (p_1, p_2, p_3)-shuffle and $r = p_1 + p_2 + p_3$

We leave the verification of the remaining axioms of the operad structure for the reader.

Our next goal is to define an auxiliary action of $\mathcal{L}_{\mathcal{O}}$ on the operad $\widetilde{\mathrm{Tw}\mathcal{O}}$. For a vector $f \in \widetilde{\mathrm{Tw}\mathcal{O}}(n)$ the action of $v \in \mathcal{L}_{\mathcal{O}}$ (6.7) on f is defined by the formula

$$v \cdot f(1_r) = -(-1)^{|v||f|} \sum_{p=1}^{r} \sum_{\sigma \in \mathrm{Sh}_{p,r-p}} \mu_{\mathbf{t}'_{\sigma,p,r-p}}(f(1_{r-p+1}) \otimes v(1^{\mathfrak{c}}_p)), \tag{6.13}$$

where $1^{\mathfrak{c}}_p$ is the generator $\mathbf{s}^{2-2p}\, 1 \in \Lambda^2 \mathsf{coCom}(p) \cong \mathbf{s}^{2-2p}\, \mathbb{K}$ and the tree $\mathbf{t}'_{\sigma,p,r-p}$ is depicted on figure 23.

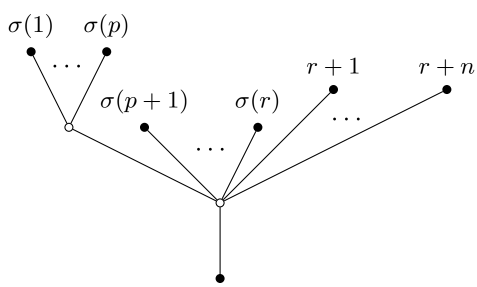


FIG. 23. Here σ is a $(p, r-p)$-shuffle

We claim that

PROPOSITION 6.3. *Formula* (6.13) *defines an action of* $\mathcal{L}_{\mathcal{O}}$ *(*6.7*) on the operad* $\widetilde{\mathrm{Tw}\mathcal{O}}$.

PROOF. A simple degree bookkeeping shows that the degree of $v \cdot f$ is $|v|+|f|$.
Then we need to check that for two homogeneous vectors $v, w \in \mathcal{L}_{\mathcal{O}}$ we have

$$[v,w]\cdot f(1_r) = (v\cdot(w\cdot f))(1_r) - (-1)^{|v||w|}(w\cdot(v\cdot f))(1_r) \tag{6.14}$$

Using the definition of the operation $\cdot$ and the associativity axiom for the operad structure on $\mathcal{O}$ we get

$$(v\cdot(w\cdot f))(1_r) - (-1)^{|v||w|}(w\cdot(v\cdot f))(1_r) = \tag{6.15}$$

$$(-1)^{|f|(|v|+|w|)+|v||w|}\sum_{p\ge 1}\sum_{q\ge 0}\sum_{\tau\in \mathrm{Sh}_{p,q,r-p-q}} \mu_{\mathbf{t}^{p,q}_{\tau}}\big(f(1_{r-p-q+1})\otimes w(1^{\mathfrak{c}}_{q+1})\otimes v(1^{\mathfrak{c}}_{p})\big)$$

$$+(-1)^{|f|(|v|+|w|)+|v||w|}\sum_{p,q\ge 1}\sum_{\tau\in \mathrm{Sh}_{p,q,r-p-q}} \mu_{\widetilde{\mathbf{t}}^{p,q}_{\tau}}\big(f(1_{r-p-q+2})\otimes w(1^{\mathfrak{c}}_{q})\otimes v(1^{\mathfrak{c}}_{p})\big)$$

$$-(-1)^{|v||w|}(v\leftrightarrow w)\,,$$

where the trees $\mathbf{t}^{p,q}_{\tau}$ and $\widetilde{\mathbf{t}}^{p,q}_{\tau}$ are depicted on figures 24 and 25 , respectively.

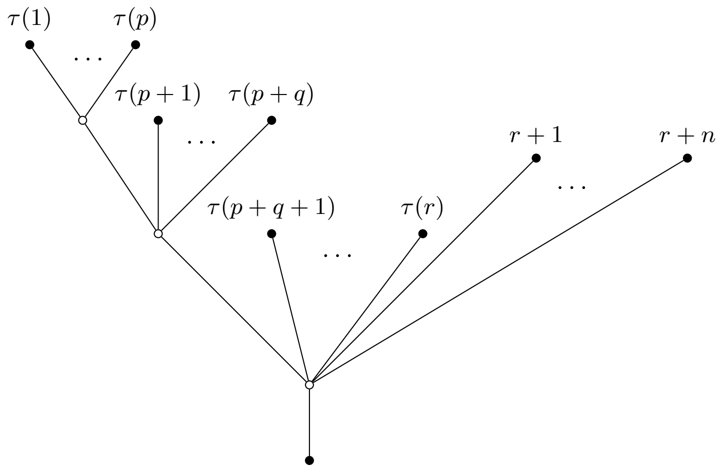


FIG. 24. The tree $\mathbf{t}^{p,q}_{\tau}$

Since $f(1_{r-p-q+2})$ is invariant with respect to the action of $S_{r-p-q+2}$ the sums involving $\mu_{\widetilde{\mathbf{t}}^{p,q}_{\tau}}$ cancel each other.

Furthermore, it is not hard to see that the sums involving $\mu_{\mathbf{t}^{p,q}_{\tau}}$ form the expression

$$[v,w]\cdot f(1_r)\,.$$

Thus equation (6.14) follows.

Due to Exercise 6.4 below, the operation $f \mapsto v\cdot f$ is an operadic derivation. Proposition 6.3 is proved. □

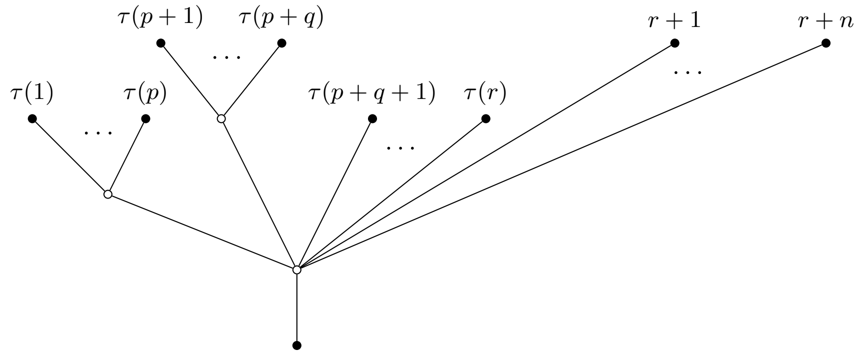


FIG. 25. The tree $\widetilde{\mathbf{t}}^{p,q}_{\tau}$

EXERCISE 6.4. Prove that for every triple of homogeneous vectors $f \in \widetilde{\mathrm{Tw}}\mathcal{O}(n)$, $g \in \widetilde{\mathrm{Tw}}\mathcal{O}(k)$, and $v \in \mathcal{L}_{\mathcal{O}}$ we have

$$v \cdot (f \circ_i g) = (v \cdot f) \circ_i g + (-1)^{|v||f|} f \circ_i (v \cdot g) \qquad \forall\ 1 \le i \le n\,.$$

6.3. The action of $\mathcal{L}_{\mathcal{O}}$ on $\widetilde{\mathbf{Tw}}\mathcal{O}$. Let us view $\widetilde{\mathrm{Tw}}\mathcal{O}(1)$ as the dg Lie algebra with the bracket being commutator.

We have an obvious degree zero map

$$\kappa : \mathcal{L}_{\mathcal{O}} \to \widetilde{\mathrm{Tw}}\mathcal{O}(1)$$

defined by the formula

$$\kappa(v)(1_r) = v(1^{\mathfrak{c}}_{r+1})\,, \tag{6.16}$$

where, as above, 1_r is the generator $\mathbf{s}^{-2r}\,1 \in \mathbf{s}^{-2r}\,\mathbb{K}$ and $1^{\mathfrak{c}}_r$ is the generator $\mathbf{s}^{2-2r}\,1 \in \Lambda^2\mathsf{coCom}(r) \cong \mathbf{s}^{2-2r}\,\mathbb{K}$.

We have the following proposition.

PROPOSITION 6.5. *Let us form the semi-direct product $\mathcal{L}_{\mathcal{O}} \ltimes \widetilde{\mathrm{Tw}}\mathcal{O}(1)$ of the dg Lie algebras $\mathcal{L}_{\mathcal{O}}$ and $\widetilde{\mathrm{Tw}}\mathcal{O}(1)$ using the action of $\mathcal{L}_{\mathcal{O}}$ on $\widetilde{\mathrm{Tw}}\mathcal{O}$ defined in Proposition* 6.3. *Then the formula*

$$\Theta(v) = v + \kappa(v) \tag{6.17}$$

defines a Lie algebra homomorphism

$$\Theta : \mathcal{L}_{\mathcal{O}} \to \mathcal{L}_{\mathcal{O}} \ltimes \widetilde{\mathrm{Tw}}\mathcal{O}(1)\,.$$

PROOF. First, let us prove that for every pair of homogeneous vectors $v, w \in \mathcal{L}_{\mathcal{O}}$ we have

$$\kappa([v,w]) = [\kappa(v), \kappa(w)] + v \cdot \kappa(w) - (-1)^{|v||w|} w \cdot \kappa(v)\,. \tag{6.18}$$

Indeed, unfolding the definition of κ we get[10]

(6.19)

$$\kappa([v,w])(1_r) = \sum_{p=1}^{r} \sum_{\tau \in \mathrm{Sh}_{p,r-p}} v_{r-p+2}\big(w_p(\tau(1), \dots, \tau(p)), \tau(p+1), \dots, \tau(r), r+1\big)$$

[10]Here we use the notation (3.9) introduced in Subsection 3.2.

$$+\sum_{p=0}^{r}\sum_{\tau\in \mathrm{Sh}_{p,r-p}} v_{r-p+1}\big(w_{p+1}(\tau(1),\dots,\tau(p),r+1),\tau(p+1),\dots,\tau(r)\big)$$

$$-(-1)^{|v||w|}(v\leftrightarrow w)\,,$$

where $v_t = v(1_t^{\mathfrak{c}})$ and $w_t = w(1_t^{\mathfrak{c}})$.

The first sum in (6.19) equals

$$-(-)^{|v||w|}(w\cdot\kappa(v))(1_r)\,.$$

Furthermore, since $v(1_t^{\mathfrak{c}})$ is invariant under the action of S_t, we see that the second sum in (6.19) equals

$$\big(\kappa(v)\circ_1\kappa(w)\big)\,(1_r)\,.$$

Thus equation (6.18) holds.

Now, using (6.18), it is easy to see that

$$[v+\kappa(v),w+\kappa(w)] = [v,w] + v\cdot\kappa(w) - (-1)^{|v||w|}w\cdot\kappa(v) + [\kappa(v),\kappa(w)] =$$

$$= [v,w] + \kappa([v,w])$$

and the statement of proposition follows. $\square$

The following corollaries are immediate consequences of Proposition 6.5

COROLLARY 6.6. *For $v\in\mathcal{L}_{\mathcal{O}}$ and $f\in\widetilde{\mathrm{Tw}}\mathcal{O}(n)$ the formula*

$$f\to v\cdot f + \delta_{\kappa(v)}(f) \tag{6.20}$$

defines an action of the Lie algebra $\mathcal{L}_{\mathcal{O}}$ on the operad $\widetilde{\mathrm{Tw}}\mathcal{O}$.

COROLLARY 6.7. *For every Maurer-Cartan element $\varphi\in\mathcal{L}_{\mathcal{O}}$, the sum*

$$\varphi+\kappa(\varphi)$$

is a Maurer-Cartan element of the Lie algebra $\mathcal{L}_{\mathcal{O}}\ltimes\widetilde{\mathrm{Tw}}\mathcal{O}(1)$.

We finally give the definition of the operad $\mathrm{Tw}\mathcal{O}$.

DEFINITION 6.8. Let $\mathcal{O}$ be an operad in $\mathsf{Ch}_{\mathbb{K}}$ and φ be a Maurer-Cartan element in $\mathcal{L}_{\mathcal{O}}$ (6.7) corresponding to an operad morphism $\widehat{\varphi}$ (6.1). Let us also denote by $\partial^{\mathcal{O}}$ the differential on $\widetilde{\mathrm{Tw}}\mathcal{O}$ coming from the one on $\mathcal{O}$. We define the operad $\mathrm{Tw}\mathcal{O}$ in $\mathsf{Ch}_{\mathbb{K}}$ by declaring that $\mathrm{Tw}\mathcal{O}=\widetilde{\mathrm{Tw}}\mathcal{O}$ as operads in $\mathsf{grVect}_{\mathbb{K}}$ and letting

$$\partial^{\mathrm{Tw}} = \partial^{\mathcal{O}} + \varphi\,\cdot + \delta_{\kappa(\varphi)} \tag{6.21}$$

be the differential on $\mathrm{Tw}\mathcal{O}$.

Corollaries 6.6 and 6.7 imply that ∂^{Tw} is indeed a differential on $\mathrm{Tw}\mathcal{O}$.

REMARK 6.9. It is easy to see that, if $\mathcal{O}(0)=\mathbf{0}$ then the cochain complexes $\mathbf{s}^{-2}\mathrm{Tw}\mathcal{O}(0)$ and $\mathcal{L}_{\mathcal{O}}$ (6.7) are tautologically isomorphic.

6.4. Algebras over Tw$\mathcal{O}$. Let us assume that V is an algebra over $\mathcal{O}$ equipped with a complete descending filtration (6.2). We also assume that the $\mathcal{O}$-algebra structure on V is compatible with this filtration.

Given a Maurer-Cartan element $\alpha \in F_1 V$ the formula

$$\partial^\alpha(v) = \partial(v) + \sum_{r=1}^{\infty} \frac{1}{r!}\varphi(1^{\mathfrak{c}}_{r+1})(\underbrace{\alpha,\dots,\alpha}_{r \text{ times}}, v) \tag{6.22}$$

defines a new (twisted) differential on V.

We will denote by V^α the cochain complex V with this new differential.

In this setting we have the following theorem

THEOREM 6.10. *If V^α is the cochain complex obtained from V via twisting the differential by α then the formula*

$$f(v_1,\dots,v_n) = \sum_{r=0}^{\infty} \frac{1}{r!} f(1_r)(\alpha,\dots,\alpha,v_1,\dots,v_n) \tag{6.23}$$

$$f \in \mathrm{Tw}\mathcal{O}(n)\,, \qquad v_i \in V\,, \qquad 1_r = \mathbf{s}^{-2r}\, 1 \in \mathbf{s}^{-2r}\,\mathbb{K}$$

defines a Tw$\mathcal{O}$*-algebra structure on* V^α.

PROOF. Let $f \in \mathrm{Tw}\mathcal{O}(n)$, $g \in \mathrm{Tw}\mathcal{O}(k)$,

$$f_r := f(1_r) \in \big(\mathcal{O}(r+n)\big)^{S_r}\,, \qquad \text{and} \qquad g_r = g(1_r) \in \big(\mathcal{O}(r+k)\big)^{S_r}\,.$$

Our first goal is to verify that

$$(-1)^{|g|(|v_i|+\dots+|v_{i-1}|)} f(v_1,\dots,v_{i-1}, g(v_i,\dots,v_{i+k-1}), v_{i+k},\dots,v_{n+k-1}) = f\circ_i g(v_1,\dots,v_{n+k-1})\,. \tag{6.24}$$

The left hand side of (6.24) can be rewritten as

$$(-1)^{|g|(|v_i|+\dots+|v_{i-1}|)} f(v_1,\dots,v_{i-1}, g(v_i,\dots,v_{i+k-1}), v_{i+k},\dots,v_{n+k-1}) =$$

$$\sum_{p,q\ge 0} \frac{(-1)^{|g|(|v_i|+\dots+|v_{i-1}|)}}{p!q!} f_p(\alpha,\dots,\alpha,v_1,\dots,v_{i-1}, g_q(\alpha,\dots,\alpha,v_i,\dots,v_{i+k-1}), v_{i+k},\dots,v_{n+k-1})\,.$$

Using the obvious combinatorial identity

$$|\mathrm{Sh}_{p,q}| = \frac{(p+q)!}{p!q!} \tag{6.25}$$

we rewrite the left hand side of (6.24) further

L.H.S. of (6.24)

$$= \sum_{p,q\ge 0} \frac{(-1)^{|g|(|v_i|+\dots+|v_{i-1}|)}}{(p+q)!} |\mathrm{Sh}_{p,q}| f_p(\alpha,\dots,\alpha,v_1,\dots,v_{i-1}, g_q(\alpha,\dots,\alpha,v_i,\dots,v_{i+k-1}), v_{i+k},\dots,v_{n+k-1})$$

$$= \sum_{r=0}^{\infty} \frac{1}{r!} \sum_{p=0}^{r} \sum_{\sigma\in \mathrm{Sh}_{p,r-p}} \sigma\circ \varrho_{r,p,i}(f_p \circ_{p+i} g_{r-p})(\underbrace{\alpha,\dots,\alpha}_{r \text{ arguments}}, v_1,\dots,v_{n+k-1}),$$

where $\varrho_{r,p,i}$ is the following permutation in S_r

$$\varrho_{r,p,i} = \begin{pmatrix} p+1 & \dots & p+i-1 & p+i & \dots & r+i-1 \\ r+1 & \dots & r+i-1 & p+1 & \dots & r \end{pmatrix}. \tag{6.26}$$

Thus

$$\text{L.H.S. of } (6.24) = f \circ_i g(v_1, \dots, v_{n+k-1})$$

and equation (6.24) holds.

Next, we need to show that

$$\partial^{\mathrm{Tw}}(f)(v_1, \dots, v_n) = \partial^{\alpha} f(v_1, \dots, v_n) \tag{6.27}$$

$$-(-1)^{|f|} \sum_{i=1}^{n} (-1)^{|v_i|+\dots+|v_{i-1}|} f(v_1, \dots, v_{i-1}, \partial^{\alpha}(v_i), v_{i+1}, \dots, v_n)$$

The right hand side of (6.27) can be rewritten as

$$\text{R.H.S. of } (6.27) =$$

$$\sum_{p\ge 0} \frac{1}{p!} \partial f_p(\alpha, \dots, \alpha, v_1, \dots, v_n) + \sum_{p\ge 0, q\ge 1} \frac{1}{p!q!} \varphi_q(\alpha, \dots, \alpha, f_p(\alpha, \dots, \alpha, v_1, \dots, v_n))$$

$$-(-1)^{|f|} \sum_{i=1}^{n} \sum_{p\ge 0} \frac{(-1)^{|v_i|+\dots+|v_{i-1}|}}{p!} f_p(\alpha, \dots, \alpha, v_1, \dots, v_{i-1}, \partial(v_i), v_{i+1}, \dots, v_n)$$

$$-(-1)^{|f|} \sum_{i=1}^{n} \sum_{p\ge 0, q\ge 1} \frac{(-1)^{|v_i|+\dots+|v_{i-1}|}}{p!q!} f_p(\alpha, \dots, \alpha, v_1, \dots, v_{i-1}, \varphi_q(\alpha, \dots, \alpha, v_i),$$

$$v_{i+1}, \dots, v_n)$$

where $f_p = f(1_p)$ and $\varphi_q = \varphi(1^{\mathfrak{c}}_q)$.

Let us now add to and subtract from the right hand side of (6.27) the sum

$$-(-1)^{|f|} \sum_{p\ge 0} \frac{1}{p!} f_{p+1}(\partial\alpha, \alpha, \dots, \alpha, v_1, \dots, v_n).$$

We get

$$\text{R.H.S. of } (6.27) =$$

$$\sum_{p\ge 0} \frac{1}{p!} \partial f_p(\alpha, \dots, \alpha, v_1, \dots, v_n) - (-1)^{|f|} \sum_{p\ge 0} \frac{1}{p!} f_{p+1}(\partial\alpha, \alpha, \dots, \alpha, v_1, \dots, v_n)$$

$$-(-1)^{|f|} \sum_{i=1}^{n} \sum_{p\ge 0} \frac{(-1)^{|v_i|+\dots+|v_{i-1}|}}{p!} f_p(\alpha, \dots, \alpha, v_1, \dots, v_{i-1}, \partial(v_i), v_{i+1}, \dots, v_n)$$

$$+(-1)^{|f|} \sum_{p\ge 0} \frac{1}{p!} f_{p+1}(\partial\alpha, \alpha, \dots, \alpha, v_1, \dots, v_n)$$

$$+ \sum_{p\ge 0, q\ge 1} \frac{1}{p!q!} \varphi_q(\alpha, \dots, \alpha, f_p(\alpha, \dots, \alpha, v_1, \dots, v_n))$$

$$-(-1)^{|f|} \sum_{i=1}^{n} \sum_{p\ge 0, q\ge 1} \frac{(-1)^{|v_i|+\dots+|v_{i-1}|}}{p!q!} f_p(\alpha, \dots, \alpha, v_1, \dots, v_{i-1}, \varphi_q(\alpha, \dots, \alpha, v_i),$$

$$v_{i+1}, \dots, v_n) = (\partial^{\mathcal{O}} f)(v_1, \dots, v_n)$$

$$+(-1)^{|f|}\sum_{p\geq 0}\frac{1}{p!}f_{p+1}(\partial\alpha,\alpha,\dots,\alpha,v_1,\dots,v_n)$$

$$+\sum_{p\geq 0,q\geq 1}\frac{1}{p!q!}\varphi_q(\alpha,\dots,\alpha,f_p(\alpha,\dots,\alpha,v_1,\dots,v_n))$$

$$-(-1)^{|f|}\sum_{i=1}^{n}\sum_{p\geq 0,q\geq 1}\frac{(-1)^{|v_i|+\dots+|v_{i-1}|}}{p!q!}f_p(\alpha,\dots,\alpha,v_1,\dots,v_{i-1},\varphi_q(\alpha,\dots,\alpha,v_i),$$
$$v_{i+1},\dots,v_n)\,.$$

Due to the Maurer-Cartan equation for α

$$\partial(\alpha)+\frac{1}{q!}\varphi_q(\alpha,\alpha,\dots,\alpha)=0$$

we have

$$+(-1)^{|f|}\sum_{p\geq 0}\frac{1}{p!}f_{p+1}(\partial\alpha,\alpha,\dots,\alpha,v_1,\dots,v_n)=$$
$$-(-1)^{|f|}\sum_{p\geq 0,q\geq 2}\frac{1}{p!q!}f_{p+1}(\varphi_q(\alpha,\dots,\alpha),\alpha,\dots,\alpha,v_1,\dots,v_n)\,.$$

Hence, using (6.25) we get

$$\text{R.H.S. of (6.27)}=$$
$$(\partial^{\mathcal{O}}f)(v_1,\dots,v_n)+(\varphi\cdot f)(v_1,\dots,v_n)$$
$$+\kappa(\varphi)\circ_1 f(v_1,\dots,v_n)-(-1)^{|f|}f\circ_1\kappa(\varphi)(v_1,\dots,v_n)\,.$$

Theorem 6.10 is proved. □

Let us now observe that the dg operad $\mathrm{Tw}\mathcal{O}$ is equipped with complete descending filtration. Namely,

$$\mathcal{F}_k\mathrm{Tw}\mathcal{O}(n)=\{f\in\mathrm{Tw}\mathcal{O}(n)\mid f(1_r)=0\quad\forall\ r<k\} \tag{6.28}$$

It is clear that the operad structure on $\mathrm{Tw}\mathcal{O}$ is compatible with this filtration.

The endomorphism operad End_V also carries a complete descending filtration since so does V.

For this reason it makes sense to give this definition:

DEFINITION 6.11. A filtered $\mathrm{Tw}\mathcal{O}$-algebra is a cochain complex V equipped with a complete descending filtration for which the operad map

$$\mathrm{Tw}\mathcal{O}\to\mathsf{End}_V$$

is compatible with the filtrations.

It is easy to see that the $\mathrm{Tw}\mathcal{O}$-algebra V^α from Theorem 6.10 is a filtered $\mathrm{Tw}\mathcal{O}$-algebra in the sense of this definition.

Thus Theorem 6.10 gives us a functor to the category of filtered $\mathrm{Tw}\mathcal{O}$-algebras from the category of pairs

$$(V,\alpha)\,,$$

where V is a filtered cochain complex equipped with an action of the operad $\mathcal{O}$ and α is a Maurer-Cartan element in $\mathcal{F}_1V$.

According to [**7**] this functor establishes an equivalence of categories.

6.5. A useful modification $\mathbf{Tw}^{\oplus}\mathcal{O}$. In practice the morphism (6.1) often comes from the map (of dg operads)

$$\mathsf{j} : \Lambda\mathsf{Lie} \to \mathcal{O}\,.$$

In this case, the above construction of twisting is well defined for the suboperad $\mathrm{Tw}^{\oplus}(\mathcal{O}) \subset \mathrm{Tw}\mathcal{O}$ with

$$\mathrm{Tw}^{\oplus}(\mathcal{O})(n) = \bigoplus_{r\ge 0} \mathbf{s}^{2r}\big(\mathcal{O}(r+n)\big)^{S_r}\,. \tag{6.29}$$

It is not hard to see that the Maurer-Cartan element

$$\varphi \in \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathcal{O})$$

corresponding to the composition

$$\mathsf{j} \circ U_{\Lambda\mathsf{Lie}} : \mathrm{Cobar}(\Lambda^2\mathsf{coCom}) \to \mathcal{O}$$

is given by the formula:

$$\varphi = \mathsf{j}(\{a_1, a_2\}) \otimes b_1 b_2\,. \tag{6.30}$$

Hence

$$\mathcal{L}^{\oplus}_{\mathcal{O}} = \bigoplus_{r\ge 0} \mathbf{s}^{2r-2}\big(\mathcal{O}(r)\big)^{S_r} \tag{6.31}$$

is a sub- dg Lie algebra of $\mathcal{L}_{\mathcal{O}}$ (6.7).

Specifying general formula (6.21) to this particular case, we see that the differential ∂^{Tw} on (6.29) is given by the equation:

$$\partial^{\mathrm{Tw}}(v) = -(-1)^{|v|} \sum_{\sigma\in \mathrm{Sh}_{2,r-1}} \sigma\big(v \circ_1 \mathsf{j}(\{a_1,a_2\})\big) + \sum_{\tau\in\mathrm{Sh}_{1,r}} \tau\big(\mathsf{j}(\{a_1,a_2\}) \circ_2 v\big) \tag{6.32}$$

$$-(-1)^{|v|} \sum_{\tau'\in \mathrm{Sh}_{r,1}} \sum_{i=1}^{n} \tau' \circ \varsigma_{r+1,r+i}\big(v \circ_{r+i} \mathsf{j}(\{a_1,a_2\})\big)\,,$$

where

$$v \in \mathbf{s}^{2r}\big(\mathcal{O}(r+n)\big)^{S_r}\,,$$

and $\varsigma_{r+1,r+i}$ is the cycle $(r+1, r+2, \dots, r+i)$.

REMARK 6.12. We should remark that, when we apply elementary insertions in the right hand side of (6.32), we view v and $\mathsf{j}(\{a_1,a_2\})$ as vectors in $\mathcal{O}(r+n)$ and $\mathcal{O}(2)$ respectively. The resulting sum in the right hand side of (6.32) is viewed as a vector in $\mathrm{Tw}\mathcal{O}(n)$.

6.6. Example: The operad TwGer. Let Ger be the operad which governs Gerstenhaber algebras (see Subsection 3.3.2). Since $\Lambda\mathsf{Lie}$ receives a quasi-isomorphism (5.22) from $\Lambda\mathsf{Lie}_\infty$ and embeds into Ger, we have a canonical map

$$\Lambda\mathsf{Lie}_\infty \to \mathsf{Ger}\,. \tag{6.33}$$

This section is devoted to the dg operad $\mathrm{Tw}\mathsf{Ger}$ which is associated to the operad Ger and the map (6.33).

According to the general procedure of twisting

$$\mathcal{L}_{\mathsf{Ger}} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Ger}) = \prod_{r=1}^{\infty} \mathbf{s}^{2r-2}\big(\mathsf{Ger}(r)\big)^{S_r} \tag{6.34}$$

and the Maurer-Cartan element $\alpha \in \mathcal{L}_{\mathsf{Ger}}$ corresponding to the map (6.33) equals

$$\alpha = \{a_1, a_2\}\,. \tag{6.35}$$

The graded vector space $\mathrm{Tw}\mathsf{Ger}(n)$ is the product

$$\mathrm{Tw}\mathsf{Ger}(n) = \prod_{r \ge 0} \mathbf{s}^{2r}\big(\mathsf{Ger}(r+n)\big)^{S_r}\,. \tag{6.36}$$

Furthermore, adapting (6.32) to this case we get

$$\partial^{\mathrm{Tw}}(v) = -(-1)^{|v|} \sum_{\sigma \in \mathrm{Sh}_{2,r-1}} \sigma\big(v \circ_1 \{a_1, a_2\}\big) + \sum_{\tau \in \mathrm{Sh}_{1,r}} \tau\big(\{a_1, a_2\} \circ_2 v\big) \tag{6.37}$$
$$-(-1)^{|v|} \sum_{\tau' \in \mathrm{Sh}_{r,1}} \sum_{i=1}^{n} \tau' \circ \varsigma_{r+1,r+i}\big(v \circ_{r+i} \{a_1, a_2\}\big)\,,$$

where

$$v \in \mathbf{s}^{2r}\big(\mathsf{Ger}(r+n)\big)^{S_r}\,,$$

and $\varsigma_{r+1,r+i}$ is the cycle $(r+1, r+2, \dots, r+i)$.

EXERCISE 6.13. Prove that for every $v \in \mathbf{s}^{2r}\mathsf{Ger}(r+n)$

$$r \le |v| + n - 1\,. \tag{6.38}$$

Similarly, prove that, for every vector $v \in \mathbf{s}^{2r-2}\mathsf{Ger}(r)$

$$r \le |v| + 1\,. \tag{6.39}$$

Inequalities (6.38) and (6.39) imply that

$$\mathrm{Tw}\mathsf{Ger}(n) = \bigoplus_{r=0}^{\infty} \mathbf{s}^{2r}\big(\mathsf{Ger}(r+n)\big)^{S_r} \tag{6.40}$$

and

$$\mathcal{L}_{\mathsf{Ger}} = \bigoplus_{r=1}^{\infty} \mathbf{s}^{2r-2}\big(\mathsf{Ger}(r)\big)^{S_r}\,. \tag{6.41}$$

In other words, $\mathrm{Tw}^{\oplus}\mathsf{Ger} = \mathrm{Tw}\mathsf{Ger}$ and $\mathcal{L}^{\oplus}_{\mathsf{Ger}} = \mathcal{L}_{\mathsf{Ger}}$.

To give a simpler description of the cochain complexes $\mathrm{Tw}\mathsf{Ger}(n)$ (6.40) we consider the free Gerstenhaber algebra

$$\mathsf{Ger}(a, a_1, \dots, a_n)$$

in n variables $a_1, \dots, a_n$ of degree zero and one additional variable a of degree 2.

We introduce the following (degree 1) derivation

$$\delta(a) = \frac{1}{2}\{a, a\}\,, \qquad \delta(a_i) = 0 \qquad \forall\ 1 \le i \le n \tag{6.42}$$

of $\mathsf{Ger}(a, a_1, \dots, a_n)$ and observe that

$$\delta^2 = 0$$

in virtue of the Jacobi identity.

Then we denote by $\mathcal{G}_n$ the subspace

$$\mathcal{G}_n \subset \mathsf{Ger}(a, a_1, \dots, a_n) \tag{6.43}$$

spanned by monomials in which each variable $a_1, a_2, \dots, a_n$ appears exactly once. It is obvious that $\mathcal{G}_n$ is a subcomplex of $\mathsf{Ger}(a, a_1, \dots, a_n)$.

We claim that

PROPOSITION 6.14. *The cochain complex* $\mathcal{G}_n$ *is isomorphic to* $\mathrm{Tw}\mathsf{Ger}(n)$ *.*

PROOF. Indeed, given a monomial $v \in \mathcal{G}_n$ of degree r in a we shift the indices of a_i up by r and replace the r factors a in v by $a_1, a_2, \dots, a_r$ in an arbitrary order. This way we get a monomial $v' \in \mathsf{Ger}(r+n)$. It is easy to see that the formula

$$f(v) = \sum_{\sigma \in S_r} \mathbf{s}^{2r} \sigma(v') \tag{6.44}$$

defines a linear map of vector spaces

$$f : \mathcal{G}_n \to \mathrm{Tw}\mathsf{Ger}(n) = \bigoplus_{r=0}^{\infty} \mathbf{s}^{2r} \big(\mathsf{Ger}(r+n)\big)^{S_r} .$$

For example,

$$f(\{a,a\}a_1) = \{a_1,a_2\}a_3 + \{a_2,a_1\}a_3 , \qquad f(\{a,a_1\}aa_2) = \{a_1,a_3\}a_2a_4 + \{a_2,a_3\}a_1a_4 .$$

It is not hard to see that f is an isomorphism of graded vector spaces. Furthermore, f is compatible with the differentials due to the following exercise.

EXERCISE 6.15. Show that the map

$$f : \mathcal{G}_n \to \mathrm{Tw}\mathsf{Ger}(n) = \bigoplus_{r=0}^{\infty} \mathbf{s}^{2r} \big(\mathsf{Ger}(r+n)\big)^{S_r}$$

defined by (6.44) is compatible with the differentials ∂^{Tw} and δ. In other words,

$$f(\delta v) = \partial^{\mathrm{Tw}} f(v) \qquad \forall\ v \in \mathcal{G}_n . \tag{6.45}$$

Thus the proposition is proved. □

Proposition 6.14 implies that every vector $v \in \mathsf{Ger}(n) \subset \mathrm{Tw}\mathsf{Ger}(n)$ is ∂-closed. Therefore, the obvious embedding

$$i : \mathsf{Ger} \to \mathrm{Tw}\mathsf{Ger} \tag{6.46}$$

is a map of dg operads.

We claim that[11]

THEOREM 6.16. *The map* (6.46) *is a quasi-isomorphism of dg operads. In particular, the dg Lie algebra* $\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Ger})$ *is acyclic.*

PROOF. Let us observe that $\Lambda\mathsf{Lie}(a, a_1, \dots, a_n)$ is a subcomplex of $\mathsf{Ger}(a, a_1, \dots, a_n)$. Moreover,

$$\mathsf{Ger}(a, a_1, \dots, a_n) = \underline{S}(\Lambda\mathsf{Lie}(a, a_1, \dots, a_n)) , \tag{6.47}$$

where $\underline{S}$ is the notation for the truncated symmetric algebra.

Let us denote by

$$\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n) \tag{6.48}$$

the subspace of $\Lambda\mathsf{Lie}(a, a_1, \dots, a_n)$ which is spanned by monomials involving each variable in the set $\{a_1, a_2, \dots, a_n\}$ at most once. It is clear that $\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)$ is a subcomplex in $\Lambda\mathsf{Lie}(a, a_1, \dots, a_n)$. Hence, the subspace

$$\mathsf{Ger}'(a, a_1, \dots, a_n) := \underline{S}(\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)) \tag{6.49}$$

is a subcomplex of $\mathsf{Ger}(a, a_1, \dots, a_n)$.

[11] This statement is also proved in [**7**].

We will prove every cocycle in $\mathsf{Ger}'(a, a_1, \dots, a_n)$ is cohomologous to a (unique) cocycle in the intersection

$$\mathsf{Ger}'(a, a_1, \dots, a_n) \;\cap\; \mathsf{Ger}(a_1, \dots, a_n)$$

and then we will deduce statements of the theorem.

Let us, first, show that every cocycle in $\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)$ is cohomologous to a cocycle in the intersection

$$\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n) \;\cap\; \Lambda\mathsf{Lie}(a_1, \dots, a_n)\,.$$

For this purpose we consider a non-empty ordered subset $\{i_1 < i_2 < \dots < i_k\}$ of $\{1, 2, \dots, n\}$ and denote by

$$\Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k}) \tag{6.50}$$

the subcomplex of $\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)$ which is spanned by $\Lambda\mathsf{Lie}$-monomials in $\Lambda\mathsf{Lie}(a, a_{i_1}, \dots, a_{i_k})$ involving each variable in the set $\{a_{i_1}, \dots, a_{i_k}\}$ exactly once.

It is clear that $\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)$ splits into the direct sum of subcomplexes:

$$\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n) = \mathbb{K}\langle a, \{a, a\}\rangle \;\oplus \bigoplus_{\{i_1 < i_2 < \dots < i_k\}} \Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k})\,, \tag{6.51}$$

where the summation runs over all non-empty ordered subsets $\{i_1 < i_2 < \dots < i_k\}$ of $\{1, 2, \dots, n\}$.

It is not hard to see that the subcomplex $\mathbb{K}\langle a, \{a, a\}\rangle$ is acyclic. Thus our goal is to show that every cocycle in $\Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k})$ is cohomologous to cocycle in the intersection

$$\Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k}) \cap \Lambda\mathsf{Lie}(a_{i_1}, \dots, a_{i_k})\,.$$

To prove this fact we consider the tensor algebra

$$T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big) \tag{6.52}$$

in the variables $\mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}$ and denote by

$$T'(\mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}) \tag{6.53}$$

the subspace of (6.52) which is spanned by monomials involving each variable from the set $\{\mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\}$ exactly once.

It is not hard to see that the formula

$$\nu(x_{j_1} \otimes x_{j_2} \otimes \dots \otimes x_{j_N}) = \{\mathbf{s}\, x_{j_1}, \{\mathbf{s}\, x_{j_2}, \{\dots \{\mathbf{s}\, x_{j_N}, a_{i_k}\}..\} \tag{6.54}$$

defines an isomorphism of the graded vector spaces

$$\nu : T'(\mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}) \xrightarrow{\cong} \Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k})\,.$$

Let us denote by δ_T a degree 1 derivation of the tensor algebra (6.52) defined by the equations

$$\delta_T(\mathbf{s}^{-1}\, a_{i_t}) = 0\,, \qquad \delta_T(\mathbf{s}^{-1}\, a) = \mathbf{s}^{-1}\, a \otimes \mathbf{s}^{-1}\, a\,. \tag{6.55}$$

It is not hard to see that $(\delta_T)^2 = 0$. Thus, δ_T is a differential on the tensor algebra (6.52).

The subspace (6.53) is obviously a subcomplex of (6.52). Furthermore, using the following consequence of Jacobi identity

$$\{a, \{a, X\}\} = -\frac{1}{2}\{\{a, a\}, X\}\,, \qquad \forall\; X \in \Lambda\mathsf{Lie}(a, a_1, \dots, a_n),$$

it is easy to show that

$$\delta \circ \nu = \nu \circ \delta_T .$$

Thus ν is an isomorphism from the cochain complex

$$\big(T'(\mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}), \delta_T\big)$$

to the cochain complex

$$\big(\Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k}), \delta\big) .$$

To compute cohomology of the cochain complex

$$\Big(T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big), \delta_T\Big) \tag{6.56}$$

we observe that the truncated tensor algebra

$$\underline{T}_{\mathbf{s}^{-1}\, a} := \underline{T}\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a\rangle\big) \tag{6.57}$$

form an acyclic subcomplex of (6.56).

We also observe that the cochain complex (6.56) splits into the direct sum of subcomplexes

(6.58)

$$T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big) = T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big) \oplus$$

$$\bigoplus_{m \ge 2,\, p_1, \dots, p_m} V_{a_\bullet}^{\otimes\, p_1} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes V_{a_\bullet}^{\otimes\, p_2} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes \dots \otimes V_{a_\bullet}^{\otimes\, p_{m-1}} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes V_{a_\bullet}^{\otimes\, p_m} ,$$

where $V_{a_\bullet}$ is the cochain complex

$$V_{a_\bullet} := \mathbb{K}\langle \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle$$

with the zero differential and the summation runs over all combinations $(p_1, \dots, p_m)$ of integers satisfying the conditions

$$p_1, p_m \ge 0, \qquad p_2, \dots, p_{m-1} \ge 1 .$$

By Künneth's theorem all the subcomplexes

$$V_{a_\bullet}^{\otimes\, p_1} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes V_{a_\bullet}^{\otimes\, p_2} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes \dots \otimes V_{a_\bullet}^{\otimes\, p_{m-1}} \otimes \underline{T}_{\mathbf{s}^{-1}\, a} \otimes V_{a_\bullet}^{\otimes\, p_m}$$

are acyclic. Hence for every cocycle c in (6.56) there exists a vector c_1 in (6.56) such that

$$c - \delta_T(c_1) \in T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big) .$$

Combining this observation with the fact that the subcomplex (6.53) is a direct summand in (6.56), we conclude that, for every cocycle c in (6.53) there exists a vector c_1 in (6.53) such that

$$c - \delta_T(c_1) \in T'(\mathbf{s}^{-1}\, a, \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}) \cap T\big(\mathbb{K}\langle \mathbf{s}^{-1}\, a_{i_1}, \mathbf{s}^{-1}\, a_{i_2}, \dots, \mathbf{s}^{-1}\, a_{i_{k-1}}\rangle\big) .$$

Since the map ν (6.54) is an isomorphism from the cochain complex (6.53) with the differential δ_T to the cochain complex (6.50) with the differential δ, we deduce that every cocycle in (6.50) is cohomologous to a unique cocycle in the intersection

$$\Lambda\mathsf{Lie}''(a, a_{i_1}, \dots, a_{i_k}) \cap \Lambda\mathsf{Lie}(a_{i_1}, \dots, a_{i_k}) .$$

Therefore every cocycle in $\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n)$ is cohomologous to a unique cocycle in the intersection

$$\Lambda\mathsf{Lie}'(a, a_1, \dots, a_n) \ \cap\ \Lambda\mathsf{Lie}(a_1, \dots, a_n) .$$

Combining the latter observation with decomposition (6.49) we conclude that every cocycle in $\mathsf{Ger}'(a, a_1, \dots, a_n)$ is cohomologous to a (unique) cocycle in the intersection

$$\mathsf{Ger}'(a, a_1, \dots, a_n) \ \cap \ \mathsf{Ger}(a_1, \dots, a_n)\,.$$

Thus, using the isomorphism

$$\mathcal{G}_n \cong \mathrm{Tw}\mathsf{Ger}(n)$$

together with the fact that the cochain complex $\mathcal{G}_n$ is a direct summand in $\mathsf{Ger}'(a, a_1, \dots, a_n)$, we deduce the first statement of Theorem 6.16.

On the other hand, since $\mathsf{Ger}(0) = \mathbf{0}$, Remark 6.9 implies that

$$\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Ger}) \cong \mathbf{s}^{-2}\,\mathrm{Tw}\mathsf{Ger}(0)\,.$$

Hence the second statement of Theorem 6.16 follows as well.

The theorem is proved. □

6.7. The dg Lie algebra $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})$. The filtration by Lie words of length 1. Let $\mathcal{O}$ be a dg operad and ι be a map (of dg operads)

$$\iota : \mathsf{Ger} \to \mathcal{O}\,. \tag{6.59}$$

(Here, we assume that $\mathcal{O}(0) = \mathbf{0}$.)

In this subsection we will describe an auxiliary construction related to the pair $(\mathcal{O}, \iota)$. In these notes, we will use this construction twice. First, we will use it in the case when $\mathcal{O} = \mathsf{Ger}$. Second, we will use it in the case when[12] $\mathcal{O} = \mathsf{Gra}$.

Restricting ι to the suboperad $\Lambda\mathsf{Lie}$ we get a morphism of dg operads

$$\mathsf{j} = \iota\Big|_{\Lambda\mathsf{Lie}} : \Lambda\mathsf{Lie} \to \mathcal{O}\,. \tag{6.60}$$

Thus, following Section 6.5, we may construct the dg operad $\mathrm{Tw}\mathcal{O}$ as well as its suboperad $\mathrm{Tw}^{\oplus}(\mathcal{O}) \subset \mathrm{Tw}\mathcal{O}$ (6.29).

On the other hand composing ι with U_{Ger} (5.20) we get a morphism

$$\iota \circ U_{\mathsf{Ger}} : \mathrm{Cobar}(\mathsf{Ger}^{\vee}) \to \mathcal{O}\,. \tag{6.61}$$

It is not hard to see that the Maurer-Cartan element $\alpha \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathcal{O})$ corresponding to the morphism (6.61) is given by the formula

$$\alpha = \iota(\{a_1, a_2\}) \otimes b_1 b_2 + \iota(a_1 a_2) \otimes \{b_1, b_2\}\,. \tag{6.62}$$

Since $\alpha \in \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})$, it makes sense to consider the cochain complex

$$\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O}) = \bigoplus_{n \ge 1} \Big(\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n} \tag{6.63}$$

with the differential

$$\partial = [\alpha, \ ]\,. \tag{6.64}$$

Let us denote by $\mathfrak{L}_1(w)$ the number of Lie words of length 1 in a monomial $w \in \Lambda^{-2}\mathsf{Ger}(n)$. For example, $\mathfrak{L}_1(b_1 b_2) = 2$ and $\mathfrak{L}_1(\{b_1, b_2\}) = 0$.

Next we consider a vector $v \in \mathcal{O}(n)$ and observe that for every vector $v_i \otimes w_i$ in the linear combination

$$\partial\left(\sum_{\sigma \in S_n} \sigma(v) \otimes \sigma(w)\right)$$

[12]The operad Gra is introduced in Section 7 below.

we have $\mathfrak{L}_1(w_i) = \mathfrak{L}_1(w)$ or $\mathfrak{L}_1(w_i) = \mathfrak{L}_1(w) + 1$.

This observation allows us to introduce the following ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O}) \subset \mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O}) \subset \dots\,, \tag{6.65}$$

where $\mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})$ consists of sums

$$\sum_i v_i \otimes w_i \in \bigoplus_n \big(\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}$$

which satisfy

$$\mathfrak{L}_1(w_i) - |\, v_i \otimes w_i \,| \le m\,, \qquad \forall\ i\,.$$

It is clear that the associated graded complex

$$\mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O}) \cong \bigoplus_{n=1}^{\infty} \Big(\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n} \tag{6.66}$$

as a graded vector space, and the differential ∂^{Gr} on $\mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})$ is obtained from the differential ∂ (6.64) on $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})$ by keeping only terms which raise the number of Lie brackets of length 1 in the second tensor factors. For example, the adjoint action

$$[\,\iota(a_1 a_2) \otimes \{b_1, b_2\},\ ]$$

of $\iota(a_1 a_2) \otimes \{b_1, b_2\}$ does not contribute to the differential ∂^{Gr} at all.

To give a convenient description of the cochain complex (6.66) we introduce the collection

$$\{\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\}_{n \ge 0} \tag{6.67}$$

where

$$\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(0) = \mathbf{s}^{-2}\mathbb{K}$$

and

$$\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n), \qquad n \ge 1$$

is the S_n-submodule of $\Lambda^{-2}\mathsf{Ger}(n)$ spanned by monomials $w \in \Lambda^{-2}\mathsf{Ger}(n)$ for which $\mathfrak{L}_1(w) = 0$.

Next, we introduce the cochain complex

$$\bigoplus_{n \ge 1} \Big(\mathrm{Tw}^{\oplus}\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n} = \bigoplus_{r \ge 0} \bigoplus_{n \ge 1} \Big(\big(\mathbf{s}^{2r}\mathcal{O}(r+n)\big)^{S_r} \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n} \tag{6.68}$$

with the differential ∂^{Tw} coming from $\mathrm{Tw}\mathcal{O}$.

We observe that the formula

$$\Upsilon_{\mathcal{O}}\left(\sum_i v_i \otimes w_i\right) := \sum_{\sigma \in \mathrm{Sh}_{r,n}} \sum_i \sigma(v_i) \otimes \sigma(b_1 \dots b_r\, w_i(b_{r+1}, \dots, b_{r+n})) \tag{6.69}$$

$$\sum_i v_i \otimes w_i \in \Big(\mathbf{s}^{2r}\mathcal{O}(r+n)^{S_r} \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n}$$

defines a morphism of graded vector spaces

$$\Upsilon_{\mathcal{O}}\ :\ \bigoplus_{n \ge 0} \Big(\mathrm{Tw}^{\oplus}\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n}\ \to\ \mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O})\,. \tag{6.70}$$

We claim that

PROPOSITION 6.17. *The map $\Upsilon_{\mathcal{O}}$ (6.70) is an isomorphism of cochain complexes.*

PROOF. It is clear that (6.66) is spanned by vectors of the form

$$\sum_{\tau \in S_{r+n}} \tau(v) \otimes \tau(b_1 \dots b_r \, w(b_{r+1}, \dots, b_{r+n}))\,, \tag{6.71}$$

where v is a vector in $\mathcal{O}(r+n)$, w is a monomial $\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)$, and numbers r, n vary within the range $r, n \geq 0$, $r+n \geq 1$.

Using the obvious identity

$$\sum_{\tau \in S_{r+n}} \tau(v) \otimes \tau(b_1 \dots b_r \, w(b_{r+1}, \dots, b_{r+n})) =$$

$$\sum_{\sigma \in \mathrm{Sh}_{r,n}} \sigma \left(\sum_{(\tau', \tau'') \in S_r \times S_n \subset S_{r+n}} (\tau', \tau'')(v) \otimes b_1 \dots b_r \, (\tau'' w)(b_{r+1}, \dots, b_{r+n}) \right)$$

we see that the formula

$$\widetilde{\Upsilon}_{\mathcal{O}} \left(\sum_{\sigma \in S_{r+n}} \sigma(v) \otimes \sigma(b_1 \dots b_r \, w(b_{r+1}, \dots, b_{r+n})) \right) = \tag{6.72}$$

$$\sum_{\tau'' \in S_n} \tau'' \left(\sum_{\tau' \in S_r} \tau'(v) \right) \otimes \tau''(w)$$

gives us a well-defined map

$$\widetilde{\Upsilon}_{\mathcal{O}} \; : \; \mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathcal{O}) \to \bigoplus_{n \geq 0} \Big(\mathrm{Tw}\mathcal{O}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n}. \tag{6.73}$$

Furthermore, it is obvious that $\widetilde{\Upsilon}_{\mathcal{O}}$ is the inverse of $\Upsilon_{\mathcal{O}}$.

Thus $\Upsilon_{\mathcal{O}}$ is an isomorphism of graded vector spaces.

Before proving that Υ is compatible with the differentials, let us recall that, for $i < j$, $\varsigma_{i,j}$ denotes the cycle $(i, i+1, \dots, j) \in S_n$ for any $n \geq j$. Furthermore, $S_{i,i+1,\dots,n}$ denotes the permutation group of the set $\{i, i+1 \dots, n\}$.

Let, as above, v be a vector in $\mathcal{O}(r+n)$ and w be a monomial in $\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)$. Due to the above consideration,

$$\sum_{\tau \in S_{r+n}} \tau(v) \otimes \tau(b_1 \dots b_r \, w(b_{r+1}, \dots, b_{r+n})) = \Upsilon_{\mathcal{O}} \left(\sum_{\lambda \in S_n} \lambda\big(\mathrm{Av}_r(v)\big) \otimes \lambda(w) \right), \tag{6.74}$$

where

$$\mathrm{Av}_r(v) = \sum_{\lambda_1 \in S_r} \lambda_1(v)$$

is viewed as a vector in $\mathrm{Tw}\mathcal{O}(n)$.

Thus our goal is to show that

$$\partial^{\mathrm{Gr}} \left(\sum_{\tau \in S_{r+n}} \tau(v) \otimes \tau(b_1 \dots b_r \, w(b_{r+1}, \dots, b_{r+n})) \right) = \tag{6.75}$$

$$\Upsilon_{\mathcal{O}}\left(\sum_{\lambda\in S_n}\partial^{\mathrm{Tw}}\circ\lambda\big(\mathrm{Av}_r(v)\big)\otimes\lambda(w)\right).$$

Collecting terms with $r+1$ Lie words of length 1 in the second tensor factors in

$$\Big[\ \iota(\{a_1,a_2\})\otimes b_1b_2\ ,\ \sum_{\tau\in S_{r+n}}\tau(v)\otimes\tau(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))\ \Big]$$

and using the obvious identity

$$\sum_{\tau\in S_{r+n}}\tau(v)\otimes\tau(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))=$$

$$\sum_{\tau'\in S_{2,3,\dots,r+n}}\sum_{i=1}^{r+n}\tau'\big(\varsigma_{1,i}(v)\otimes\varsigma_{1,i}(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))\big)$$

we get

$$\partial^{\mathrm{Gr}}\left(\sum_{\tau\in S_{r+n}}\tau(v)\otimes\tau(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))\right)= \tag{6.76}$$

$$=\sum_{\sigma\in\mathrm{Sh}_{r+n,1}}\sum_{\tau\in S_{r+n}}\sigma\big(\iota(\{a_1,a_2\})\circ_1\tau(v)\big)\otimes\sigma\big(\tau(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))b_{r+n+1}\big)$$

$$-(-1)^{|v|}\sum_{\substack{\tau'\in S_{3,4,\dots,r+n+1}\\ \lambda\in\mathrm{Sh}_{2,r+n-1}}}\sum_{i=1}^{r}\lambda\Big(\tau'\circ\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$

$$\otimes b_1b_2b_{\tau'(3)}\dots b_{\tau'(r+1)}\ w(b_{\tau'(r+2)},\dots,b_{\tau'(r+1+n)})\Big)$$

$$-(-1)^{|v|}\sum_{\substack{\tau'\in S_{3,4,\dots,r+n+1}\\ \lambda\in\mathrm{Sh}_{2,r+n-1}}}\sum_{i=r+1}^{r+n}\lambda\Big(\tau'\circ\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$

$$\otimes b_{\tau'(3)}\dots b_{\tau'(r+2)}b_1\ w(b_{\tau'(r+3)},\dots,b_{\tau'(i+1)},b_2,b_{\tau'(i+2)},\dots,b_{\tau'(r+1+n)})\Big)$$

$$-(-1)^{|v|}\sum_{\substack{\tau'\in S_{3,4,\dots,r+n+1}\\ \lambda\in\mathrm{Sh}_{2,r+n-1}}}\sum_{i=r+1}^{r+n}\lambda\Big(\tau'\circ\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$

$$\otimes b_{\tau'(3)}\dots b_{\tau'(r+2)}b_2\ w(b_{\tau'(r+3)},\dots,b_{\tau'(i+1)},b_1,b_{\tau'(i+2)},\dots,b_{\tau'(r+1+n)})\Big),$$

where θ_i is the following permutation in S_{r+1+n}

$$\theta_i=\begin{pmatrix}1 & 2 & \dots & i-1 & i & i+1\\ 3 & 4 & \dots & i+1 & 1 & 2\end{pmatrix}. \tag{6.77}$$

The first sum in the right hand side of (6.76) can be simplified as follows.

$$\sum_{\sigma\in\mathrm{Sh}_{r+n,1}}\sum_{\tau\in S_{r+n}}\sigma\big(\iota(\{a_1,a_2\})\circ_1\tau(v)\big)\otimes\sigma\big(\tau(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n}))b_{r+n+1}\big)=$$

$$\sum_{\lambda\in S_{r+1+n}}\lambda\big(\iota(\{a_1,a_2\})\circ_1 v\big)\otimes\lambda(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n})b_{r+n+1})= \tag{6.78}$$

$$\sum_{\lambda\in S_{r+1+n}}\lambda\circ\varsigma_{1,r+1+n}\big(\iota(\{a_1,a_2\})\circ_1 v\big)\otimes\lambda\circ\varsigma_{1,r+1+n}(b_1\dots b_r\,w(b_{r+1},\dots,b_{r+n})b_{r+n+1})=$$

$$\sum_{\lambda\in S_{r+1+n}} \lambda\big(\iota(\{a_1,a_2\})\circ_2 v\big)\otimes\lambda(b_1\dots b_{r+1}\,w(b_{r+2},\dots,b_{r+1+n})) =$$

$$\sum_{\sigma\in \mathrm{Sh}_{r+1,n}}^{(\lambda_1,\lambda_2)\in S_{r+1}\times S_n} \sigma\Big((\lambda_1,\lambda_2)\big(\iota(\{a_1,a_2\})\circ_2 v\big)\otimes\lambda_2(b_1\dots b_{r+1}\,w(b_{r+2},\dots,b_{r+1+n}))\Big) =$$

$$\sum_{\sigma\in \mathrm{Sh}_{r+1,n}}^{\tau\in\mathrm{Sh}_{1,r}} \sum_{\lambda'\in S_{2,\dots,r+1}}^{\lambda''\in S_{r+2,\dots,r+1+n}} \sigma\big(\tau\circ\lambda'\circ\lambda''\big(\iota(\{a_1,a_2\})\circ_2 v\big)$$
$$\otimes\, b_1\dots b_{r+1}\,w(b_{\lambda''(r+2)},\dots,b_{\lambda''(r+1+n)})\big)\,.$$

Thus

(6.79) The first sum in the R.H.S. of (6.76) =

$$\Upsilon_{\mathcal{O}}\left(\sum_{\tau\in\mathrm{Sh}_{1,r}}\sum_{\lambda\in S_n}\tau\big(\{a_1,a_2\}\circ_2\lambda(\mathrm{Av}_r(v))\big)\otimes\lambda(w)\right)\,.$$

Using the symmetry of the bracket $\{\ ,\ \}$, we rewrite the second sum in the right hand side of (6.76) as follows

$$-(-1)^{|v|}\sum_{\lambda\in\mathrm{Sh}_{2,r+n-1}}^{\tau'\in S_{3,4,\dots,r+n+1}}\sum_{i=1}^{r}\lambda\Big(\tau'\circ\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$

$$\otimes\, b_1b_2b_{\tau'(3)}\dots b_{\tau'(r+1)}\ w(b_{\tau'(r+2)},\dots,b_{\tau'(r+1+n)})\Big) = \tag{6.80}$$

$$-\frac{(-1)^{|v|}}{2}\sum_{\lambda\in S_{r+1+n}}\sum_{i=1}^{r}\lambda\Big(\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)\otimes b_1b_2b_3\dots b_{r+1}\ w(b_{r+2},\dots,b_{r+1+n})\Big) =$$

$$-\frac{(-1)^{|v|}}{2}\sum_{\sigma\in\mathrm{Sh}_{r+1,n}}\sum_{\lambda'\in S_{r+1}}^{\lambda''\in S_{r+2,\dots,r+1+n}}\sum_{i=1}^{r}\sigma\circ\lambda''\circ\lambda'\Big(\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$
$$\otimes\, b_1b_2\dots b_{r+1}\ w(b_{r+2},\dots,b_{r+1+n})\Big) =$$

$$-(-1)^{|v|}\sum_{\sigma\in\mathrm{Sh}_{r+1,n}}^{\tau\in\mathrm{Sh}_{2,r-1}}\sum_{\lambda_1\in S_{3,\dots,r+1}}^{\lambda_2\in S_{r+2,\dots,r+1+n}}\sum_{i=1}^{r}\sigma\circ\tau\Big(\lambda_2\circ\lambda_1\circ\theta_i\big(v\circ_i\iota(\{a_1,a_2\})\big)$$
$$\otimes\, b_1b_2\dots b_{r+1}\ w(b_{\lambda_2(r+2)},\dots,b_{\lambda_2(r+1+n)})\Big) =$$

$$-(-1)^{|v|}\sum_{\sigma\in\mathrm{Sh}_{r+1,n}}^{\tau\in\mathrm{Sh}_{2,r-1}}\sum_{\lambda_1'\in S_r}^{\lambda_2\in S_{r+2,\dots,r+1+n}}\sigma\Big(\lambda_2\circ\tau\big(\lambda_1'(v)\circ_1\{a_1,a_2\}\big)$$
$$\otimes\, b_1b_2\dots b_{r+1}\ w(b_{\lambda_2(r+2)},\dots,b_{\lambda_2(r+1+n)})\Big)\,,$$

where θ_i is defined in (6.77).

Thus

(6.81) The second sum in the R.H.S. of (6.76) =

$$-(-1)^{|v|}\Upsilon_{\mathcal{O}}\left(\sum_{\tau\in\mathrm{Sh}_{2,r-1}}\sum_{\lambda\in S_n}\lambda\Big(\tau\big(\mathrm{Av}_r(v)\circ_1\{a_1,a_2\}\big)\Big)\otimes\lambda(w)\right)\,,$$

where $\mathrm{Av}_r(v)$ is viewed as a vector in $\mathcal{O}(r+n)$ and $\tau\big(\mathrm{Av}_r(v) \circ_1 \{a_1, a_2\}\big)$ is viewed as a vector in $\mathrm{Tw}\mathcal{O}(n)$.

Due to Exercise 6.18 below,

(6.82) The combination of the last two sums in the R.H.S. of (6.76) =

$$-(-1)^{|v|} \Upsilon_{\mathcal{O}} \left(\sum_{\tau \in \mathrm{Sh}_{r,1}} \sum_{\lambda \in S_n} \sum_{i=1}^{n} \lambda\Big(\tau \circ \varsigma_{r+1,r+i}\big(\mathrm{Av}_r(v) \circ_{r+i} \iota(\{a_1, a_2\})\big)\Big) \otimes \lambda(w) \right),$$

where $\mathrm{Av}_r(v)$ is viewed as a vector in $\mathcal{O}(r+n)$ and

$$\tau \circ \varsigma_{r+1,r+i}\big(\mathrm{Av}_r(v) \circ_{r+i} \iota(\{a_1, a_2\})\big)$$

is viewed as a vector in $\mathrm{Tw}\mathcal{O}(n)$.

Comparing (6.79), (6.81), and (6.82) with the second, the first and the third sums, respectively, in the right hand side of equation (6.32) from Section 6.5, we see that the equation (6.75) indeed holds.

Proposition 6.17 is proved. □

EXERCISE 6.18. Let v be a vector in $\mathcal{O}(r+n)$ and w be a monomial in $\Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)$. Prove that

(6.83) The combination of the last two sums in the R.H.S. of (6.76) =

$$-(-1)^{|v|} \Upsilon_{\mathcal{O}} \left(\sum_{\tau \in \mathrm{Sh}_{r,1}} \sum_{\lambda \in S_n} \sum_{i=1}^{n} \lambda\Big(\tau \circ \varsigma_{r+1,r+i}\big(\mathrm{Av}_r(v) \circ_{r+i} \iota(\{a_1, a_2\})\big)\Big) \otimes \lambda(w) \right),$$

where

$$\mathrm{Av}_r(v) = \sum_{\lambda_1 \in S_r} \lambda_1(v).$$

Hint for Exercise 6.18*:* Using the symmetry of the bracket $\{\ ,\ \}$ we can rewrite the combination of the last two sums in the right hand side of (6.76) as follows:

(6.84) The combination of the last two sums in the R.H.S. of (6.76) =

$$-(-1)^{|v|} \sum_{i=r+1}^{r+n} \sum_{\lambda \in S_{r+1+n}} \lambda\Big(\theta_i\big(v \circ_i \iota(\{a_1, a_2\})\big)$$

$$\otimes\, b_2 b_3 \dots b_{r+2}\, w(b_{r+3}, \dots, b_{i+1}, b_1, b_{i+2}, \dots, b_{r+1+n})\Big) =$$

$$-(-1)^{|v|} \sum_{i=r+1}^{r+n} \sum_{\lambda \in S_{r+1+n}} \lambda \circ \varsigma^{-1}_{1,i+1}\Big(\theta_i\big(v \circ_i \iota(\{a_1, a_2\})\big)$$

$$\otimes\, b_2 b_3 \dots b_{r+2}\, w(b_{r+3}, \dots, b_{i+1}, b_1, b_{i+2}, \dots, b_{r+1+n})\Big) =$$

$$-(-1)^{|v|} \sum_{i=r+1}^{r+n} \sum_{\lambda \in S_{r+1+n}} \lambda\Big(\varsigma_{1,i}\big(v \circ_i \iota(\{a_1, a_2\})\big) \otimes b_1 b_2 \dots b_{r+1}\, w(b_{r+2}, \dots, b_{r+1+n})\Big) =$$

$$-(-1)^{|v|} \sum_{i=r+1}^{r+n} \sum_{\lambda \in S_{r+1+n}} \lambda \circ \varsigma^{-1}_{1,r+1}\Big(\varsigma_{1,i}\big(v \circ_i \iota(\{a_1, a_2\})\big) \otimes b_1 b_2 \dots b_{r+1}\, w(b_{r+2}, \dots, b_{r+1+n})\Big) =$$

$$-(-1)^{|v|} \sum_{i=r+1}^{r+n} \sum_{\lambda \in S_{r+1+n}} \lambda\Big(\varsigma_{r+1,i}\big(v \circ_i \iota(\{a_1, a_2\})\big) \otimes b_1 b_2 \dots b_{r+1}\, w(b_{r+2}, \dots, b_{r+1+n})\Big).$$

7. The operad Gra and its link to the operad Ger

Let us recall from [**42**] the operad (in $\mathsf{grVect}_{\mathbb{K}}$) of labeled graphs Gra.

To define the space $\mathsf{Gra}(n)$ (for $n \geq 1$) we introduce an auxiliary set gra_n. An element of gra_n is a labelled graph Γ with n vertices and with the additional piece of data: the set of edges of Γ is equipped with a total order. An example of an element in gra_4 is shown on figure 26. We will often use Roman numerals to specify

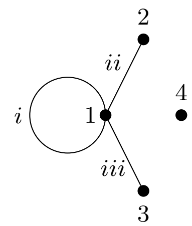


FIG. 26. The Roman numerals indicate that we chose the total order on the set of edges $(1,1) < (1,2) < (1,3)$

total orders on sets of edges. Thus the Roman numerals on figure 26 indicate that we chose the total order $(1,1) < (1,2) < (1,3)$.

The space $\mathsf{Gra}(n)$ (for $n \geq 1$) is spanned by elements of gra_n, modulo the relation $\Gamma^{\sigma} = (-1)^{|\sigma|}\Gamma$ where the elements Γ^{σ} and Γ correspond to the same labelled graph but differ only by permutation σ of edges. We also declare that the degree of a graph Γ in $\mathsf{Gra}(n)$ equals $-e(\Gamma)$, where $e(\Gamma)$ is the number of edges in Γ. For example, the graph Γ on figure 26 has 3 edges. Thus its degree is -3.

Finally, we set

$$\mathsf{Gra}(0) = \mathbf{0}\,. \tag{7.1}$$

REMARK 7.1. It clear that, if a graph $\Gamma \in \mathsf{gra}_n$ has multiple edges, then

$$\Gamma = -\Gamma$$

in $\mathsf{Gra}(n)$. Thus for every graph $\Gamma \in \mathsf{gra}_n$ with multiple edges $\Gamma = 0$ in $\mathsf{Gra}(n)$.

We will now define elementary insertions for the collection $\{\mathsf{Gra}(n)\}_{n \geq 0}$.

Let Γ and $\widetilde{\Gamma}$ be graphs representing vectors in $\mathsf{Gra}(n)$ and $\mathsf{Gra}(m)$, respectively. Let $1 \leq i \leq m$.

The vector $\widetilde{\Gamma} \circ_i \Gamma \in \mathsf{Gra}(n+m-1)$ is represented by the sum of graphs $\Gamma_{\alpha} \in \mathsf{gra}_{n+m-1}$

$$\widetilde{\Gamma} \circ_i \Gamma = \sum_{\alpha} \Gamma_{\alpha}\,, \tag{7.2}$$

where Γ_{α} is obtained by "plugging in" the graph Γ into the i-th vertex of the graph $\widetilde{\Gamma}$ and reconnecting the edges incident to the i-th vertex of $\widetilde{\Gamma}$ to vertices of Γ in all possible ways. (The index α refers to a particular way of connecting the edges incident to the i-th vertex of $\widetilde{\Gamma}$ to vertices of Γ.) After reconnecting edges we label vertices of Γ_{α} as follows:

- we leave the same labels on the first $i-1$ vertices of $\widetilde{\Gamma}$;
- we shift all labels on vertices of Γ up by $i-1$;
- finally, we shift the labels on the last $m-i$ vertices of $\widetilde{\Gamma}$ up by $n-1$.

To define the total order on edges of the graph Γ_α we declare that all edges of $\widetilde{\Gamma}$ are smaller than all edges of the graph Γ.

EXAMPLE 7.2. Let $\widetilde{\Gamma}$ (resp. Γ) be the graph depicted on figure 27 (resp. figure 28). The vector $\widetilde{\Gamma} \circ_2 \Gamma$ is shown on figure 29.

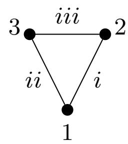


FIG. 27. A graph $\widetilde{\Gamma} \in \mathsf{gra}_3$

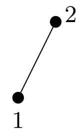


FIG. 28. A graph $\Gamma \in \mathsf{gra}_2$

$\widetilde{\Gamma} \circ_2 \Gamma \quad =$

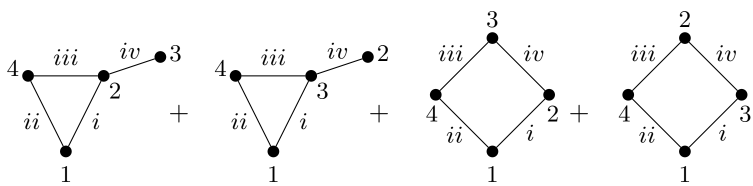
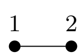
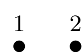


FIG. 29. The vector $\widetilde{\Gamma} \circ_2 \Gamma \in \mathsf{Gra}(4)$

The symmetric group S_n acts on $\mathsf{Gra}(n)$ in the obvious way by rearranging the labels on vertices. It is not hard to see that insertions (7.2) together with this action of S_n give on Gra an operad structure with the identity element being the unique graph in gra_1 with no edges.

It is clear that if two graphs $\widetilde{\Gamma}$ and Γ representing vectors in Gra do not have loops (i.e. cycles of length 1) then each graph in the linear combination $\widetilde{\Gamma} \circ_i \Gamma$ does not have loops either. Thus, by discarding graphs with loops, we arrive at a suboperad $\mathsf{Gra}_{\not\circlearrowleft}$ of Gra.

The graphs depicted below represent vectors in $\mathsf{Gra}_{\not\circlearrowleft}(2)$ and in $\mathsf{Gra}(2)$.

(7.3) $\qquad \Gamma_{\bullet\!-\!\bullet} =$  $\qquad \Gamma_{\bullet\,\bullet} =$ 

Later they will play a special role.

7.1. "Graphical" interpretation of the operad Ger. Since Ger is generated by the monomials $a_1 a_2$, $\{a_1, a_2\} \in \mathsf{Ger}(2)$, any map of operads

$$f : \mathsf{Ger} \to \mathcal{O}$$

is uniquely determined by its values on $a_1 a_2$ and $\{a_1, a_2\}$.

EXERCISE 7.3. Let $\Gamma_{\bullet\!-\!\bullet}$ and $\Gamma_{\bullet\,\bullet}$ be the vectors in $\mathsf{Gra}(2)$ introduced in (7.3). Prove that the assignment

$$\iota(a_1 a_2) = \Gamma_{\bullet\,\bullet}, \qquad \iota(\{a_1, a_2\}) = \Gamma_{\bullet\!-\!\bullet} \tag{7.4}$$

defines a map of operads (in $\mathsf{grVect}_{\mathbb{K}}$)

$$\iota : \mathsf{Ger} \to \mathsf{Gra}\,. \tag{7.5}$$

Notice that, one only has to check that

$$\iota\big((a_1 a_2)a_3 - a_1(a_2 a_3)\big) = 0\,,$$

$$\iota\big(\{a_1, a_2 a_3\} - \{a_1, a_2\}a_3 + a_2\{a_1, a_3\}\big) = 0\,,$$

$$\iota\big(\{\{a_1, a_2\}, a_3\} + \{\{a_2, a_3\}, a_1\} + \{\{a_3, a_1\}, a_2\}\big) = 0\,.$$

We claim that

PROPOSITION 7.4. *The map of operads $\iota : \mathsf{Ger} \to \mathsf{Gra}$ is injective.*

PROOF. Recall that due to Exercise 3.12, the monomials

$$\{a_{i_{11}}, \dots, \{a_{i_{1(p_1-1)}}, a_{i_{1p_1}}\}..\} \dots \{a_{i_{t1}}, \dots, \{a_{i_{t(p_t-1)}}, a_{i_{tp_t}}\}..\} \tag{7.6}$$

corresponding to the ordered partitions (3.30) form a basis of $\mathsf{Ger}(n)$.

Let us observe that for every ordered partition (3.30) the graph depicted on figure 30 enters the linear combination

$$\iota\big(\{a_{i_{11}}, \dots, \{a_{i_{1(p_1-1)}}, a_{i_{1p_1}}\}..\} \dots \{a_{i_{t1}}, \dots, \{a_{i_{t(p_t-1)}}, a_{i_{tp_t}}\}..\}\big)$$

with the coefficient 1.

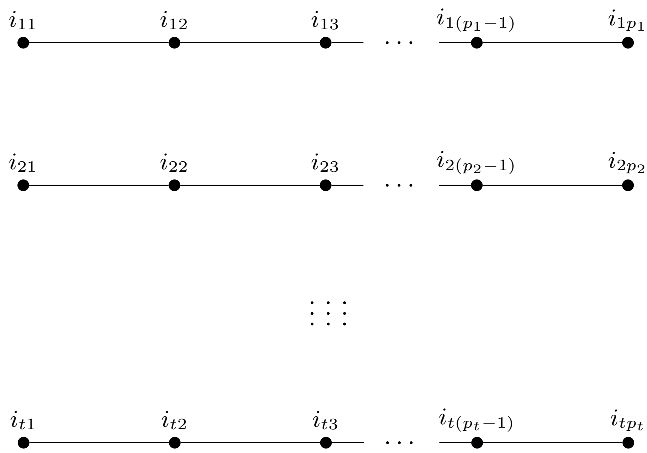


FIG. 30. The edges are ordered "left to right", "top to bottom"

Since such graphs are linearly independent in $\mathsf{Gra}(n)$, we conclude that ι is indeed injective.

The proposition is proved. □

8. The full graph complex fGC: the first steps

Let $\Gamma_{\bullet\!-\!\bullet}$ and $\Gamma_{\bullet\bullet}$ be the vectors in $\mathsf{Gra}(2)$ introduced in (7.3). Following Exercise 7.3 and Proposition 7.3 the formulas

$$\iota(\{a_1, a_2\}) = \Gamma_{\bullet\!-\!\bullet}\,, \qquad \iota(a_1 a_2) = \Gamma_{\bullet\bullet}$$

define an embedding ι of the operad Ger into the operad Gra.

Thus, restricting ι to the suboperad $\Lambda\mathsf{Lie} \subset \mathsf{Ger}$ we get an embedding

$$\Lambda\mathsf{Lie} \hookrightarrow \mathsf{Gra}\,.$$

Hence we have a canonical map of (dg) operads

$$\varphi_{\mathsf{Gra}} : \Lambda\mathsf{Lie}_\infty \to \mathsf{Gra}\,. \tag{8.1}$$

Applying the general procedure of twisting (see Section 6) to the pair $(\mathsf{Gra}, \varphi_{\mathsf{Gra}})$ we get a dg operad $\mathrm{Tw}\mathsf{Gra}$ and a dg Lie algebra

$$\mathcal{L}_{\mathsf{Gra}} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra}) \tag{8.2}$$

which acts on the operad $\mathrm{Tw}\mathsf{Gra}$.

Following [**42**] we denote the dg Lie algebra $\mathcal{L}_{\mathsf{Gra}}$ by fGC. In other words,

$$\mathsf{fGC} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra}) \tag{8.3}$$

The vector

$$\Gamma_{\bullet\!-\!\bullet} \in \mathsf{fGC} \tag{8.4}$$

is a Maurer-Cartan element in fGC and the differential on fGC is given by the formula:

$$\partial = \mathrm{ad}_{\Gamma_{\bullet\!-\!\bullet}}\,. \tag{8.5}$$

DEFINITION 8.1. The cochain complex fGC (8.3) with the differential (8.5) is called the *full graph complex*.

In this subsection we take a first few steps towards analyzing the cochain complex fGC.

Unfolding the definition of the convolution Lie algebra we get

$$\mathsf{fGC} = \prod_{n=1}^{\infty} \mathrm{Hom}_{S_n}\big(\Lambda^2\mathsf{coCom}(n), \mathsf{Gra}(n)\big) = \prod_{n=1}^{\infty} \mathrm{Hom}_{S_n}\big(\mathbf{s}^{2-2n}\mathbb{K}, \mathsf{Gra}(n)\big) = \tag{8.6}$$

$$= \prod_{n=1}^{\infty} \mathbf{s}^{2n-2}\big(\mathsf{Gra}(n)\big)^{S_n}\,.$$

In other words, vectors in fGC are infinite sums

$$\gamma = \sum_{n=1}^{\infty} \gamma_n \tag{8.7}$$

of S_n-invariant vectors $\gamma_n \in \mathbf{s}^{2n-2}\mathsf{Gra}(n)$.

The vector space

$$\mathbf{s}^{2n-2}\big(\mathsf{Gra}(n)\big)^{S_n} \tag{8.8}$$

is spanned by vectors of the form

$$\mathrm{Av}(\Gamma) = \sum_{\sigma \in S_n} \sigma(\Gamma) \tag{8.9}$$

where Γ is an element in gra_n. In other words, formula (8.9) defines a surjective $\mathbb{K}$-linear map

$$\mathrm{Av} : \mathbb{K}\langle \mathsf{gra}_n \rangle \twoheadrightarrow \mathbf{s}^{2n-2}\big(\mathsf{Gra}(n)\big)^{S_n}. \tag{8.10}$$

To describe the kernel of the map Av, we observe that $\mathrm{Av}(\Gamma) = 0$ if and only if the underlying unlabeled graph has an automorphism which induces an odd permutation on the set of edges. In this case we say that the element $\Gamma \in \mathsf{gra}_n$ is *odd*. Otherwise, we say that the element $\Gamma \in \mathsf{gra}_n$ is *even*. For example, the square depicted on figure 31 is odd and the pentagon depicted on figure 32 is even. It is obvious that the property of being even or odd depends only on the isomorphism class of the underlying unlabeled graph.

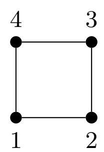


FIG. 31. We choose this order on the set of edges: $(1, 2) < (2, 3) < (3, 4) < (4, 1)$

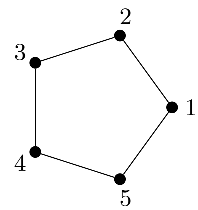


FIG. 32. We choose this order on the set of edges: $(1, 2) < (2, 3) < (3, 4) < (4, 5) < (5, 1)$

Let us consider a pair of even elements $\Gamma, \Gamma' \in \mathsf{gra}_n$ whose underlying unlabeled graphs are isomorphic. Any isomorphism of the underlying unlabeled graphs gives us a bijection from the set $E(\Gamma)$ of edges of Γ to the set $E(\Gamma')$ of edges of Γ'. Since both sets $E(\Gamma)$ and $E(\Gamma')$ are totally ordered, this bijection determines a permutation $\sigma \in S_m$ where $m = |E(\Gamma)|$. Furthermore, since Γ and Γ' are even, such permutations σ are either all even or all odd. In the later case, we say that even elements Γ and Γ' are *opposite* and the former case we say that even elements Γ and Γ' are *concordant*.

It is clear that

PROPOSITION 8.2. *The kernel of the map* Av (8.10) *is spanned by vectors of the form*

$$\Gamma, \qquad \Gamma_1 - \Gamma_2, \qquad \Gamma'_1 + \Gamma'_2, \tag{8.11}$$

where Γ is odd, (Γ_1, Γ_2) is a pair of concordant (even) graphs, and (Γ'_1, Γ'_2) is a pair of opposite (even) graphs. $\square$

In view of Proposition 8.2, we may identify the vector space (8.8) with the quotient of $\mathbb{K}\langle \mathsf{gra}_n \rangle$ by the subspace spanned by vectors (8.11).

The following proposition gives us a convenient description of the differential (8.5) on fGC:

PROPOSITION 8.3. *For every (even) element* $\Gamma \in \mathsf{gra}_n$ *we have*

$$\partial\big(\mathrm{Av}(\Gamma)\big) = \mathrm{Av}(\Gamma_{\bullet\!-\!\bullet} \circ_1 \Gamma) - (-1)^{e(\Gamma)}\, \frac{1}{2} \sum_{i=1}^{n} \mathrm{Av}(\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}) \tag{8.12}$$

where $e(\Gamma)$ *is the number of edges of* Γ *. Moreover, if* Γ *is a connected (even) graph in* gra_n *with at least one edge, then*

$$\partial\big(\mathrm{Av}(\Gamma)\big) = -\frac{(-1)^{e(\Gamma)}}{2} \sum_{i=1}^{n} \mathrm{Av}(\Gamma'_i) \tag{8.13}$$

where Γ'_i *is obtained from* $\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}$ *by discarding all graphs in which either vertex* i *or vertex* $i+1$ *has valency* 1 *.*

PROOF. It is straightforward to verify the first claim by unfolding the definition of the Lie bracket on $\mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra})$. The second claim follows from the observation that

$$\mathrm{Av}(\Gamma_{\bullet\!-\!\bullet} \circ_1 \Gamma) = \frac{(-1)^{e(\Gamma)}}{2} \sum_{i=1}^{n} \mathrm{Av}(\widetilde{\Gamma}_i)$$

where $\widetilde{\Gamma}_i$ is obtained from the linear combination $\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}$ by keeping only the graphs in which either vertex i or vertex $i+1$ has valency 1 . $\square$

EXERCISE 8.4. Let $\Gamma_{\bullet}$ be the graph in gra_1 which consists of a single vertex. Show that

$$\partial \Gamma_{\bullet} = \Gamma_{\bullet\!-\!\bullet}\,. \tag{8.14}$$

Let $\Gamma_{\circlearrowleft}$ be the graph in gra_1 with consists of a single loop. Show that

$$\partial \Gamma_{\circlearrowleft} = 0\,.$$

Thus $\Gamma_{\circlearrowleft}$ represents a degree -1 (non-trivial) cocycle in fGC.

8.1. The subcomplex of cables. Let us denote by $\mathcal{K}_{-}$ the subspace of fGC which is spanned by vectors

$$\mathrm{Av}(\Gamma)$$

where Γ is either the single vertex graph $\Gamma_{\bullet}$ or a graph Γ^{-}_{l} depicted on figure 33 for $l \ge 2$. For example, $\Gamma^{-}_{2} = \Gamma_{\bullet\!-\!\bullet}$.

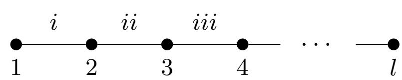


FIG. 33. The graph Γ^{-}_{l}

It is easy to see that the vectors $\mathrm{Av}(\Gamma^{-}_{l})$ have degrees

$$|\mathrm{Av}(\Gamma^{-}_{l})| = l-1\,,$$

$$\mathrm{Av}(\Gamma^{-}_{l}) = 0\,, \qquad \text{if} \qquad l = 0, 3 \bmod 4,$$

and

$$\mathrm{Av}(\Gamma^{-}_{l}) \neq 0\,, \qquad \text{if} \qquad l = 1, 2 \bmod 4\,.$$

Furthermore, due to Exercise 8.6 below,

$$\partial \mathrm{Av}(\Gamma^{-}_{4k+1}) = \mathrm{Av}(\Gamma^{-}_{4k+2})$$

for all $k \geq 1$.

Combining these observations with equation (8.14) we conclude that

PROPOSITION 8.5. *The subspace $\mathcal{K}_{\text{—}}$ is subcomplex of* fGC. *Moreover $\mathcal{K}_{\text{—}}$ is acyclic.* □

We call $\mathcal{K}_{\text{—}}$ the subcomplex of *cables.*

EXERCISE 8.6. Let Γ_l^- be the family of graphs for $l \geq 2$ defined on figure 33. Prove that for every $k \geq 1$

$$\partial \mathrm{Av}(\Gamma^-_{4k+1}) = \mathrm{Av}(\Gamma^-_{4k+2}).$$

8.2. The subcomplex of polygons. Let us denote by $\mathcal{K}_{\diamond}$ the subspace of fGC which is spanned by vectors of the form

$$\mathrm{Av}(\Gamma^{\diamond}_m),$$

where $\Gamma^{\diamond}_m$ is the element of gra_m depicted on figure 34. For example, $\Gamma^{\diamond}_1$ is the

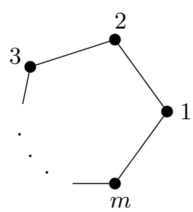


FIG. 34. The edges are ordered as follows $(1,2) < (2,3) < \cdots < (m-1,m) < (m,1)$

graph $\Gamma_{\circlearrowleft}$ in gra_1 which consists of a single loop.

Due to Exercise 8.7 below, $\mathcal{K}_{\diamond}$ is a subcomplex of fGC with

$$H^{\bullet}(\mathcal{K}_{\diamond}) \cong \bigoplus_{q \geq 1} \mathbf{s}^{4q-1}\,\mathbb{K}. \tag{8.15}$$

We call $\mathcal{K}_{\diamond}$ the subcomplex of polygons.

EXERCISE 8.7. Show that the graph $\Gamma^{\diamond}_m$ is odd if $m \neq 1 \bmod 4$ and even if $m = 1 \bmod 4$. Using equation (8.13), prove that for every $q \geq 0$

$$\mathrm{Av}(\Gamma^{\diamond}_{4q+1})$$

is a non-trivial cocycle of fGC of degree $4q-1$.

8.3. The connected part $\mathsf{fGC}_{\mathbf{conn}}$ of fGC. Let us denote by $\mathsf{fGC}_{\mathrm{conn}}$ the subspace of fGC which consists of infinite sums

$$\gamma = \sum_{n=1}^{\infty} \gamma_n\,, \qquad \gamma_n \in \mathbf{s}^{2n-2}\big(\mathsf{Gra}(n)\big)^{S_n}$$

where γ_n is a linear combination of connected graphs in $\mathsf{Gra}(n)$.

It is clear that $\mathsf{fGC}_{\mathrm{conn}}$ is a Lie subalgebra of fGC and hence a subcomplex. It is also clear that

$$\mathsf{fGC} = \mathbf{s}^{-2}\widehat{S}\big(\mathbf{s}^2\,\mathsf{fGC}_{\mathrm{conn}}\big)\,, \tag{8.16}$$

where $\widehat{S}$ denotes the completed symmetric algebra. Thus the question of computing cohomology of fGC reduces to the question of computing cohomology of its connected part $\mathsf{fGC}_{\mathrm{conn}}$.

9. Analyzing the dg operad $\mathrm{Tw}\mathsf{Gra}$

According to the general procedure of twisting

$$\mathrm{Tw}\mathsf{Gra}(n) = \prod_{r=0}^{\infty} \mathbf{s}^{2r} \Big(\mathsf{Gra}(r+n)\Big)^{S_r} . \tag{9.1}$$

In other words, vectors in $\mathrm{Tw}\mathsf{Gra}(n)$ are infinite linear combinations

$$\gamma = \sum_{r=0}^{\infty} \gamma_r \,, \tag{9.2}$$

where γ_r is an S_r invariant vector in $\mathbf{s}^{2r}\mathsf{Gra}(r+n)$.

It is clear that the first r vertices and the last n vertices in graphs of γ_r play different roles. We call the first r vertices *neutral* and the remaining n vertices *operational*. It is convenient to represent neutral (reps. operational) vertices on figures by small black circles (reps. small white circles). In this way, the same element of gra_m may be treated as a vector in different spaces of the operad $\mathrm{Tw}\mathsf{Gra}$. For example, the graph on figure 35 represents a vector in $\mathrm{Tw}\mathsf{Gra}(0)$, the graph on figure 36 represents a vector in $\mathrm{Tw}\mathsf{Gra}(1)$, and the graph on figure 37 represents a vector in $\mathrm{Tw}\mathsf{Gra}(2)$.

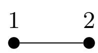


FIG. 35. A vector in $\mathrm{Tw}\mathsf{Gra}(0)$

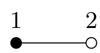


FIG. 36. A vector in $\mathrm{Tw}\mathsf{Gra}(1)$

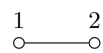


FIG. 37. A vector in $\mathrm{Tw}\mathsf{Gra}(2)$

It is obvious that the vector space

$$\mathbf{s}^{2r}\big(\mathsf{Gra}(r+n)\big)^{S_r} \subset \mathrm{Tw}\mathsf{Gra}(n)$$

is spanned by vectors of the form

$$\mathrm{Av}_r(\Gamma) = \sum_{\sigma \in S_r} \sigma(\Gamma) \,, \tag{9.3}$$

where Γ is an element in gra_{r+n} .

In other words, equation (9.3) defines a surjective map

$$\mathrm{Av}_r : \mathbb{K}\langle \mathsf{gra}_{r+n} \rangle \twoheadrightarrow \mathbf{s}^{2r}\big(\mathsf{Gra}(r+n)\big)^{S_r} . \tag{9.4}$$

For an element $\Gamma \in \mathsf{gra}_{r+n}$ we denote by Γ^{oub} the partially labeled graph which is obtained from Γ by forgetting labels on neutral vertices and shifting labels on operational vertices down by r . Note that, since Γ^{oub} has unlabeled vertices, it may have non-trivial automorphisms.

It is obvious that $\mathrm{Av}_r(\Gamma) = 0$ if and only if Γ^{oub} has an automorphism which induces an odd permutation on the set of edges. In this case, we say that an element $\Gamma \in \mathsf{gra}_{r+n}$ is *r-odd*. Otherwise, we say that Γ is *r-even*.

Let us consider two r-even elements $\Gamma, \Gamma' \in \mathsf{gra}_{r+n}$ whose underlying partially labeled graphs Γ^{oub} and $(\Gamma')^{\mathrm{oub}}$ are isomorphic. Any isomorphism from Γ^{oub} to $(\Gamma')^{\mathrm{oub}}$ gives us a bijection from the set $E(\Gamma)$ of edges of Γ to the set $E(\Gamma')$ of edges of Γ'. Since both sets $E(\Gamma)$ and $E(\Gamma')$ are totally ordered, this bijection determines a permutation $\sigma \in S_e$ where $e = |E(\Gamma)|$. Furthermore, since Γ and Γ' are r-even, such permutations σ are either all even or all odd. In the latter case, we say that r-even elements Γ and Γ' are *r-opposite* and in the former case we say that even elements Γ and Γ' are *r-concordant*.

It is clear that

Proposition 9.1. *The kernel of the map* Av_r (9.4) *is spanned by vectors of the form*

$$\Gamma, \qquad \Gamma_1 - \Gamma_2, \qquad \Gamma'_1 + \Gamma'_2\,, \tag{9.5}$$

where Γ is r-odd, (Γ_1, Γ_2) is a pair of r-concordant (r-even) graphs, and (Γ'_1, Γ'_2) is a pair of r-opposite (r-even) graphs in gra_{r+n}. $\square$

In the following proposition we give a convenient formula for the differential on $\mathrm{Tw}\mathsf{Gra}$.

Proposition 9.2. *Let Γ be an r-even element in gra_{r+n}. Then*

$$\partial^{\mathrm{Tw}}\mathrm{Av}_r(\Gamma) = \mathrm{Av}_{r+1}\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma\big) - (-1)^{e(\Gamma)}\mathrm{Av}_{r+1}\Big(\sum_{i=1}^{n} \varsigma_{r+1,r+i}\big(\Gamma \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big)\Big) \tag{9.6}$$

$$-\frac{(-1)^{e(\Gamma)}}{2}\sum_{i=1}^{r}\mathrm{Av}_{r+1}\big(\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}\big)\,,$$

where $e(\Gamma)$ is the number of edges of Γ, $\Gamma_{\bullet\!-\!\bullet}$ is defined in (7.3), *and $\varsigma_{r+1,r+i}$ is the cycle $(r+1, r+2, \dots, r+i) \in S_{r+1+n}$.*

Remark 9.3. We should remark that the vector $\partial^{\mathrm{Tw}}\mathrm{Av}_r(\Gamma)$ is a linear combination of graphs in gra_{r+1+n} in which the first $r+1$ vertices are treated as neutral. Thus vertices with labels $r+1$ and $r+i+1$ in graphs in $\varsigma_{r+1,r+i}\big(\Gamma \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big)$ come from $\Gamma_{\bullet\!-\!\bullet}$. The vertex with label $r+1$ is treated as neutral and the vertex with label $r+i+1$ is treated as operational.

Proof. Adapting general formula (6.32) to the case when $\mathcal{O} = \mathsf{Gra}$ we get

$$\partial^{\mathrm{Tw}}\mathrm{Av}_r(\Gamma) = \sum_{\tau \in \mathrm{Sh}_{1,r}} \tau\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \mathrm{Av}_r(\Gamma)\big)$$

$$-(-1)^{e(\Gamma)} \sum_{\tau' \in \mathrm{Sh}_{r,1}} \sum_{i=1}^{n} \tau' \circ \varsigma_{r+1,r+i}\big(\mathrm{Av}_r(\Gamma) \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big) \tag{9.7}$$

$$-(-1)^{e(\Gamma)} \sum_{\lambda \in \mathrm{Sh}_{2,r-1}} \big(\mathrm{Av}_r(\Gamma) \circ_1 \Gamma_{\bullet\!-\!\bullet}\big)\,.$$

Using the obvious identity

$$\mathrm{Av}_r(\Gamma) = \sum_{i=1}^{r} \sum_{\sigma' \in S_{2,\dots,r}} \sigma' \circ \varsigma_{1,i}(\Gamma),$$

axioms of operad, and S_2-invariance of $\Gamma_{\bullet\!-\!\bullet}$ we rewrite the last sum in (9.7) as follows

$$\sum_{\lambda \in \mathrm{Sh}_{2,r-1}} \lambda\big(\mathrm{Av}_r(\Gamma) \circ_1 \Gamma_{\bullet\!-\!\bullet}\big) = \sum_{i=1}^{r} \sum_{\sigma \in S_{r+1}}^{\sigma(i)<\sigma(i+1)} \sigma\big(\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}\big) =$$

$$\frac{1}{2} \sum_{\sigma \in S_{r+1}} \sum_{i=1}^{r} \sigma\big(\Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}\big) = \frac{1}{2}\mathrm{Av}_{r+1}\Big(\sum_{i=1}^{r} \Gamma \circ_i \Gamma_{\bullet\!-\!\bullet}\Big).$$

The first sum in the right hand side of (9.7) can be rewritten as

$$\sum_{\tau \in \mathrm{Sh}_{1,r}} \tau\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \mathrm{Av}_r(\Gamma)\big) = \sum_{\tau \in \mathrm{Sh}_{1,r}} \sum_{\sigma' \in S_{2,\dots,r+1}} \tau \circ \sigma'\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma\big) =$$

$$\sum_{\sigma \in S_{r+1}} \big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma\big) = \mathrm{Av}_{r+1}\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma\big),$$

where $S_{2,\dots,r+1}$ denotes the permutation group of the set $\{2,\dots,r+1\}$.

As for the second sum in the right hand side of (9.7), we have

$$\sum_{\tau' \in \mathrm{Sh}_{r,1}} \sum_{i=1}^{n} \tau' \circ \varsigma_{r+1,r+i}\big(\mathrm{Av}_r(\Gamma) \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big) =$$

$$\sum_{\tau' \in \mathrm{Sh}_{r,1}} \sum_{i=1}^{n} \sum_{\sigma' \in S_r} \tau' \circ \varsigma_{r+1,r+i} \circ \sigma'\big(\Gamma \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big) =$$

$$\sum_{i=1}^{n} \sum_{\sigma \in S_{r+1}} \sigma \circ \varsigma_{r+1,r+i}\big(\Gamma \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big) = \mathrm{Av}_{r+1}\Big(\sum_{i=1}^{n} \varsigma_{r+1,r+i}\big(\Gamma \circ_{r+i} \Gamma_{\bullet\!-\!\bullet}\big)\Big).$$

Thus, equation (9.6) indeed holds. □

9.1. The Euler characteristic trick. Let us consider sums (9.2) satisfying

PROPERTY 9.4. For every $r \ge 0$, each graph in the linear combination γ_r has Euler characteristic χ.

Using equation (9.6), it is not hard to see that the subspace of such sums is a subcomplex in $\mathrm{Tw}\mathsf{Gra}(n)$. We denote this subcomplex by

$$\mathrm{Tw}\mathsf{Gra}_\chi(n). \tag{9.8}$$

We claim that

PROPOSITION 9.5. *For every triple of integers $n \ge 0, m$ and χ the subspace $\mathrm{Tw}\mathsf{Gra}_\chi(n)^m$ of degree m vectors in $\mathrm{Tw}\mathsf{Gra}_\chi(n)$ is spanned by graphs with*

$$e = 2(n-\chi) + m \tag{9.9}$$

edges and

$$r = n + m - \chi \tag{9.10}$$

neutral vertices. In particular, the subspace $\mathrm{Tw}\mathsf{Gra}_\chi(n)^m$ is finite dimensional.

PROOF. Recall that for every graph $\Gamma \in \mathsf{gra}_{r+n}$ the vector $\mathrm{Av}_r(\Gamma) \in \mathrm{Tw}\mathsf{Gra}(n)$ has degree

$$\big|\mathrm{Av}_r(\Gamma)\big| = 2r - e\,,$$

where e is the number of edges of Γ.

Hence, if $\mathrm{Av}_r(\Gamma) \in \mathrm{Tw}\mathsf{Gra}_\chi(n)^m$ then

$$m = 2r - e\,, \tag{9.11}$$

and

$$\chi = n + r - e\,. \tag{9.12}$$

Subtracting (9.11) from (9.12), we get

$$\chi - m = n - r\,.$$

Therefore,

$$r = n + m - \chi$$

and

$$e = 2n - 2\chi + m\,.$$

Thus the proposition follows from the fact that the number of graphs with a fixed number of vertices and a fixed number of edges is finite. □

Proposition 9.5 has the following useful corollary.

COROLLARY 9.6. *The cochain complex* $\mathrm{Tw}\mathsf{Gra}(n)$ *decomposes into the product of sub-complexes*

$$\mathrm{Tw}\mathsf{Gra}(n) = \prod_{\chi\in\mathbb{Z}} \mathrm{Tw}\mathsf{Gra}_\chi(n)\,. \tag{9.13}$$

PROOF. Let

$$\gamma = \sum_{r=1}^{\infty} \gamma_r\,, \qquad \gamma_r \in \mathbf{s}^{2r}\big(\mathsf{Gra}(r+n)\big)^{S_r}$$

be a vector of degree m.

Equations (9.11) and (9.12) imply that for every r

$$\gamma_r \in \mathrm{Tw}\mathsf{Gra}_\chi(n)$$

where

$$\chi = n + m - r\,.$$

Thus

$$\mathrm{Tw}\mathsf{Gra}(n) \subset \prod_{\chi\in\mathbb{Z}} \mathrm{Tw}\mathsf{Gra}_\chi(n)\,.$$

The inclusion

$$\prod_{\chi\in\mathbb{Z}} \mathrm{Tw}\mathsf{Gra}_\chi(n) \subset \mathrm{Tw}\mathsf{Gra}(n)$$

is proved in a similar way. □

REMARK 9.7. We will often need to prove that any cocycle in $\mathrm{Tw}\mathsf{Gra}(n)$ or a similar cochain complex is cohomologous to a cocycle satisfying a certain property. Proposition 9.5 and Corollary 9.6 (or its corresponding versions) will allow us to reduce such questions to the corresponding questions for finite sums of graphs. We will refer to this maneuver as *the Euler characteristic trick*.

9.2. The suboperads $\mathsf{Graphs}^\sharp \subset \mathsf{Graphs}^\sharp \subset \mathbf{Tw}\mathsf{Gra}$. Let us denote by $\mathsf{fGraphs}^\sharp(n)$ the subspace of $\mathrm{Tw}\mathsf{Gra}(n)$ which consists of linear combinations (9.2) satisfying

PROPERTY 9.8. If a connected component of a graph in γ_r for some $r > 0$ has no operational vertices then this connected component has at least one vertex of valency ≥ 3.

REMARK 9.9. It is not hard to see that, if all vertices of a connected graph Γ have valencies ≤ 2 then Γ is isomorphic to one of the graphs in the list: $\Gamma_\bullet$, Γ^-_l (see figure 33), or $\Gamma^\diamond_m$ (see figure 34). In other words, $\mathsf{fGraphs}^\sharp(n)$ is obtained from $\mathrm{Tw}\mathsf{Gra}(n)$ by "throwing away" graphs which have connected components $\Gamma_\bullet$, Γ^-_l (see figure 33), or $\Gamma^\diamond_m$ (see figure 34) with all neutral vertices.

Let us also denote by $\mathsf{Graphs}^\sharp(n)$ the subspace of $\mathrm{Tw}\mathsf{Gra}(n)$ which consists of linear combinations (9.2) whose neutral vertices all have valencies ≥ 3.

We claim that

PROPOSITION 9.10. *Both* $\mathsf{Graphs}^\sharp(n)$ *and* $\mathsf{fGraphs}^\sharp(n)$ *are subcomplexes of* $\mathrm{Tw}\mathsf{Gra}(n)$

$$\mathsf{Graphs}^\sharp(n) \subset \mathsf{fGraphs}^\sharp(n) \subset \mathrm{Tw}\mathsf{Gra}(n)\,. \tag{9.14}$$

Moreover, the collections

$$\{\mathsf{Graphs}^\sharp(n)\}_{n\geq 0}\,, \qquad \{\mathsf{fGraphs}^\sharp(n)\}_{n\geq 0}$$

are suboperads of $\mathrm{Tw}\mathsf{Gra}$.

PROOF. The only non-obvious statement in this proposition is that the subspace $\mathsf{Graphs}^\sharp(n)$ is closed with respect to the differential ∂^{Tw}.

So let us denote by Γ an r-even graph in gra_{r+n} whose neutral vertices all have valencies ≥ 3 and analyze the right hand side of (9.6).

All graphs in the first linear combination in the right hand side of (9.6) have a univalent neutral vertex. However, it is not hard to see that they cancel with the corresponding terms in the second and the third linear combinations in the right hand side of (9.6).

Graphs with bivalent neutral vertices come from both the second and third linear combinations of the right hand side of (9.6). Again, it is not hard to see that these contributions cancel each other.

The proposition is proved. □

The goal of this subsection is to prove that

PROPOSITION 9.11. *The embedding*

$$\mathrm{emb}^\sharp_1 : \mathsf{Graphs}^\sharp(n) \hookrightarrow \mathsf{fGraphs}^\sharp(n) \tag{9.15}$$

is a quasi-isomorphism.

PROOF. Let us denote by $\mathsf{graphs}^\sharp(n)$ (resp. $\mathsf{fgraphs}^\sharp(n)$) the subcomplex of $\mathsf{Graphs}^\sharp(n)$ (resp. $\mathsf{fGraphs}^\sharp(n)$) which consists of **finite** linear combinations (9.2) in $\mathsf{Graphs}^\sharp(n)$ (resp. $\mathsf{fGraphs}^\sharp(n)$). In other words,

(9.16)
$$\mathsf{graphs}^\sharp(n) := \mathsf{Graphs}^\sharp(n) \cap \mathrm{Tw}^{\oplus}\mathsf{Gra}\,, \qquad \mathsf{fgraphs}^\sharp(n) := \mathsf{fGraphs}^\sharp(n) \cap \mathrm{Tw}^{\oplus}\mathsf{Gra}\,.$$

Next we observe that the cochain complexes $\mathsf{Graphs}^\sharp(n)$ and $\mathsf{fGraphs}^\sharp(n)$ admit decompositions with respect to the Euler characteristic

$$\mathsf{Graphs}^\sharp(n) = \prod_{\chi\in\mathbb{Z}} \mathsf{Graphs}^\sharp(n) \cap \mathrm{TwGra}_\chi(n)\,,$$

$$\mathsf{fGraphs}^\sharp(n) = \prod_{\chi\in\mathbb{Z}} \mathsf{fGraphs}^\sharp(n) \cap \mathrm{TwGra}_\chi(n)$$

and Proposition 9.5 implies that the subspace of elements of fixed degree in $\mathsf{Graphs}^\sharp(n) \cap \mathrm{TwGra}_\chi(n)$ and in $\mathsf{fGraphs}^\sharp(n) \cap \mathrm{TwGra}_\chi(n)$ is spanned by a finite number of graphs.

Thus, in virtue of Remark 9.7, it suffices to prove that the embedding

$$\mathsf{graphs}^\sharp(n) \hookrightarrow \mathsf{fgraphs}^\sharp(n) \tag{9.17}$$

is a quasi-isomorphism.

Let Γ be an element in gra_{r+n} such that $\mathrm{Av}_r(\Gamma)$ represents a vector in $\mathsf{fgraphs}^\sharp(n)$. Let us denote by $\nu_2(\Gamma)$ the number of neutral vertices having valency 2.

It is clear that the linear combination

$$\partial^{\mathrm{Tw}}\mathrm{Av}(\Gamma)$$

may involve only graphs Γ' with $\nu_2(\Gamma') = \nu_2(\Gamma)$ or $\nu_2(\Gamma') = \nu_2(\Gamma) + 1$.

Thus we may introduce on the complex $\mathsf{fgraphs}^\sharp(n)$ an ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}\mathsf{fgraphs}^\sharp(n) \subset \mathcal{F}^{m}\mathsf{fgraphs}^\sharp(n) \subset \mathcal{F}^{m+1}\mathsf{fgraphs}^\sharp(n) \subset \dots \tag{9.18}$$

where $\mathcal{F}^{m}\mathsf{fgraphs}^\sharp(n)$ consists of vectors $\gamma \in \mathsf{fgraphs}^\sharp(n)$ which only involve graphs Γ satisfying the inequality

$$\nu_2(\Gamma) - |\gamma| \le m\,.$$

It is clear that

$$\mathcal{F}^{m}\mathsf{fgraphs}^\sharp(n)$$

does not have non-zero vectors in degree $< -m$. Therefore, the filtration (9.18) is locally bounded from the left. Furthermore, since $\mathsf{fgraphs}^\sharp(n)$ consists of finite sums of graphs,

$$\mathsf{fgraphs}^\sharp(n) = \bigcup_m \mathcal{F}^{m}\mathsf{fgraphs}^\sharp(n)\,.$$

In other words, the filtration (9.18) is cocomplete.

It is also clear that the differential ∂^{Gr} on the associated graded complex

$$\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n)) = \bigoplus_m \mathcal{F}^{m}\mathsf{fgraphs}^\sharp(n) \Big/ \mathcal{F}^{m-1}\mathsf{fgraphs}^\sharp(n)\,. \tag{9.19}$$

is obtained from ∂^{Tw} by keeping only the terms which raise the number of the bivalent neutral vertices.

Thus, since $\mathsf{graphs}^\sharp(n)$ is a subcomplex of $\mathsf{fgraphs}^\sharp(n)$, we conclude that

$$\mathsf{graphs}^\sharp(n)^k \subset \mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)^k \;\cap\; \ker \partial^{\mathrm{Gr}}\,,$$

where $\mathsf{graphs}^\sharp(n)^k$ (resp. $\mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)^k$) denotes the subspace of degree k vectors in $\mathsf{graphs}^\sharp(n)$ (resp. in $\mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)$).

To complete the proof of the proposition, we need the following technical lemma which is proved in Subsection 9.2.2 below.

LEMMA 9.12. *For the filtration* (9.18) *on* $\mathsf{fgraphs}^\sharp(n)$ *we have*

$$H^k\Big(\mathcal{F}^m\mathsf{fgraphs}^\sharp(n)/\mathcal{F}^{m-1}\mathsf{fgraphs}^\sharp(n)\Big) = 0 \tag{9.20}$$

for all $m > -k$ *. Moreover,*

$$\mathsf{graphs}^\sharp(n)^k = \mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)^k \ \cap\ \ker\partial^{\mathrm{Gr}}\,. \tag{9.21}$$

It is easy to see that the restriction of (9.18) to the subcomplex $\mathsf{graphs}^\sharp(n)$ gives us the "silly" filtration:

$$\mathcal{F}^m\mathsf{graphs}^\sharp(n)^k = \begin{cases}\mathsf{graphs}^\sharp(n)^k & \text{if } m\ge -k\,,\\ \mathbf{0} & \text{otherwise}\,.\end{cases} \tag{9.22}$$

The associated graded complex $\mathrm{Gr}(\mathsf{graphs}^\sharp(n))$ for this filtration has the zero differential.

Since

$$\mathcal{F}^m\mathsf{fgraphs}^\sharp(n)^k = \mathbf{0} \qquad \forall\ m < -k\,,$$

we have

$$\mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)^k\ \cap\ \ker\partial^{\mathrm{Gr}} = H^k\big(\mathcal{F}^{-k}\mathsf{fgraphs}^\sharp(n)/\mathcal{F}^{-k-1}\mathsf{fgraphs}^\sharp(n)\big)\,.$$

Thus, Lemma 9.12 implies that, the embedding (9.17) induces a quasi-isomorphism of cochain complexes

$$\mathrm{Gr}(\mathsf{graphs}^\sharp(n)) \xrightarrow{\sim} \mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))\,.$$

On the other hand, both filtrations (9.18) and (9.22) are locally bounded from the left and cocomplete.

Therefore the embedding (9.17) satisfies all the conditions of Lemma A.3 from Appendix A and Proposition 9.11 follows. □

9.2.1. *An alternative description of* $\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))$. In order to prove Lemma 9.12 we need a convenient description of the associated graded complex (9.19).

For this purpose we introduce three cochain complexes:

- The first cochain complex is the tensor algebra

$$T_a = T(\mathbb{K}\langle a\rangle) \tag{9.23}$$

  in a single variable a carrying degree 1 with the differential δ defined by the formula

$$\delta(a) = a^2\,. \tag{9.24}$$

- The second cochain complex is the truncation of the above tensor algebra

$$\underline{T}_a = \underline{T}(\mathbb{K}\langle a\rangle) = \mathbb{K}\langle a\rangle \oplus \mathbb{K}\langle a\otimes a\rangle \oplus \mathbb{K}\langle a\otimes a\otimes a\rangle \oplus \dots \tag{9.25}$$

  with the same differential (9.24).
- Finally, the third cochain complex

$$L = \mathbb{K}\langle\{\mathfrak{l}_n\}_{n>0,\ n=0,1 \bmod 4}\rangle \tag{9.26}$$

  has the basis vector $\{\mathfrak{l}_n\}_{n>0,\ n=0,1 \bmod 4}$ carrying degrees

$$|\mathfrak{l}_n| = n-2\,.$$

  The differential on (9.26) is given by the formulas:

$$\delta(\mathfrak{l}_{4k}) = -\mathfrak{l}_{4k+1}\,, \qquad \delta(\mathfrak{l}_{4k+1}) = 0\,. \tag{9.27}$$

It is easy to see that the cochain complex $(\underline{T}_a, \delta)$ is acyclic,

$$H^{\bullet}(T_a) = \begin{cases} \mathbb{K} & \text{if } \bullet = 0\,, \\ \mathbf{0} & \text{otherwise}\,, \end{cases} \tag{9.28}$$

and

$$H^{\bullet}(L) = \begin{cases} \mathbb{K} & \text{if } \bullet = -1\,, \\ \mathbf{0} & \text{otherwise}\,. \end{cases} \tag{9.29}$$

Moreover $H^0(T_a, \delta)$ is spanned by the class of 1 and $H^{-1}(L)$ is spanned by the cohomology class of $\mathfrak{l}_1$.

Next, to every pair of non-negative integers r, n satisfying $r + n > 0$ we assign an auxiliary groupoid $\mathsf{Frame}_{r,n}$. An object of this groupoid is a labeled <u>directed</u> graph $\gimel$ with $r + n$ vertices and with an additional piece of data: the set $E(\gimel)$ of edges of $\gimel$ is equipped with a total order. For our purposes, we call the first r vertices neutral and the last n vertices operational. (On figures we use small black circles (resp. small white circles) for neural (resp. operational) vertices.) Each object $\gimel \in \mathsf{Frame}_{r,n}$ obeys the following properties:

- $\gimel$ does not have bivalent neutral vertices;
- $\gimel$ does not have a connected component which consists of a single neutral vertex;
- $\gimel$ does not have a connected component which consists of a single edge which connects two neutral vertices;
- each edge adjacent to a univalent neutral vertex (if any) of $\gimel$ originates at this univalent neutral vertex;
- the set $E(\gimel)$ is ordered in such a way that edges adjacent to univalent neutral vertices (if any) are smaller than all the remaining edges;
- finally, loops of $\gimel$ (if any) are bigger than all the remaining edges.

Objects of the groupoid $\mathsf{Frame}_{r,n}$ are called *frames.*

A morphism from a frame $\gimel$ to a frame $\gimel'$ is an isomorphism of the underlying graphs which respects labels only on the operational vertices and respects neither labels on neutral vertices, nor the total order on the set of edges, nor the directions of edges.

EXAMPLE 9.13. Let $\gimel$ be the frame in $\mathsf{Frame}_{3,4}$ depicted on figure 38. Let g_1 be the automorphism of $\gimel$ which swaps the first edge with the second edge and g_2 be the automorphism of $\gimel$ which swaps the fifth edge with the sixth edge. It is obvious that $\mathrm{Aut}(\gimel)$ is generated by g_1 and g_2. Moreover, $\mathrm{Aut}(\gimel) \cong S_2 \times S_2$.

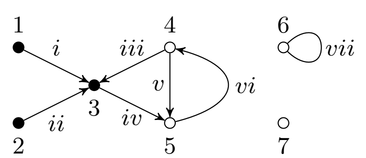


FIG. 38. As above, we use Roman numerals to specify the total order on the set of edges

The total number of edges e of any frame $\gimel$ splits into the sum

$$e = e_{\bullet} + e_{\circ} + e_{-}\,,$$

where $e_{\bullet}$ is the number of edges of $\gimel$ adjacent to univalent neutral vertices (if any), $e_{\circ}$ is the number of loops of $\gimel$ and e_{-} is the number of the remaining edges. Thus, for the frame $\gimel$ on figure 38 we have $e_{\bullet} = 2$, $e_{\circ} = 1$, and $e_{-} = 4$.

For every frame $\gimel \in \mathsf{Frame}_{r,n}$ we construct a linear map

$$F_{\gimel} : \mathbf{s}^{2r-2e_{\bullet}} \big(\underline{T}_a\big)^{\otimes\, e_{\bullet}} \ \otimes\ \big(\mathbf{s}^{-1}\, T_a\big)^{\otimes\, e_{-}} \ \otimes\ L^{\otimes\, e_{\circ}} \to \mathrm{Gr}(\mathsf{fgraphs}^{\sharp}(n))\,. \tag{9.30}$$

Namely, given a collection of monomials $a^{k_1}, a^{k_2}, \dots, a^{k_{e_{\bullet}+e_{-}}}$ with $k_i > 0$ for all $i \le k_{e_{\bullet}}$ and vectors $\mathfrak{l}_{k_{e_{\bullet}+e_{-}+1}}, \dots, \mathfrak{l}_{k_e}$ in L we form a graph $\Gamma \in \mathsf{gra}_{(r+r')+n}$ with

$$r' = \sum_{i=1}^{e_{\bullet}} (k_i - 1) + \sum_{i=e_{\bullet}+1}^{e_{\bullet}+e_{-}} k_i + \sum_{i=e_{\bullet}+e_{-}+1}^{e} (k_i - 1)$$

following these steps:

- first, for each $1 \le i \le e_{\bullet}$, we divide the i-th edge into k_i sub-edges;
- second, for each $e_{\bullet} < i \le e_{\bullet} + e_{-}$, we divide the i-th edge into $k_i + 1$ sub-edges;
- third, for each $e_{\bullet} + e_{-} < i \le e$, we divide the i-th edge[13] into k_i sub-edges;
- we declare that the additional r' vertices obtained in the above steps are neutral, label them by numbers $r+1, r+2, \dots, r+r'$ in an arbitrary possible way and shift labels on all operational vertices up by r';
- we order the set $E(\Gamma)$ of edges of Γ in the following way[14]: if $s_1, s_2 \in E(\Gamma)$ are parts of different edges of $\gimel$ then $s_1 < s_2$ provided s_1 is a part of a smaller edge; if $s_1, s_2 \in E(\Gamma)$ are parts of the same edge of $\gimel$ which is not a loop then $s_1 < s_2$ provided s_1 is closer to the origin of its edge; finally, we order sub-edges of each loop of $\gimel$ by choosing one of the two possible directions of walking around the loop.

We will refer to this graph Γ as the graph *reconstructed* from the monomial

$$(a^{k_1}, a^{k_2}, \dots, a^{k_{e_{\bullet}+e_{-}}}, \mathfrak{l}_{k_{e_{\bullet}+e_{-}+1}},\ \dots, \mathfrak{l}_{k_e}) \in \mathbf{s}^{2r-2e_{\bullet}} \big(\underline{T}_a\big)^{\otimes\, e_{\bullet}} \otimes \big(\mathbf{s}^{-1}\, T_a\big)^{\otimes\, e_{-}} \otimes L^{\otimes\, e_{\circ}}$$

using the frame $\gimel$.

It is not hard to see that the equation

$$F_{\gimel}(a^{k_1}, a^{k_2}, \dots, a^{k_{e_{\bullet}+e_{-}}}, \mathfrak{l}_{k_{e_{\bullet}+e_{-}+1}},\ \dots, \mathfrak{l}_{k_e}) = \sum_{\sigma \in S_{r+r'}} \sigma(\Gamma) \tag{9.31}$$

defines a (degree zero) map of graded vector spaces (9.30).

EXAMPLE 9.14. Let $\gimel$ be the frame in $\mathsf{Frame}_{3,4}$ depicted on figure 38. Then

$$F_{\gimel}(a, a^3, a, 1, 1, a, \mathfrak{l}_4) = \sum_{\sigma \in S_{10}} \sigma(\Gamma)\,,$$

where Γ is the element in gra_{10+4} depicted on figure 39.

[13]Note that, for each $e_{\bullet} + e_{-} < i \le e$ the i-th edge is necessarily a loop.

[14]The order on the set $E(\Gamma)$ is defined up to an even permutation.

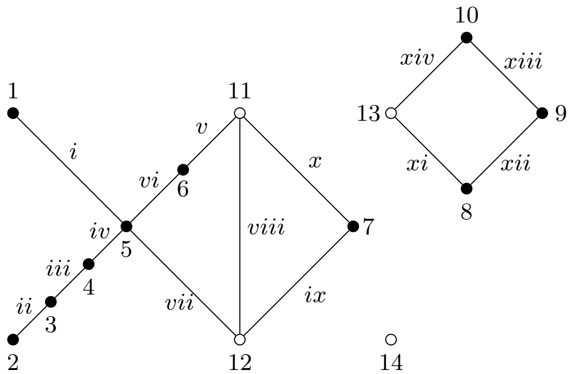


FIG. 39. The graph $\Gamma \in \mathsf{gra}_{10+4}$ corresponding to the vector $(a, a^3, a, 1, 1, a, \mathfrak{l}_4)$

Let us denote by ∂^{Gr} the differential on the associated graded complex $\mathrm{Gr}(\mathsf{fgraphs}^{\sharp}(n))$. It is clear from the definition of the filtration (9.18) on $\mathsf{fgraphs}^{\sharp}(n)$ that ∂^{Gr} is obtained from ∂^{Tw} by keeping only the terms which raise the number of the bivalent neutral vertices. Hence the image of the map $F_{\gimel}$ is closed with respect to the action the differential ∂^{Gr}. Furthermore, going through the steps of the definition of $F_{\gimel}$, it is not hard to verify that

$$\partial^{\mathrm{Gr}} \circ F_{\gimel} = F_{\gimel} \circ \delta\,. \tag{9.32}$$

Our next goal is to describe the kernel of the map $F_{\gimel}$. For this purpose, we introduce the semi-direct product

$$S_e \ltimes \big(S_2\big)^e \tag{9.33}$$

of the groups S_e and $\big(S_2\big)^e$ with the multiplication rule:

$$(\tau; \sigma_1, \dots, \sigma_e) \cdot (\lambda; \sigma'_1, \dots, \sigma'_e) = (\tau\lambda; \sigma_{\lambda(1)}\sigma'_1, \dots, \sigma_{\lambda(e)}\sigma'_e)\,. \tag{9.34}$$

Next we observe that the group $\mathrm{Aut}(\gimel)$ admits an obvious homomorphism to the subgroup

$$\Big(S_{e_\bullet} \times S_{e_-} \times S_{e_\circ}\Big) \ltimes \Big(\{\mathsf{id}\}^{e_\bullet} \times \big(S_2\big)^{e_-} \times \{\mathsf{id}\}^{e_\circ}\Big) \tag{9.35}$$

of (9.33), where $\{\mathsf{id}\}$ denotes the trivial group. Namely, this homomorphism assigns to an element $g \in \mathrm{Aut}(\gimel)$ the string

$$(\tau; \sigma_1, \dots, \sigma_e), \qquad \tau \in S_e,\ \sigma_1, \dots, \sigma_e \in S_2$$

in (9.33) according to this rule: $\tau(i) = j$ if the automorphism g sends the i-th edge to the j-th edge; σ_i is non-trivial (for $e_\bullet + 1 \le i \le e_\bullet + e_-$) if g sends the i-th edge to the j-th edge and the directions of these edges are opposite. It is clear that this homomorphism lands in the subgroup (9.35).

The group (9.35) acts on the graded vector space

$$\mathbf{s}^{2r - 2e_\bullet} \big(\underline{T}_a\big)^{\otimes e_\bullet} \ \otimes\ \big(\mathbf{s}^{-1}\, T_a\big)^{\otimes e_-} \ \otimes\ L^{\otimes e_\circ}\,. \tag{9.36}$$

Namely, if σ is the non-trivial element of S_2 and $e_\bullet < i \le e_\bullet + e_-$ then

$$(1, \dots, 1, \underbrace{\ \sigma\ }_{i\text{-th spot}}, 1, \dots 1)(a^{k_1}, a^{k_2}, \dots, a^{k_{e_\bullet + e_-}}, \mathfrak{l}_{k_{e_\bullet + e_- + 1}}, \dots, \mathfrak{l}_{k_e}) =$$

$$(-1)^{\frac{k_i(k_i+1)}{2}}(a^{k_1}, a^{k_2}, \dots, a^{k_{e_\bullet+e_-}}, \mathfrak{l}_{k_{e_\bullet+e_-+1}}, \dots, \mathfrak{l}_{k_e})\,.$$

Furthermore, for every $\tau \in S_{e_\bullet} \times S_{e_-} \times S_{e_\circ}$ we set

$$\tau(a^{k_1}, a^{k_2}, \dots, a^{k_{e_\bullet+e_-}}, \mathfrak{l}_{k_{e_\bullet+e_-+1}}, \dots, \mathfrak{l}_{k_e}) =$$

$$(-1)^{\varepsilon(\tau,k_1,\dots,k_e)}(a^{k_{\tau^{-1}(1)}}, a^{k_{\tau^{-1}(2)}}, \dots, a^{k_{\tau^{-1}(e_\bullet+e_-)}}, \mathfrak{l}_{k_{\tau^{-1}(e_\bullet+e_-+1)}}, \dots, \mathfrak{l}_{k_{\tau^{-1}(e)}})\,,$$

where the sign factor $(-1)^{\varepsilon(\tau,k_1,\dots,k_e)}$ is determined by the usual Koszul rule.

Thus the graded vector space (9.36) is equipped with a left action of the group $\mathrm{Aut}(\daleth)$.

EXAMPLE 9.15. Let us consider the frame $\daleth$ depicted on figure 38. Let g_1 be the generator of $\mathrm{Aut}(\daleth)$ which swaps the first edge with the second edge and let g_2 be the generator of $\mathrm{Aut}(\daleth)$ which swaps the fifth edge with the sixth edge. Then for the vector $(a, a^3, a, 1, 1, a, \mathfrak{l}_4)$ we have

$$g_1(a, a^3, a, 1, 1, a, \mathfrak{l}_4) = -(a^3, a, a, 1, 1, a, \mathfrak{l}_4)\,, \tag{9.37}$$

and

$$g_2(a, a^3, a, 1, 1, a, \mathfrak{l}_4) = -(a, a^3, a, 1, a, 1, \mathfrak{l}_4)\,. \tag{9.38}$$

The sign in (9.37) comes from the fact that a "jumps" over a^3 and the sign in (9.38) appears due to the fact that the fifth edge and the sixth edge carry opposite directions.

We can now describe the kernel of the map $F_\daleth$ (9.30).

CLAIM 9.16. *Let $\daleth \in \mathsf{Frame}_{r,n}$ be a frame with e edges*

$$e = e_\bullet + e_- + e_\circ\,,$$

where $e_\bullet$ is the number of edges of $\daleth$ adjacent to univalent neutral vertices, $e_\circ$ is the number of loops and $e_- = e - e_\bullet - e_\circ$. Then the kernel of $F_\daleth$ is spanned by vectors of the form

$$X - g(X)\,, \tag{9.39}$$

where X is a vector in (9.36) *and g is an automorphism of $\daleth$ in $\mathsf{Frame}_{r,n}$.*

PROOF. Let Y be a monomial in (9.36) such that

$$F_\daleth(Y) = 0\,. \tag{9.40}$$

The latter means that the graph $\Gamma \in \mathsf{gra}_{(r+r')+n}$ which is constructed from the monomial Y using the frame $\daleth$ is $(r+r')$-odd.

In other words, there exists an automorphism $\widetilde{g}$ of Γ which respects labels only on operational vertices and induces an odd permutation on the set of edges of Γ.

It is clear that $\widetilde{g}$ induces an automorphism g of the frame $\daleth$. Furthermore, since $\widetilde{g}$ induces an odd permutation on the set of edges of Γ we have

$$Y = -g(Y)\,.$$

Hence,

$$Y = \frac{1}{2}(Y - g(Y))\,. \tag{9.41}$$

Thus every monomial Y in (9.36) satisfying equation (9.40) belongs to the span of vectors of the form (9.39).

Let us now consider a linear combination

$$c_1 Y_1 + c_2 Y_2 + \dots + c_m Y_m\,, \qquad c_i \in \mathbb{K} \tag{9.42}$$

of monomials $Y_1, \dots, Y_m$ in (9.36) such that

$$\sum_i c_i F_{\daleth}(Y_i) = 0\,. \tag{9.43}$$

Due to the above observation about monomials satisfying (9.40) we may assume, without loss of generality, that

$$F_{\daleth}(Y_i) \neq 0 \qquad \forall\ 1 \le i \le m\,.$$

We may also assume, without loss of generality, that the graphs $\{\Gamma_i\}_{1 \le i \le m}$ reconstructed from the monomial $\{Y_i\}_{1 \le i \le m}$ have the same number of neutral vertices $r + r'$.

Thus, for every $1 \le i \le m$, the graph $\Gamma_i \in \mathsf{gra}_{(r+r')+n}$ is $(r+r')$-even.

Combining this observation with Proposition 9.1 we conclude that the number m is even and the set of graphs $\{\Gamma_i\}_{1 \le i \le m}$ splits into pairs

$$(\Gamma_{i_t}, \Gamma_{i'_t})\,, \qquad t \in \{1, \dots, m/2\}$$

such that for every t the graphs Γ_{i_t} and $\Gamma_{i'_t}$ are either $(r+r')$-opposite or $(r+r')$-concordant. For every pair $(\Gamma_{i_t}, \Gamma_{i'_t})$ of $(r+r')$-opposite graphs we have

$$c_{i_t} = c_{i'_t}\,. \tag{9.44}$$

For every pair $(\Gamma_{i_t}, \Gamma_{i'_t})$ of $(r+r')$-concordant graphs we have

$$c_{i_t} = -c_{i'_t}\,. \tag{9.45}$$

Let e_t denote the number of edges of Γ_{i_t} (or $\Gamma_{i'_t}$) and let $\widetilde{g}_t$ be the isomorphism from Γ_{i_t} to $\Gamma_{i'_t}$ which induces an odd or even permutation in S_{e_t} depending on whether Γ_{i_t} and $\Gamma_{i'_t}$ are $(r+r')$-opposite or $(r+r')$-concordant. Let g_t be the automorphism of the frame $\daleth$ which is induced by the isomorphism $\widetilde{g}_t$.

Equations (9.44) and (9.45) imply that

$$\sum_{i=1}^{m} c_i Y_i = \sum_{t=1}^{m/2} c_{i_t} (Y_{i_t} - g_t(Y_{i_t}))\,.$$

In other words, the linear combination (9.42) belongs to the span of vectors of the form (9.39) and the claim follows. $\square$

Now we are ready to give a convenient description of the associated graded complex $\mathrm{Gr}(\mathsf{fgraphs}^{\sharp}(n))$.

CLAIM 9.17. *Let us choose a representative* $\daleth_z$ *for every isomorphism class* $z \in \pi_0(\mathsf{Frame}_{r,n})$. *Let* $e^z_{\bullet}$ *be the number of edges of* $\daleth_z$ *adjacent to univalent neutral vertices,* $e^z_{\circ}$ *be the number of loops of* $\daleth_z$ *and*

$$e^z_{-} = |E(\daleth_z)| - e^z_{\bullet} - e^z_{\circ}\,.$$

Then the cochain complex $\mathrm{Gr}(\mathsf{fgraphs}^{\sharp}(n))$ *splits into the direct sum*

$$\mathrm{Gr}(\mathsf{fgraphs}^{\sharp}(n)) \cong$$

$$\bigoplus_{r \ge 0} \ \bigoplus_{z \in \pi_0(\mathsf{Frame}_{r,n})} \mathbf{s}^{2r - 2e^z_{\bullet}} \Big(\big(\underline{T}_a\big)^{\otimes e^z_{\bullet}} \otimes \big(\mathbf{s}^{-1} T_a\big)^{\otimes e^z_{-}} \otimes L^{\otimes e^z_{\circ}} \Big)_{\mathrm{Aut}(\daleth_z)}\,. \tag{9.46}$$

.

PROOF. Let us recall that the map $F_{\daleth_z}$ (9.30) is a morphism from the cochain complex

$$\mathbf{s}^{2r-2e^z_\bullet} \, (\underline{T}_a)^{\otimes e^z_\bullet} \;\otimes\; (\mathbf{s}^{-1} T_a)^{\otimes e^z_-} \;\otimes\; L^{\otimes e^z_\circ}$$

with the differential δ to $\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))$.

Thus, Claim 9.16 implies that $F_{\daleth_z}$ induces an isomorphism from the cochain complex of coinvariants

$$\mathbf{s}^{2r-2e^z_\bullet} \Big((\underline{T}_a)^{\otimes e^z_\bullet} \;\otimes\; (\mathbf{s}^{-1} T_a)^{\otimes e^z_-} \;\otimes\; L^{\otimes e^z_\circ} \Big)_{\mathrm{Aut}(\daleth_z)}$$

to the subcomplex

$$\mathrm{Im}(F_{\daleth_z}) \subset \mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))\,.$$

On the other hand, the cochain complex $\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))$ is obviously the direct sum

$$\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n)) = \bigoplus_{r \ge 0} \; \bigoplus_{z \in \pi_0(\mathsf{Frame}_{r,n})} \mathrm{Im}(F_{\daleth_z})\,. \tag{9.47}$$

Thus, the desired statement follows. $\square$

9.2.2. *Proof of Lemma* 9.12. We will now use the above description of the cochain complex $\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))$ to prove Lemma 9.12.

First, we observe that, since the cochain complex $\underline{T}_a$ is acyclic, the direct summand

$$\mathrm{Im}(F_{\daleth}) \tag{9.48}$$

of $\mathrm{Gr}(\mathsf{fgraphs}^\sharp(n))$ is acyclic for every frame $\daleth$ with at least one univalent neutral vertex.

So let us consider a frame $\daleth$ with $e_\bullet = 0$.

It is easy to see that the cochain complex

$$\mathbf{s}^{2r} \Big((\mathbf{s}^{-1} T_a)^{\otimes e_-} \;\otimes\; L^{\otimes e_\circ} \Big)_{\mathrm{Aut}(\daleth)} \tag{9.49}$$

is concentrated is degrees

$$\ge 2r - e_- - e_\circ\,.$$

Furthermore, using (9.28), (9.29), Künneth's theorem, and the fact that the cohomology functor commutes with taking coinvariants, we conclude that every cocycle X in (9.49) of degree $> 2r - e_- - e_\circ$ is trivial and the space

$$H^{2r-e_- - e_\circ}\Big(\mathbf{s}^{2r} \Big((\mathbf{s}^{-1} T_a)^{\otimes e_-} \;\otimes\; L^{\otimes e_\circ} \Big)_{\mathrm{Aut}(\daleth)} \Big) = \mathbb{K} \tag{9.50}$$

is spanned by the class of the vector

$$\mathbf{s}^{2r}\, (\mathbf{s}^{-1}\, 1)^{\otimes e_-} \;\otimes\; (\mathfrak{l}_1)^{\otimes e_\circ}\,. \tag{9.51}$$

Since images of cocycles X in (9.49) of degrees $> 2r - e_- - e_\circ$ lie in

$$\Big(\mathcal{F}^m \mathsf{fgraphs}^\sharp(n) / \mathcal{F}^{m-1} \mathsf{fgraphs}^\sharp(n) \Big)^k$$

for $m > -k$ and images of the vectors (9.51) belong to $\mathsf{graphs}^\sharp(n)^{2r-e_- - e_\circ}$, Lemma 9.12 follows from Claim 9.17.

9.3. We are getting rid of loops. Let us denote by $\mathsf{Graphs}^{\sharp}_{\not\circlearrowleft}(n)$ the subspace of $\mathsf{Graphs}^{\sharp}(n)$ which consists of vectors in $\mathsf{Graphs}^{\sharp}(n)$ involving exclusively graphs without loops.

Since the differential ∂^{Tw} "does not create" loops, the subspace $\mathsf{Graphs}^{\sharp}_{\not\circlearrowleft}(n)$ is a subcomplex of $\mathsf{Graphs}^{\sharp}(n)$ for every n. Moreover the collection

$$\mathsf{Graphs}^{\sharp}_{\not\circlearrowleft} = \{\mathsf{Graphs}^{\sharp}_{\not\circlearrowleft}(n)\}_{n \geq 0} \tag{9.52}$$

is obviously a suboperad $\mathsf{Graphs}^{\sharp}$.

The goal of this section is to prove that

PROPOSITION 9.18. *The embedding*

$$\mathrm{emb}^{\sharp}_{2} : \mathsf{Graphs}^{\sharp}_{\not\circlearrowleft} \hookrightarrow \mathsf{Graphs}^{\sharp} \tag{9.53}$$

is a quasi-isomorphism (of dg operads).

PROOF. Let us introduce the subcomplex $\mathsf{graphs}^{\sharp}_{\not\circlearrowleft}(n)$ of $\mathsf{Graphs}^{\sharp}_{\not\circlearrowleft}$ which consists of finite sums of graphs, i.e.

$$\mathsf{graphs}^{\sharp}_{\not\circlearrowleft}(n) := \mathsf{Graphs}^{\sharp}_{\not\circlearrowleft}(n) \cap \mathrm{Tw}^{\oplus}\mathsf{Gra}(n)\,. \tag{9.54}$$

We will prove that the embedding

$$\mathsf{graphs}^{\sharp}_{\not\circlearrowleft}(n) \hookrightarrow \mathsf{graphs}^{\sharp}(n) \tag{9.55}$$

is a quasi-isomorphism of cochain complexes. Then the desired statement can be easily deduced from this fact using the Euler characteristic trick (see Remark 9.7).

Let Γ be a r-even graph in gra_{r+n} whose first r vertices have valency ≥ 3. Let us denote by $\mathrm{tp}_r(\Gamma)$ the number of loops (if any) of Γ which are based on a trivalent vertex whose label $\leq r$. For example, the graph $\Gamma \in \mathsf{gra}_{3+3}$ depicted on figure 40 has $\mathrm{tp}_3(\Gamma) = 1$. Indeed, the vertex with label 1 supports a loop but it has valency 4; the vertex with label 2 does not support a loop; finally, the vertex with label 3 supports a loop and has valency 3.

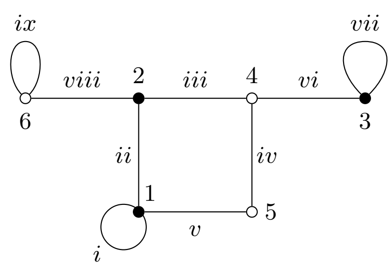


FIG. 40. It is the vertex with label 3 which contributes to $\mathrm{tp}_3(\Gamma)$

It is obvious that the expression

$$\partial^{\mathrm{Tw}}(\mathrm{Av}_r(\Gamma))$$

involves graphs $\Gamma' \in \mathsf{gra}_{(r+1)+n}$ with $\mathrm{tp}_{r+1}(\Gamma') = \mathrm{tp}_r(\Gamma)$ or $\mathrm{tp}_{r+1}(\Gamma') = \mathrm{tp}_r(\Gamma) + 1$.

Thus the cochain complex $\mathsf{graphs}^{\sharp}(n)$ carries the following ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}\mathsf{graphs}^{\sharp}(n) \subset \mathcal{F}^{m}\mathsf{graphs}^{\sharp}(n) \subset \mathcal{F}^{m+1}\mathsf{graphs}^{\sharp}(n) \subset \dots \tag{9.56}$$

where $\mathcal{F}^m \mathsf{graphs}^\sharp(n)$ is spanned by vectors in $\mathsf{graphs}^\sharp(n)$ of the form

$$\mathrm{Av}_r(\Gamma) \qquad \Gamma \in \mathsf{gra}_{r+n}$$

with

$$\mathrm{tp}_r(\Gamma) - |\mathrm{Av}_r(\Gamma)| \le m\,.$$

It is clear that the differential ∂^{tp} on the associated graded complex

$$\mathrm{Gr}(\mathsf{graphs}^\sharp(n)) = \bigoplus_m \mathcal{F}^m \mathsf{graphs}^\sharp(n) \big/ \mathcal{F}^{m-1}\mathsf{graphs}^\sharp(n) \tag{9.57}$$

is obtained from ∂^{Tw} by keeping only terms which raise the number of loops based on trivalent neutral vertices.

It is also clear that the restriction of (9.56) to the subcomplex $\mathsf{graphs}^\sharp_{\not\circlearrowleft}(n)$ gives us the "silly" filtration

$$\mathcal{F}^m \mathsf{graphs}^\sharp_{\not\circlearrowleft}(n)^k = \begin{cases} \mathsf{graphs}^\sharp_{\not\circlearrowleft}(n)^k & \text{if } m \ge -k\,, \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{9.58}$$

with the associated graded complex $\mathrm{Gr}(\mathsf{graphs}^\sharp_{\not\circlearrowleft}(n))$ carrying the zero differential.

It is not hard to see that the cochain complex $\mathrm{Gr}(\mathsf{graphs}^\sharp(n))$ splits into the direct sum of subcomplexes

$$\mathrm{Gr}(\mathsf{graphs}^\sharp(n)) \cong \mathrm{Gr}(\mathsf{graphs}^\sharp_{\not\circlearrowleft}(n)) \;\oplus\; \mathsf{graphs}^\sharp{}_{\circlearrowleft}(n)\,, \tag{9.59}$$

where $\mathsf{graphs}^\sharp{}_{\circlearrowleft}(n)$ is spanned by vectors in $\mathsf{graphs}^\sharp(n)$ of the form

$$\mathrm{Av}_r(\Gamma)\,, \qquad \Gamma \in \mathsf{gra}_{r+n}$$

with Γ having at least one loop.

Let Γ be graph in gra_{r+n} for which $\mathrm{Av}_r(\Gamma) \in \mathsf{graphs}^\sharp{}_{\circlearrowleft}(n)$ and let $V^r_{\circlearrowleft}(\Gamma)$ denote the following subset of vertices of Γ

$$V^r_{\circlearrowleft}(\Gamma) =$$

(9.60)

$$\big\{ v \in V(\Gamma) \;\big|\; v \text{ carries label } \le r, \text{ has valency } > 3, \text{ and supports a loop} \big\} \cup$$

$$\big\{ v \in V(\Gamma) \;\big|\; v \text{ carries label } > r \text{ and supports a loop} \big\}\,.$$

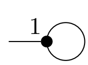
For example, if Γ is the graph depicted on figure 40 then $V^3_{\circlearrowleft}(\Gamma)$ consists of vertices labeled by 1 and 6 .

Collecting terms in (9.6) which raise the number of loops based on trivalent neutral vertices we see that

$$\partial^{\mathrm{tp}}(\mathrm{Av}_r(\Gamma)) = \begin{cases} -\displaystyle\sum_{v \in V_{\circlearrowleft}(\Gamma)} \mathrm{Av}_{r+1}\big(\mathrm{Tp}_v(\Gamma)\big)\,, & \text{if } V_{\circlearrowleft}(\Gamma) \text{ is non-empty} \\ \qquad 0 & \text{if } V_{\circlearrowleft}(\Gamma) = \emptyset\,, \end{cases} \tag{9.61}$$

where $\mathrm{Tp}_v(\Gamma)$ is a graph in $\mathsf{gra}_{(r+1)+n}$ obtained from Γ by

- shifting labels on all vertices of Γ up by 1;
- removing the loop based at the vertex v;
- attaching to v the piece



- declaring that the loop based at first neutral vertex takes the spot of the removed loop in $E(\Gamma)$ and the edge connecting the first neutral vertex to v is the smallest in $E(\mathrm{Tp}_v(\Gamma))$.

Let Γ be graph in gra_{r+n} for which $\mathrm{Av}_r(\Gamma) \in \mathsf{graphs}^{\sharp}{}_{\circlearrowleft}(n)$ and let $V^r_{\mathrm{tp}}(\Gamma)$ denote the set of trivalent vertices (if any) which support loops and carry labels $\le r$. We denote by h the linear map of degree -1

$$h : \mathsf{graphs}^{\sharp}{}_{\circlearrowleft}(n) \to \mathsf{graphs}^{\sharp}{}_{\circlearrowleft}(n)$$

defined by formula

$$h(\mathrm{Av}_r(\Gamma)) := \begin{cases} -\sum\limits_{v \in V^r_{\circlearrowleft}(\Gamma)} \mathrm{Av}_{r-1}\big(\mathrm{Tp}^*_v(\Gamma)\big), & \text{if } V^r_{\mathrm{tp}}(\Gamma) \text{ is non-empty} \\ 0 & \text{if } V^r_{\mathrm{tp}}(\Gamma) = \emptyset, \end{cases} \tag{9.62}$$

where $\mathrm{Tp}^*_v(\Gamma)$ is a vector in $\mathsf{Gra}(r-1+n)$ obtained from Γ by

- switching the label on v with the label 1 on the first vertex of Γ (provided v is not the first vertex);
- changing the order of the edges of Γ such that the single edge e_v connecting v to another vertex becomes the smallest one (this step may produce the sign factor (-1) in front of Γ);
- removing the edge e_v together with the vertex v and attaching the vacated loop to the other end of e_v;
- shifting labels on all the remaining vertices down by 1.

For example, if Γ is the graph depicted on figure 40 then

$$h(\mathrm{Av}_3(\Gamma)) = -\mathrm{Av}_2(\Gamma'), \tag{9.63}$$

where Γ' is the vector in $\mathsf{Gra}(5)$ depicted on figure 41.

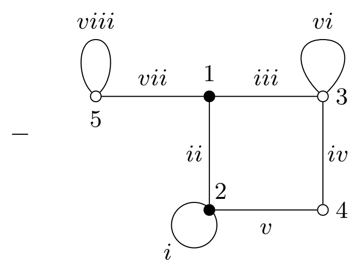


FIG. 41. The vector Γ' defining $h(\mathrm{Av}_3(\Gamma))$

Figure 42 illustrates intermediate steps in the construction of Γ'.

Let Γ be a graph in gra_{r+n} such that $\mathrm{Av}_r(\Gamma) \in \mathsf{graphs}^{\sharp}{}_{\circlearrowleft}(n)$. Using the fact that Γ has no bivalent neutral vertices, it is not hard to show that the operations ∂^{tp} and h satisfy the identity

$$\partial^{\mathrm{tp}} \circ h(\mathrm{Av}_r(\Gamma)) + h \circ \partial^{\mathrm{tp}}(\mathrm{Av}_r(\Gamma)) = \lambda_\Gamma \mathrm{Av}_r(\Gamma), \tag{9.64}$$

where λ_Γ is the number of loops of Γ.

Therefore, the cochain complex

$$(\mathsf{graphs}^{\sharp}{}_{\circlearrowleft}(n), \partial^{\mathrm{tp}})$$

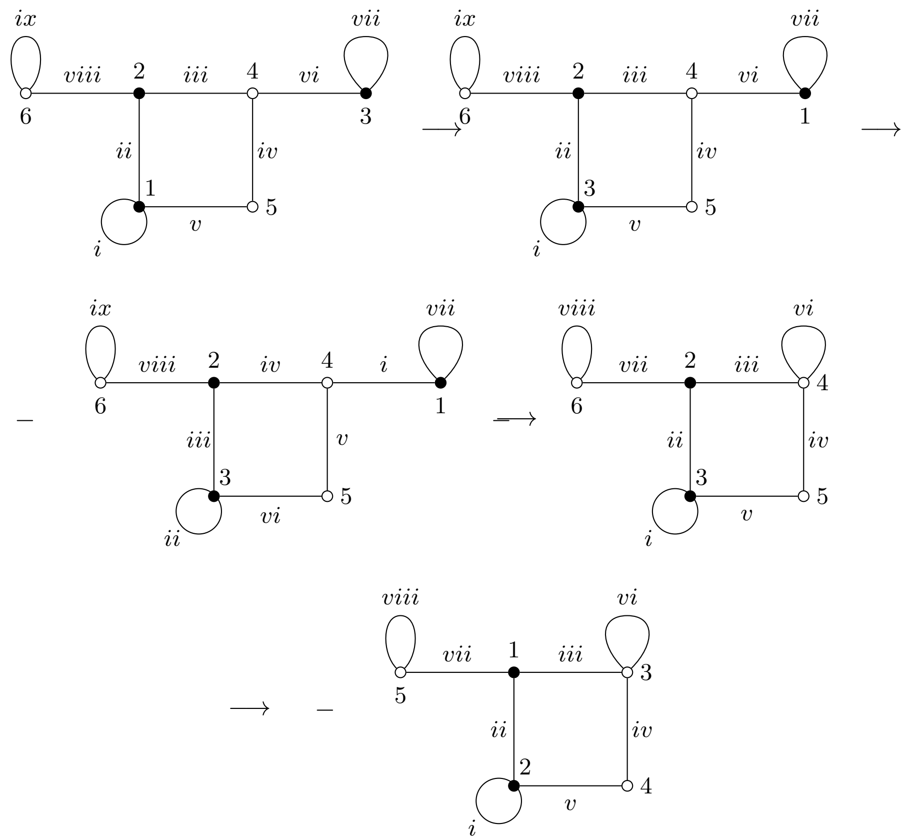


FIG. 42. Intermediate steps in the construction of Γ'

is acyclic and hence the embedding (9.55) induces a quasi-isomorphism of cochain complexes:

$$\mathrm{Gr}(\mathsf{graphs}^{\sharp}_{\emptyset}(n)) \xrightarrow{\sim} \mathrm{Gr}(\mathsf{graphs}^{\sharp}(n))\,.$$

On the other hand, both filtrations (9.56) and (9.58) are cocomplete and locally bounded from the left. Thus Lemma A.3 from Appendix A implies that the embedding (9.55) is a quasi-isomorphism. Hence so is the embedding

$$\mathsf{Graphs}^{\sharp}_{\emptyset}(n) \hookrightarrow \mathsf{Graphs}^{\sharp}(n)\,.$$

Proposition 9.18 is proved. $\square$

9.4. The suboperads $\mathsf{Graphs}_{\emptyset} \subset \mathsf{Graphs} \subset \mathsf{fGraphs} \subset \mathbf{Tw}\mathsf{Gra}$. In this subsection we introduce yet another series of suboperads of $\mathrm{Tw}\mathsf{Gra}$

$$\mathsf{Graphs}_{\emptyset} \subset \mathsf{Graphs} \subset \mathsf{fGraphs} \subset \mathrm{Tw}\mathsf{Gra}\,.$$

We will show that the embeddings

$$\mathsf{Graphs}_{\emptyset} \hookrightarrow \mathsf{Graphs}\,,$$
$$\mathsf{Graphs} \hookrightarrow \mathsf{fGraphs}$$

are quasi-isomorphisms of dg operads.

We denote by $\mathsf{fGraphs}(n)$ the subspace of $\mathrm{Tw}\mathsf{Gra}(n)$ which consists of linear combinations (9.2) satisfying

PROPERTY 9.19. For every r, each graph in the linear combination γ_r has no connected components which involve exclusively neutral vertices.

For example, it means that

$$\mathsf{fGraphs}(0) = \mathbf{0}\,.$$

We denote by $\mathsf{Graphs}(n)$ the subspace of $\mathsf{fGraphs}(n)$ which consists of sums of graphs with neutral vertices having valencies ≥ 3.

Finally, $\mathsf{Graphs}_{\not{\circlearrowleft}}(n)$ is the subspace of $\mathsf{Graphs}(n)$ which consists of sums of graphs without loops.

It is easy to see that for every n, $\mathsf{Graphs}_{\not{\circlearrowleft}}(n)$, $\mathsf{Graphs}(n)$, and $\mathsf{fGraphs}(n)$ are subcomplexes of $\mathrm{Tw}\mathsf{Gra}(n)$. Moreover, collections

$$\mathsf{fGraphs} = \{\mathsf{fGraphs}(n)\}_{n\geq 0}\,, \tag{9.65}$$

$$\mathsf{Graphs} = \{\mathsf{Graphs}(n)\}_{n\geq 0}\,, \tag{9.66}$$

and

$$\mathsf{Graphs}_{\not{\circlearrowleft}} = \{\mathsf{Graphs}_{\not{\circlearrowleft}}(n)\}_{n\geq 0} \tag{9.67}$$

are suboperads of $\mathrm{Tw}\mathsf{Gra}$.

We claim that

PROPOSITION 9.20. *The embeddings*

$$\mathrm{emb}_1 : \mathsf{Graphs} \hookrightarrow \mathsf{fGraphs} \tag{9.68}$$

and

$$\mathrm{emb}_2 : \mathsf{Graphs}_{\not{\circlearrowleft}} \hookrightarrow \mathsf{Graphs} \tag{9.69}$$

are quasi-isomorphisms of dg operads.

PROOF. It clear that the cone

$$\mathrm{Cone}(\mathrm{emb}_1) = \mathsf{Graphs}\ \oplus\ \mathbf{s}\,\mathsf{fGraphs}$$

of the embedding (9.68) is a direct summand in the cone $\mathrm{Cone}(\mathrm{emb}_1^\sharp)$ of

$$\mathrm{emb}_1^\sharp : \mathsf{Graphs}^\sharp \hookrightarrow \mathsf{fGraphs}^\sharp\,.$$

Thus the desired statement about the embedding emb_1 follows from Proposition 9.11 above and Claim A.1 given in Appendix A.

Similarly the cone

$$\mathrm{Cone}(\mathrm{emb}_2) = \mathsf{Graphs}_{\not{\circlearrowleft}}\ \oplus\ \mathsf{Graphs}$$

of the embedding (9.69) is a direct summand in the cone $\mathrm{Cone}(\mathrm{emb}_2^\sharp)$ of

$$\mathrm{emb}_2^\sharp : \mathsf{Graphs}^\sharp_{\not{\circlearrowleft}} \hookrightarrow \mathsf{Graphs}^\sharp\,.$$

Thus, using Proposition 9.18 above and Claim A.1 given in Appendix A, it is easy to prove that emb_2 is a quasi-isomorphism. $\square$

9.5. The master diagram for the dg operad $\mathrm{Tw}\mathsf{Gra}$. Let $\mathcal{O}$ be a (dg) operad which receives a morphism from $\Lambda\mathsf{Lie}_\infty$. Let us observe that we have the obvious embedding

$$\mathrm{emb}_{\mathcal{O}} : \mathcal{O} \hookrightarrow \mathrm{Tw}\mathcal{O} \tag{9.70}$$

$$\mathrm{emb}_{\mathcal{O}}(v)(\mathbf{s}^{-2r}\,1) = \begin{cases} v & \text{if } r = 0\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

which is compatible with the operad structure but may not be compatible with the differentials.

We denote by $\Gamma_{\circ\!-\!\circ} \in \mathrm{Tw}\mathsf{Gra}(2)$ (resp. $\Gamma_{\circ\,\circ} \in \mathrm{Tw}\mathsf{Gra}(2)$) the images of $\Gamma_{\bullet\!-\!\bullet}$ and $\Gamma_{\bullet\,\bullet}$ with respect to the embedding

$$\mathrm{emb}_{\mathsf{Gra}} : \mathsf{Gra} \to \mathrm{Tw}\mathsf{Gra}\,.$$

Namely,

$$\Gamma_{\circ\!-\!\circ} = \overset{1}{\circ}\!\!-\!\!\!-\!\!\overset{2}{\circ}\ , \tag{9.71}$$

and

$$\Gamma_{\circ\,\circ} = \overset{1}{\circ}\quad\overset{2}{\circ}\ . \tag{9.72}$$

Although $\mathrm{emb}_{\mathsf{Gra}}$ is not compatible with the differential ∂^{Tw}, the vectors $\Gamma_{\circ\!-\!\circ}, \Gamma_{\circ\,\circ} \in \mathrm{Tw}\mathsf{Gra}(2)$ are ∂^{Tw}-closed (see Exercise 9.26 below).

Therefore, the composition of embeddings ι (7.5) and $\mathrm{emb}_{\mathsf{Gra}}$

$$\iota' = \mathrm{emb}_{\mathsf{Gra}} \circ \iota : \mathsf{Ger} \hookrightarrow \mathrm{Tw}\mathsf{Gra} \tag{9.73}$$

is a morphism of dg operads. Furthermore, it is obvious that ι' lands in the suboperad $\mathsf{Graphs}_{\varnothing}$.

It turns out that the map ι' satisfies the following remarkable property[15]:

THEOREM 9.21 (M. Kontsevich, [**24**], Section 3.3.4). *The embedding*

$$\iota' : \mathsf{Ger} \hookrightarrow \mathsf{Graphs}_{\varnothing} \tag{9.74}$$

induces an isomorphism

$$\mathsf{Ger} \cong H^\bullet(\mathsf{Graphs}_{\varnothing})\,.$$

REMARK 9.22. It is obvious that the map (9.74) lands in the suboperad $\mathsf{graphs}_{\varnothing} \subset \mathsf{Graphs}_{\varnothing}$ with

$$\mathsf{graphs}_{\varnothing}(n) = \mathsf{Graphs}_{\varnothing}(n) \cap \mathrm{Tw}^{\oplus}\mathsf{Gra}(n)\,.$$

The arguments given in [**24**, Section 3.3.4] or [**26**] allow us to prove that the embedding

$$\iota' : \mathsf{Ger} \hookrightarrow \mathsf{graphs}_{\varnothing}$$

is a quasi-isomorphism of dg operads. The desired statement about (9.74) can be easily deduced from this fact using the Euler characteristic trick.

We now assemble all the above results about suboperads of $\mathrm{Tw}\mathsf{Gra}$ into the following theorem:

[15]For a more detailed proof of this fact we refer the reader to paper [**26**] by P. Lambrechts and I. Volic.

THEOREM 9.23. *The suboperads* $\mathsf{fGraphs}^\sharp$, $\mathsf{Graphs}^\sharp$, $\mathsf{Graphs}^\sharp_\varnothing$, $\mathsf{fGraphs}$, Graphs *and* $\mathsf{Graphs}_\varnothing$ *of* $\mathrm{Tw}\mathsf{Gra}$ *introduced in Sections* 9.2 *and* 9.4 *fit into the following commutative diagram:*

$$
\begin{tikzcd}
 & \mathsf{Graphs}^\sharp_\varnothing \arrow[r, hook, "\sim"] & \mathsf{Graphs}^\sharp \arrow[r, hook, "\sim"] & \mathsf{fGraphs}^\sharp \arrow[r, hook] & \mathrm{Tw}\mathsf{Gra} \\
\mathsf{Ger} \arrow[r, hook, "\sim"] & \mathsf{Graphs}_\varnothing \arrow[r, hook, "\sim"] \arrow[u, hook] & \mathsf{Graphs} \arrow[r, hook, "\sim"] \arrow[u, hook] & \mathsf{fGraphs} \arrow[u, hook] &
\end{tikzcd} \tag{9.75}
$$

Here the arrow $\hookrightarrow$ *denotes an embedding and the arrow* $\overset{\sim}{\hookrightarrow}$ *denotes an embedding which induces an isomorphism on the level of cohomology.* $\square$

We refer to (9.75) as the master diagram for the dg operad $\mathrm{Tw}\mathsf{Gra}$.

Theorem 9.23 has the following obvious corollary

COROLLARY 9.24. *The embedding*

$$\iota' : \mathsf{Ger} \hookrightarrow \mathsf{fGraphs} \tag{9.76}$$

induces an isomorphism on the level of cohomology. $\square$

Let us observe that the map ι' lands in the suboperad $\mathsf{fgraphs} \subset \mathsf{fGraphs}$ for which

$$\mathsf{fgraphs}(n) := \mathsf{fGraphs}(n) \cap \mathrm{Tw}^{\oplus}\mathsf{Gra}(n)\,. \tag{9.77}$$

Furthermore, using the Euler characteristic trick, it is not hard to deduce from Corollary 9.24 that

COROLLARY 9.25. *The embedding*

$$\iota' : \mathsf{Ger} \hookrightarrow \mathsf{fgraphs} \tag{9.78}$$

induces an isomorphism on the level of cohomology. $\square$

EXERCISE 9.26. Prove that the vectors $\Gamma_{\circ\!-\!\circ}, \Gamma_{\circ\,\circ} \in \mathrm{Tw}\mathsf{Gra}(2)$ defined in (9.71) and (9.72), respectively, satisfy the conditions

$$\partial^{\mathrm{Tw}}\Gamma_{\circ\!-\!\circ} = \partial^{\mathrm{Tw}}\Gamma_{\circ\,\circ} = 0\,.$$

10. The full graph complex fGC revisited

Let us recall that $\mathsf{Gra}(0) = \mathbf{0}$. Hence, due to Remark 6.9, we have a tautological isomorphism

$$\mathsf{fGC} \cong \mathbf{s}^{-2}\mathrm{Tw}\mathsf{Gra}(0) \tag{10.1}$$

between the full graph complex fGC (8.3) and the cochain complex $\mathbf{s}^{-2}\mathrm{Tw}\mathsf{Gra}(0)$.

Here we will use the isomorphism (10.1) together with the results of Sections 9.2 and 9.3 to deduce various useful facts about the full graph complex fGC.

Recall that vectors of fGC are infinite sums

$$\gamma = \sum_{n=1}^{\infty} \gamma_n \tag{10.2}$$

of S_n-invariant vectors $\gamma_n \in \mathbf{s}^{2n-2}\mathsf{Gra}(n)$.

We denote by

$$\mathsf{fGC}_{\geq 3} \subset \mathsf{fGC} \tag{10.3}$$

the subspace of sums (10.2) satisfying

PROPERTY 10.1. For every n, each connected component of a graph in γ_n has at least one vertex of valency ≥ 3.

We also denote by GC the subspace of sums (9.2) which involve exclusively graphs whose vertices all have valencies ≥ 3. It is obvious that $\mathsf{GC} \subset \mathsf{fGC}_{\geq 3}$.

Comparing $\mathsf{fGC}_{\geq 3}$ and GC with the suboperads $\mathsf{fGraphs}^{\sharp}$ and $\mathsf{Graphs}^{\sharp}$ from Section 9.2 we see that

$$\mathsf{fGC}_{\geq 3} \cong \mathbf{s}^{-2}\mathsf{fGraphs}^{\sharp}(0)\,, \qquad \mathsf{GC} \cong \mathbf{s}^{-2}\mathsf{Graphs}^{\sharp}(0)\,. \tag{10.4}$$

In particular, GC and $\mathsf{fGC}_{\geq 3}$ are subcomplexes of fGC.

Let us denote by $\mathsf{GC}_{\varnothing}$ the subspace of vectors in GC involving exclusively graphs without loops. It is clear that $\mathsf{GC}_{\varnothing}$ is a subcomplex of GC. Moreover,

$$\mathsf{GC}_{\varnothing} \cong \mathbf{s}^{-2}\mathsf{Graphs}^{\sharp}_{\varnothing}(0)\,, \tag{10.5}$$

where $\mathsf{Graphs}^{\sharp}_{\varnothing}$ is the suboperad of $\mathrm{Tw}\mathsf{Gra}$ introduced in Section 9.3.

Thus, Propositions 9.11, 9.18 imply that

COROLLARY 10.2. *The embeddings*

$$\mathrm{emb}_{\mathsf{GC}} \ : \ \mathsf{GC} \hookrightarrow \mathsf{fGC}_{\geq 3} \tag{10.6}$$

and

$$\mathrm{emb}_{\varnothing} \ : \ \mathsf{GC}_{\varnothing} \hookrightarrow \mathsf{GC} \tag{10.7}$$

are quasi-isomorphisms of cochain complexes. □

Let us denote by $\mathsf{fGC}_{\geq 3,\,\mathrm{conn}}$, $\mathsf{GC}_{\mathrm{conn}}$, and $\mathsf{GC}_{\varnothing,\,\mathrm{conn}}$ the "connected" versions of the subcomplexes $\mathsf{fGC}_{\geq 3}$, GC and $\mathsf{GC}_{\varnothing}$, respectively. Namely,

$$\begin{gathered} \mathsf{fGC}_{\geq 3,\,\mathrm{conn}} := \mathsf{fGC}_{\geq 3} \cap \mathsf{fGC}_{\mathrm{conn}}\,, \\ \mathsf{GC}_{\mathrm{conn}} := \mathsf{GC} \cap \mathsf{fGC}_{\mathrm{conn}}\,, \\ \mathsf{GC}_{\varnothing\,\mathrm{conn}} := \mathsf{GC}_{\varnothing} \cap \mathsf{fGC}_{\mathrm{conn}}\,, \end{gathered} \tag{10.8}$$

where $\mathsf{fGC}_{\mathrm{conn}}$ is the subcomplex of fGC introduced in Section 8.3.

For the subcomplexes (10.8) we have

PROPOSITION 10.3. *The embeddings*

$$\mathrm{emb}_{\mathsf{GC},\,\mathrm{conn}} \ : \ \mathsf{GC}_{\mathrm{conn}} \hookrightarrow \mathsf{fGC}_{\geq 3,\mathrm{conn}} \tag{10.9}$$

and

$$\mathrm{emb}_{\varnothing,\,\mathrm{conn}} \ : \ \mathsf{GC}_{\varnothing,\,\mathrm{conn}} \hookrightarrow \mathsf{GC}_{\mathrm{conn}} \tag{10.10}$$

are quasi-isomorphisms of cochain complexes.

PROOF. It is easy to see that the cone of the embedding $\mathrm{emb}_{\mathsf{GC},\,\mathrm{conn}}$ (resp. $\mathrm{emb}_{\varnothing,\,\mathrm{conn}}$) is a direct summand in the cone of the embedding $\mathrm{emb}_{\mathsf{GC}}$ (resp. $\mathrm{emb}_{\varnothing}$).

Thus the desired statements follow from Corollary 10.2 above and Claim A.1 from Appendix A. □

Let us observe that, if all vertices of a connected graph Γ have valencies ≤ 2, then Γ is isomorphic to one of the graphs in the list: $\Gamma_{\bullet}$, Γ^{-}_{l} (see figure 33), or $\Gamma^{\diamond}_{m}$ (see figure 34). Hence, $\mathsf{fGC}_{\mathrm{conn}}$ decomposes as

$$\mathsf{fGC}_{\mathrm{conn}} = \mathsf{fGC}_{\ge 3,\,\mathrm{conn}} \,\oplus\, \mathcal{K}_{\diamond} \,\oplus\, \mathcal{K}_{-}\,, \tag{10.11}$$

where $\mathcal{K}_{-}$ (resp. $\mathcal{K}_{\diamond}$) is the subcomplex of cables (resp. polygons) introduced in Subsection 8.1 (resp. Subsection 8.2).

Therefore, using Proposition 8.5 and isomorphism (8.15) we deduce that

$$H^{\bullet}(\mathsf{fGC}_{\mathrm{conn}}) \cong H^{\bullet}(\mathsf{fGC}_{\ge 3,\,\mathrm{conn}}) \,\oplus\, \bigoplus_{q\ge 1} \mathbf{s}^{4q-1}\,\mathbb{K}\,. \tag{10.12}$$

Thus we arrive at the main result of this section.

THEOREM 10.4 (T. Willwacher, [**42**]). *Let $\mathsf{fGC}_{\mathrm{conn}}$ be the "connected part" of the full graph complex fGC (8.3). Moreover, let $\mathsf{GC}_{\varnothing,\,\mathrm{conn}}$ be the subcomplex of vectors in $\mathsf{fGC}_{\mathrm{conn}}$ involving exclusively graphs Γ satisfying these two properties:*

- *Γ does not have loops;*
- *each vertex of Γ has valency ≥ 3.*

Then

$$H^{\bullet}(\mathsf{fGC}) \cong \mathbf{s}^{-2}\widehat{S}\big(\mathbf{s}^{2}\,H^{\bullet}(\mathsf{fGC}_{\mathrm{conn}})\big)\,, \tag{10.13}$$

and

$$H^{\bullet}(\mathsf{fGC}_{\mathrm{conn}}) \cong H^{\bullet}(\mathsf{GC}_{\varnothing,\,\mathrm{conn}}) \,\oplus\, \bigoplus_{q\ge 1} \mathbf{s}^{4q-1}\,\mathbb{K}\,, \tag{10.14}$$

where $\widehat{S}$ is the notation for the completed symmetric algebra.

PROOF. The first decomposition (10.13) is obtained by applying the Künneth theorem to (8.16). The second decomposition (10.14) is obtained by applying Proposition 10.3 to the isomorphism (10.12). □

EXERCISE 10.5. Using equation (8.13), prove that for every even trivalent graph $\Gamma \in \mathsf{gra}_n$ the vector

$$\mathrm{Av}(\Gamma) = \sum_{\sigma\in S_n} \sigma(\Gamma) \tag{10.15}$$

is a cocycle in fGC. Show that the tetrahedron depicted on figure 43 represents a non-trivial (degree zero) cocycle in fGC.

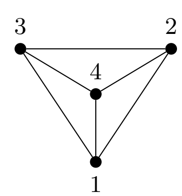


FIG. 43. We may choose this order on the set of edges: $(1,2) < (1,3) < (1,4) < (2,3) < (2,4) < (3,4)$

11. Deformation complex of Ger

Let us consider the following graded Lie algebra

$$\mathrm{Conv}(\mathsf{Ger}_\circ^\vee, \mathsf{Ger})\,. \tag{11.1}$$

Due to (5.17) we have

$$\mathrm{Conv}(\mathsf{Ger}_\circ^\vee, \mathsf{Ger}) = \prod_{n\ge 2} \big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}\,. \tag{11.2}$$

The operad $\Lambda^{-2}\mathsf{Ger}$ is generated by the vectors $b_1 b_2$ and $\{b_1, b_2\}$ in $\Lambda^{-2}\mathsf{Ger}(2)$. Moreover, the vectors $b_1 b_2$ and $\{b_1, b_2\}$ carry the degrees 2 and 1, respectively:

$$|b_1 b_2| = 2\,, \qquad |\{b_1, b_2\}| = 1\,. \tag{11.3}$$

Following Section 5.2, the canonical map $\mathrm{Cobar}(\mathsf{Ger}^\vee) \to \mathsf{Ger}$ (5.20) corresponds to the Maurer-Cartan element[16]

$$\alpha = a_1 a_2 \otimes \{b_1, b_2\} + \{a_1, a_2\} \otimes b_1 b_2 \in \mathrm{Conv}(\mathsf{Ger}_\circ^\vee, \mathsf{Ger})\,. \tag{11.4}$$

Thus, using this Maurer-Cartan element, we can equip the graded Lie algebra (11.2) with the differential

$$\partial = [\alpha, \ ]\,. \tag{11.5}$$

According to [**34**], the cochain complex (11.2) with the differential (11.5) "governs" deformations of the operad structure on Ger. So we refer to (11.2) as the *deformation complex* of the operad Ger.

EXERCISE 11.1. Verify the identity

$$[\alpha, \alpha] = 0$$

by a direct computation.

For our purposes it is convenient to extend the deformation complex of Ger to

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) = \prod_{n\ge 1} \big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}\,. \tag{11.6}$$

Vectors in the cochain complex (11.6) are formal infinite sum

$$\sum_{n=1}^{\infty} \gamma_n\,, \tag{11.7}$$

where each γ_n is an S_n-invariant vector in $\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$. For example,

$$a_1 a_2 \otimes \{b_1, b_2\}$$

is a degree 1 vector in (11.6).

It is obvious that

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) = \mathbb{K}\langle a_1 \otimes b_1 \rangle \ \oplus \ \mathrm{Conv}(\mathsf{Ger}_\circ^\vee, \mathsf{Ger})$$

and

$$\partial(a_1 \otimes b_1) = \alpha\,.$$

Thus

$$H^\bullet\Big(\mathrm{Conv}(\mathsf{Ger}_\circ^\vee, \mathsf{Ger})\Big) = H^\bullet\Big(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})\Big) \ \oplus \ \mathbf{s}\,\mathbb{K}\,, \tag{11.8}$$

[16] From now on we omit the subscript Ger in the notation for the Maurer-Cartan element α_{Ger}.

where the additional degree 1 class is represented by the Maurer-Cartan element (11.4).

Using the map ι (7.5), we embed $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ into the vector space

$$\prod_{n \ge 1} \mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Gra}(n) \tag{11.9}$$

and represent vectors in (11.9) by formal linear combinations of labeled graphs with two types of edges: solid edges for left tensor factors and dashed edges for right tensor factors.

For example, the Maurer-Cartan element (11.4) corresponds to the linear combination of graphs depicted on figure 44 and the vector

$$\{a_1, a_2\} \otimes \{b_1, b_2\}$$

corresponds to the graph depicted on figure 45

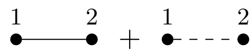


FIG. 44. The Maurer-Cartan element in the deformation complex of Ger

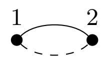


FIG. 45. The graph corresponding to the vector $\{a_1, a_2\} \otimes \{b_1, b_2\}$

DEFINITION 11.2. We say that a monomial $X \in \mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$ is connected if its image in (11.9) is a linear combination of connected graphs. We denote by $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ the subspace of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ which consists of sums (11.7) involving exclusively connected monomials.

EXAMPLE 11.3. According to the above definition the monomials

$$\{a_1, a_2\} \otimes \{b_1, b_2\}, \qquad a_1 a_2 \otimes \{b_1, b_2\}, \qquad \{a_1, a_2\} \otimes b_1 b_2, \qquad a_2\{a_1, a_3\} \otimes b_1\{b_2, b_3\}$$

are connected while the monomials

$$a_1 a_2 \otimes b_1 b_2, \qquad a_2\{a_1, a_3\} \otimes b_2\{b_1, b_3\}$$

are disconnected.

It is not hard to see that $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ is a subcomplex of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$. Furthermore, we have

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) = \mathbf{s}^{-2}\widehat{S}\big(\mathbf{s}^2 \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}\big)\,, \tag{11.10}$$

where $\widehat{S}$ stands for the completed symmetric algebra.

REMARK 11.4. A simple degree bookkeeping shows that for every monomial $X \in \mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$

$$|X| \ge 0\,.$$

Thus,

$$H^{<0}(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}) = H^{<0}(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})) = \mathbf{0}. \tag{11.11}$$

11.1. Decomposition of Conv$(\mathsf{Ger}^{\vee}, \mathsf{Ger})$ with respect to the Euler characteristic. Let us denote by $\mathfrak{b}(v)$ the total number of Lie brackets in the Gerstenhaber monomial $v \in \mathsf{Ger}(n)$ or $v \in \Lambda^{-2}\mathsf{Ger}(n)$. Using the embedding of $\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$ into $\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Gra}(n)$ we introduce the notion of Euler characteristic for monomials in $\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$:

DEFINITION 11.5. Let v and w be monomials in $\mathsf{Ger}(n)$ and $\Lambda^{-2}\mathsf{Ger}(n)$, respectively. We call the number

$$\chi(v \otimes w) := n - \mathfrak{b}(v) - \mathfrak{b}(w)$$

the *Euler characteristic* of the monomial $v \otimes w \in \mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$.

We observe that for every sum

$$\sum_i v_i \otimes w_i \in \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}$$

of monomials with the same Euler characteristic χ, each monomial in the linear combination

$$\partial\left(\sum_i v_i \otimes w_i\right)$$

also has Euler characteristic χ. Thus sums (11.7) in which each γ_n is a linear combination of monomials of Euler characteristic χ form a subcomplex of $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$. We denote this subcomplex by

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\chi}\,. \tag{11.12}$$

We claim that

PROPOSITION 11.6. *For every pair of integers m, χ the subspace* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})^m_{\chi}$ *of degree m vectors in* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\chi}$ *is a subspace in*

$$\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$$

where

$$n = m - \chi + 2\,. \tag{11.13}$$

In particular, $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})^m_{\chi}$ *is finite dimensional.*

PROOF. Let $v \otimes w$ be a monomial in $\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$ of degree m and Euler characteristic χ.

Let t_v (resp. t_w) be the number of Lie words in the monomial v (resp. in the monomial w). For example, if $v = \{a_2, a_4\}a_1 a_7 \{\{a_3, a_5\}, a_6\}$ then $t_v = 4$.

It is not hard to see that

$$|v| = t_v - n, \qquad |w| = n + t_w - 2\,, \tag{11.14}$$

and

$$\mathfrak{b}(v) = n - t_v\,, \qquad \mathfrak{b}(w) = n - t_w\,. \tag{11.15}$$

Hence

$$m = t_v + t_w - 2\,. \tag{11.16}$$

and

$$\chi = t_v + t_w - n\,. \tag{11.17}$$

Using equations (11.16) and (11.17) we deduce that

$$n = m - \chi + 2\,.$$

Thus a combination "degree and Euler characteristic" determines the arity n uniquely via equation (11.13). Furthermore, since $\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)$ is finite dimensional, so is $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})^m_\chi$.

The proposition is proved. □

Proposition 11.6 has the following useful corollary.

COROLLARY 11.7. *The cochain complex* $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ *splits into the following product of its subcomplexes:*

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) = \prod_{\chi \in \mathbb{Z}} \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_\chi\,. \tag{11.18}$$

PROOF. Let

$$\gamma = \sum_{n=1}^{\infty} \gamma_n, \qquad \gamma_n \in \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}$$

be a homogeneous vector of degree m in $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$.

Equation (11.13) implies that for every n

$$\gamma_n \in \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_\chi$$

with $\chi = m + 2 - n$. Thus,

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) \subset \prod_{\chi \in \mathbb{Z}} \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_\chi\,.$$

The other inclusion

$$\prod_{\chi \in \mathbb{Z}} \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_\chi \subset \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$$

is proved similarly. □

The combination of Proposition 11.6 and Corollary 11.7 will allow us to reduce questions about cocycles in $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ to the corresponding questions about cocycles in its subcomplex[17] $\mathrm{Conv}^\oplus(\mathsf{Ger}^\vee, \mathsf{Ger})$.

11.2. We are getting rid of Lie words of length 1. Let us recall that, for a monomial $w \in \Lambda^{-2}\mathsf{Ger}(n)$, the notation $\mathfrak{L}_1(w)$ is reserved for the number of Lie words in w of length $= 1$. For example, $\mathfrak{L}_1(b_1 b_2) = 2$ and $\mathfrak{L}_1(\{b_1, b_2\}) = 0$.

Let us also recall that for the collection $\{\Lambda^{-2}\mathsf{Ger}^\heartsuit(n)\}_{n \ge 0}$ (6.67)

$$\Lambda^{-2}\mathsf{Ger}^\heartsuit(0) = \mathbf{s}^{-2}\mathbb{K}$$

and

$$\Lambda^{-2}\mathsf{Ger}^\heartsuit(n), \qquad n \ge 1$$

is the S_n-submodule of $\Lambda^{-2}\mathsf{Ger}(n)$ spanned by monomials $w \in \Lambda^{-2}\mathsf{Ger}(n)$ for which $\mathfrak{L}_1(w) = 0$.

Using this collection, we introduce the subspace of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$

$$\Xi := \prod_{n \ge 2} \Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}^\heartsuit(n)\Big)^{S_n}, \tag{11.19}$$

[17]The functor $\mathrm{Conv}^\oplus$ was introduced in Section 4.1.

which will play an important role in establishing a link between the deformation complex (11.1) of Ger and the full graph complex fGC (8.3).

We reserve the notation Ξ_{conn} for the "connected part" of Ξ:

$$\Xi_{\mathrm{conn}} := \Xi \cap \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}} \,. \tag{11.20}$$

Furthermore,

$$\Xi^{\oplus} := \Xi \cap \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \tag{11.21}$$

and

$$\Xi^{\oplus}_{\mathrm{conn}} := \Xi_{\mathrm{conn}} \cap \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \cap \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}} \tag{11.22}$$

We claim that

PROPOSITION 11.8. *The subspaces Ξ, Ξ_{conn}, $\Xi^{\oplus}$, and $\Xi^{\oplus}_{\mathrm{conn}}$ are subcomplexes of*

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \,.$$

PROOF. Let

$$X = \sum_{n=2}^{\infty} v_n \otimes w_n \tag{11.23}$$

be a vector in Ξ.

The bracket

$$\big[\, a_1 a_2 \otimes \{b_1, b_2\}\,,\, X \,\big]$$

is obviously a vector in Ξ. So we need to prove that the vector

$$\big[\, \{a_1, a_2\} \otimes b_1 b_2 \,,\, X \,\big] \tag{11.24}$$

belongs Ξ.

We have

$$\big[\, \{a_1, a_2\} \otimes b_1 b_2 \,,\, v_n \otimes w_n \,\big] =$$

$$\sum_{i=1}^{n+1} \varsigma_{i,n+1}\big(\{v_n, a_{n+1}\}\big) \otimes \varsigma_{i,n+1}\big(w_n\, b_{n+1}\big) - \tag{11.25}$$

$$(-1)^{|v_n|} \sum_{\sigma \in \mathrm{Sh}_{2,n-1}} \sigma\big(v_n \circ_1 \{a_1, a_2\}\big) \otimes \sigma\big(w_n \circ_1 b_1 b_2\big) \,,$$

where $\varsigma_{i,n+1}$ is the cycle $(i, i+1, \dots, n+1)$ in S_{n+1}.

Using the defining identities of the Gerstenhaber algebra, it is not hard to prove that unwanted terms in (11.25) cancel each other. □

We will need the following Theorem.

THEOREM 11.9. *The embeddings*

$$\Xi \hookrightarrow \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \,, \tag{11.26}$$

$$\Xi_{\mathrm{conn}} \hookrightarrow \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}} \,, \tag{11.27}$$

and

$$\Xi^{\oplus} \hookrightarrow \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \tag{11.28}$$

are quasi-isomorphisms of cochain complexes.

Proof. We will prove that the embedding (11.28) is a quasi-isomorphism of cochain complexes. Then we will deduce that the embeddings (11.26) and (11.27) are also quasi-isomorphisms.

Let us recall from Section 6.7 that $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$ has the following ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \subset \mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \subset \dots\,, \tag{11.29}$$

where $\mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$ consists of sums

$$\sum_i v_i \otimes w_i \in \bigoplus_n \big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}$$

which satisfy

$$\mathfrak{L}_1(w_i) - |\, v_i \otimes w_i \,| \le m\,, \qquad \forall\ i\,.$$

The restriction of (11.29) to the subcomplex $\Xi^{\oplus}$ gives us the "silly" filtration

$$\mathcal{F}^{m}\big(\Xi^{\oplus}\big)^{k} = \begin{cases} \big(\Xi^{\oplus}\big)^{k} & \text{if } m \ge -k\,, \\ \mathbf{0} & \text{if } m < -k \end{cases} \tag{11.30}$$

with the zero differential on the associated graded complex

$$\mathrm{Gr}\,\Xi^{\oplus} \;\cong\; \bigoplus_{n\ge 2}\Big(\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n}\,. \tag{11.31}$$

Due to Proposition 6.17 in Section 6.7, the formula[18]

$$\Upsilon_{\mathsf{Ger}}\left(\sum_i v_i \otimes w_i\right) := \sum_{\sigma\in \mathrm{Sh}_{r,n}} \sum_i \sigma(v_i(a_1,\dots,a_{r+n}))\otimes\sigma(b_1\dots b_r\, w_i(b_{r+1},\dots,b_{r+n})) \tag{11.32}$$

$$\sum_i v_i \otimes w_i \in \Big(\mathbf{s}^{2r}\mathsf{Ger}(r+n)^{S_r} \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n}$$

defines an isomorphism of cochain complexes

$$\Upsilon_{\mathsf{Ger}}\ :\ \bigoplus_{n\ge 1}\Big(\mathrm{Tw}\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n} \ \to\ \mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger})\,, \tag{11.33}$$

where the differential on the source comes from the differential ∂^{Tw} on $\mathrm{Tw}\mathsf{Ger}$.

It is easy to see that the natural map

$$\mathrm{Gr}\,\Xi^{\oplus} \hookrightarrow \bigoplus_{n\ge 1}\Big(\mathrm{Tw}\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n} \tag{11.34}$$

induced by the embedding (6.46) fits into the commutative diagram

$$\begin{tikzcd} \mathrm{Gr}\,\Xi^{\oplus} \arrow[d, hook] \arrow[dr, hook] & \\ \bigoplus_{n\ge 1}\Big(\mathrm{Tw}\mathsf{Ger}(n) \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n} \arrow[r, "\Upsilon_{\mathsf{Ger}}"] & \mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger})\,, \end{tikzcd}$$

[18]Recall that, due to Exercise 6.13, $\mathrm{Tw}\mathsf{Ger} = \mathrm{Tw}^{\oplus}\mathsf{Ger}$.

where the slanted arrow is the canonical embedding of $\mathrm{Gr}\,\Xi^{\oplus}$ into $\mathrm{Gr}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$.

On the other hand, using Künneth's theorem and Theorem 6.16 together with the fact that, in characteristic zero, the cohomology commutes with taking invariants we deduce that the embedding (11.34) is a quasi-isomorphism of cochain complexes.

Therefore the embedding (11.28) induces a quasi-isomorphism of the associated graded complexes.

Thus, since the filtrations (11.29) and (11.30) are locally bounded and cocomplete, we deduce from Lemma A.3 that the embedding (11.28) is also a quasi-isomorphism of cochain complexes.

Combining this fact with Proposition 11.6 and Corollary 11.7 we conclude that the embedding (11.26) is a quasi-isomorphism of cochain complexes.

Since the cone of the map (11.27) is the direct summand in the cone of the map (11.26), the embedding (11.27) is also a quasi-isomorphism by Claim A.1.

Theorem 11.9 is proved. □

12. Tamarkin's rigidity in the stable setting

Let us consider the Lie algebra

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) = \prod_{n \ge 1} \big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}. \tag{12.1}$$

The map of operads (7.5) induces a homomorphism of Lie algebras

$$\iota_* : \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger}) \to \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}). \tag{12.2}$$

In particular, the vector[19]

$$\iota_*(\alpha) = \Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2 + \Gamma_{\bullet\,\bullet} \otimes \{b_1, b_2\} \tag{12.3}$$

is a Maurer-Cartan element in (12.1) and the formula

$$\partial = [\iota_*(\alpha), \ ] \tag{12.4}$$

defines a differential on the Lie algebra (12.1).

Using map (7.5) once again we can embed (12.1) into (11.9). Thus, by analogy with (11.10), we have

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) = \mathbf{s}^{-2}\widehat{S}(\mathbf{s}^2\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}) \tag{12.5}$$

where $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$ is the subcomplex of $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ which consists of formal linear combinations of connected monomials in $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$.

The goal of this section is to prove the following theorem[20]

THEOREM 12.1. *For the cooperad* $\mathsf{Ger}^{\vee}$ *and the operad* Gra *we have*

$$H^m\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})\big) = \begin{cases} \mathbb{K} & \text{if} \quad m = 1 \\ \mathbf{0} & \quad \text{otherwise}. \end{cases} \tag{12.6}$$

Furthermore, $H^1\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})\big)$ *is spanned by the cohomology class of the vector* $\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2$.

---

[19]The vectors $\Gamma_{\bullet\!-\!\bullet}, \Gamma_{\bullet\,\bullet} \in \mathsf{Gra}(2)$ are defined in (7.3).

[20]Another version of this theorem is proved in [**5**].

The proof of this theorem is given below in Subsection 12.6. It is based on auxiliary constructions which are described in Subsections 12.2, 12.3, 12.4, and 12.5.

Before proceeding to these constructions, we will give a couple of useful corollaries of Theorem 12.1 and discuss its relation to Tamarkin's rigidity from [**20**, Subsection 5.4.5.].

First, we claim that

COROLLARY 12.2. *For the cochain complex* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$ *we have*

$$H^m\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}\big) = \begin{cases} \mathbb{K} & \text{if} \quad m = 1 \\ \mathbf{0} & \qquad \text{otherwise}\,. \end{cases} \tag{12.7}$$

Furthermore, $H^1\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}\big)$ *is spanned by the cohomology class of the vector* $\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2$.

PROOF. Due to Theorem 12.1, every cocycle $c \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ is cohomologous to a cocycle of the form

$$\lambda\, \Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2\,, \qquad \lambda \in \mathbb{K} \tag{12.8}$$

On the other hand, the subcomplex $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$ is a direct summand in $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$. Therefore every cocycle $c \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$ is cohomologous to a cocycle of the form (12.8).

Thus, since the cocycle

$$\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2 \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$$

is non-trivial the desired statement about cohomology of $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\mathrm{conn}}$ follows. □

Following the terminology of Section 4.4 the Lie algebra $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ is equipped with the descending filtration "by arity":

$$\mathcal{F}_m\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) := \{ f \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \mid f(w) = 0 \quad \forall\ w \in \mathsf{Ger}(n),\ n \le m \}\,. \tag{12.9}$$

In other words,

$$\mathcal{F}_m\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) = \prod_{n \ge m+1} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}\,. \tag{12.10}$$

Furthermore, since the Maurer-Cartan element (12.3) belongs to $\mathcal{F}_1\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$, the differential (12.4) is compatible with the filtration (12.9).

Theorem 12.1 implies that

COROLLARY 12.3. *The cochain complex*

$$\mathcal{F}_2\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) = \prod_{n \ge 3} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}$$

with the differential (12.4) *is acyclic.*

PROOF. Let c be a cocycle in $\mathcal{F}_2\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$.

Due to Theorem 12.1 there exists a vector $c_1 \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ and a scalar $\lambda \in \mathbb{K}$ such that

$$c = \lambda\, \Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2 + \partial(c_1)\,. \tag{12.11}$$

On the other hand, it is easy to see that $\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2$ represents a non-trivial cocycle in the quotient

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \;/\; \mathcal{F}_2 \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})\,.$$

Hence $\lambda = 0$ and c is exact. □

12.1. Why rigidity? Let PV_d be the graded vector space of polyvector fields on the affine space $\mathbb{K}^d$. This graded vector space carries the canonical structure of a Gerstenhaber algebra. The multiplication is the exterior multiplication of polyvector fields and the Lie bracket is the well-known Schouten bracket.

Let recall from [**42**] or [**4**, Section 3.5] that the operad Gra acts on PV_d. Moreover, the vector $\Gamma_{\bullet\!-\!\bullet}$ (resp. $\Gamma_{\bullet\bullet}$) gives us the Schouten bracket (resp. the exterior multiplication) on PV_d.

Let us suppose that we are interested in Ger_{∞}-structures $\mathcal{Q}$ on PV_d which satisfy these two properties:

- $\mathcal{Q}$ factors through the canonical map $\mathsf{Gra} \to \mathsf{End}_{\mathrm{PV}_d}$;
- the binary operations of $\mathcal{Q}$ on PV_d coincide with the Schouten bracket and the exterior multiplication.

Using Corollary 12.3, it is not hard to prove that any Ger_{∞}-structure $\mathcal{Q}$ on PV_d satisfying the above properties is homotopy equivalent to the canonical Gerstenhaber algebra structure on PV_d.

This property is an analog of the rigidity[21] of the Gerstenhaber algebra PV_d of polyvector fields in the homotopy category. We refer the reader to [**5**] for more details.

12.2. Decomposition of Conv($\mathsf{Ger}^{\vee}, \mathsf{Gra}$) **with respect to the Euler characteristic.** Let χ be an integer and let c be a vector

$$c \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$$

for which the image

$$1 \otimes \iota(c) \in \prod_{n \ge 1} \mathsf{Gra}(n) \otimes \Lambda^{-2} \mathsf{Gra}(n)$$

is a (possibly infinite) sum of graphs whose Euler characteristic[22] equals χ. We denote by

$$\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\chi}$$

the subspace of such vectors. For example, both summands in $\iota \otimes \iota(\alpha)$ have Euler characteristic 1. Hence

$$\iota_*(\alpha) \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_1\,.$$

It is not hard to see that, for every integer χ, the subspace $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\chi}$ is a subcomplex of $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$.

Let us recall that we represent vectors in the space

$$\mathsf{Gra}(n) \otimes \Lambda^{-2} \mathsf{Gra}(n) \tag{12.12}$$

by linear combinations of labeled graphs with two types of edges: solid edges for left tensor factors and dashed edges for right tensor factors.

[21] This rigidity property is one of the corner stones of Tamarkin's proof [**20**], [**37**] of Kontsevich's formality theorem [**22**].

[22] As above, both solid and dashed edges enter with the same weight.

Let us denote by

$$\big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Gra}(n)\big)_e \tag{12.13}$$

the subspace of (12.12) which is spanned by graphs whose total number of edges (solid and dashed) equals e. It is obvious that the subspace (12.13) is finite dimensional.

We have the following proposition.

PROPOSITION 12.4. *For every pair of integers m, χ the subspace* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^m_{\chi}$ *of degree m vectors in* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})_{\chi}$ *is isomorphic to the subspace of* (12.13) *with*

$$n = m - \chi + 2 \tag{12.14}$$

and

$$e = m - 2\chi + 2\,. \tag{12.15}$$

In particular, $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^m_{\chi}$ *is finite dimensional.*

PROOF. Let Γ be an graph in gra_n representing a vector in $\mathsf{Gra}(n)$ and w be a monomial in $\Lambda^{-2}\mathsf{Ger}(n)$. As above, t_w denotes the total number of Lie monomials and $\mathfrak{b}(w)$ denotes the the total number of brackets in w.

Let us suppose that $\Gamma \otimes w$ carries degree m and $\Gamma \otimes \iota(w)$ has Euler characteristic χ. In other words,

$$m = -e_{\Gamma} + |w|$$

and

$$\chi = n - e_{\Gamma} - \mathfrak{b}(w)\,, \tag{12.16}$$

where e_{Γ} is the number of edges of Γ.

Due to equations (11.14) and (11.15)

$$|w| = n + t_w - 2\,, \qquad t_w = n - \mathfrak{b}(w)\,. \tag{12.17}$$

Therefore,

$$m = n + t_w - 2 - e_{\Gamma} = 2n - \mathfrak{b}(w) - e_{\Gamma} - 2 = n + \chi - 2\,.$$

Thus

$$n = m - \chi + 2\,. \tag{12.18}$$

Using (12.16) and (12.18) we deduce that

$$e_{\Gamma} + \mathfrak{b}(w) = m - 2\chi + 2\,.$$

Hence $\Gamma \otimes \iota(w)$ is a vector in (12.13) with numbers n and e given by equations (12.14) and (12.15).

Thus, the proposition follows from the fact that the map

$$\iota : \Lambda^{-2}\mathsf{Ger}(n) \to \Lambda^{-2}\mathsf{Gra}(n)$$

is injective. □

Proposition 12.4 has the following useful corollary.

COROLLARY 12.5. *The cochain complex* $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$ *splits into the following product of its subcomplexes:*

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra}) = \prod_{\chi \in \mathbb{Z}} \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_\chi \,. \tag{12.19}$$

PROOF. The proof of this statement is very similar to that of Corollary 11.7. So we leave it as an exercise for the reader. □

EXERCISE 12.6. Prove Corollary 12.5.

Just as for $\mathrm{Tw}\mathsf{Gra}$ and $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$, the combination of Proposition 12.4 and Corollary 12.5 will allow us to reduce questions about cocycles in $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})$ to the corresponding questions about cocycles in its subcomplex

$$\mathrm{Conv}^{\oplus}(\mathsf{Ger}^\vee, \mathsf{Gra}) := \bigoplus_{n \ge 1} \big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n} \,. \tag{12.20}$$

12.3. An alternative description of $\mathsf{Gra}(n)$. Let e be a positive integer and

$$\{\rho_1, \rho'_1, \rho_2, \rho'_2, \dots, \rho_e, \rho'_e\} \tag{12.21}$$

be a set of auxiliary variables with degrees $|\rho_i| = -1$ and $|\rho'_i| = 0$.

We will need the symmetric algebra

$$S(V_e) = \mathbb{K} \oplus V_e \oplus S^2(V_e) \oplus S^3(V_e) \oplus \dots \tag{12.22}$$

of the vector space

$$V_e = \mathbb{K}\langle \rho_1, \rho'_1, \rho_2, \rho'_2, \dots, \rho_e, \rho'_e \rangle \,, \tag{12.23}$$

spanned by elements on the set (12.21).

We view $S(V_e)$ as the cocommutative coalgebra with the standard comultiplication.

Let us denote by $T_n\big(S(V_e)\big)$ the subspace

$$T_n\big(S(V_e)\big) \subset \big(S(V_e)\big)^{\otimes n} \tag{12.24}$$

of $\big(S(V_e)\big)^{\otimes n}$ which is spanned by monomials

$$X_1 \otimes X_2 \otimes \dots \otimes X_n \tag{12.25}$$

in which each variable from the set (12.21) appears exactly once.

For example, if $e = 3$ then

$$\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3 \in T_4\big(S(V_e)\big) \,, \qquad \rho'_1\rho_2\rho'_3 \otimes \rho'_2\rho_1\rho_3 \in T_2\big(S(V_e)\big) \,,$$

and

$$\rho_1\rho_2\rho_3 \otimes 1 \otimes \rho'_2\rho_1 \otimes \rho_3\rho'_3 \notin T_4\big(S(V_e)\big) \,, \qquad \rho_1\rho_2\rho'_3 \otimes \rho'_1\rho_3 \notin T_2\big(S(V_e)\big) \,.$$

It makes sense to include the degenerate case $e = 0$ in our consideration. If $e = 0$ then the set (12.21) is empty,

$$S(V_e) = \mathbb{K} \,,$$

and

$$T_n\big(S(V_e)\big) = \mathbb{K}^{\otimes n} \cong \mathbb{K} \,.$$

Given a monomial (12.25) we form a labeled graph Γ' with n vertices and e <u>directed</u> edges following these two steps:

- we declare that edge i originates at the j-th vertex if the factor X_j involves the variable ρ_i;
- we declare that edge i terminates at the k-th vertex if the factor X_k involves the variable ρ'_i.

Since each variable in the set (12.21) appears in the monomial (12.25) exactly once, these two steps give us a labeled graph with n vertices and with e directed edges.

Notice that we use indices of the variables (12.21) to keep track of edges of Γ'. This bijection between the set of edges of Γ' and natural numbers $1, 2, \dots, e$ plays a purely auxiliary role and we do not keep it for Γ' as a piece of additional data.

It is more important to observe that the set $E(\Gamma')$ of edges of Γ' is equipped with an order up to an even permutation. This order is defined by the following rule:

- if the initial vertex of edge i_1 carries a smaller label than the initial vertex of edge i_2 then we set $i_1 < i_2$;
- if edges i_1 and i_2 originate from the same vertex (say, vertex j) and ρ_{i_1} stands to the left from ρ_{i_2} in the factor X_j then we also set $i_1 < i_2$.

For example, the monomial $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ corresponds to the labeled directed graph Γ' depicted on figure 46.

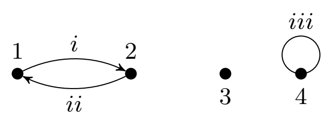


FIG. 46. The directed labeled graph corresponding to the monomial $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$

Let us denote by Γ the undirected graph (with an order on the set of edges up to an even permutation) which is obtained from Γ' by forgetting the directions. It is clear that the formula

$$\Theta(X_1 \otimes X_2 \otimes \cdots \otimes X_n) = \Gamma \tag{12.26}$$

defines a surjective map of graded vector spaces

$$\Theta : \bigoplus_{e\ge 0} T_n\big(S(V_e)\big) \twoheadrightarrow \mathsf{Gra}(n)\,. \tag{12.27}$$

For example, $\Theta(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3) = 0$ because the graph Γ corresponding to the monomial $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ has double edges.

To describe the kernel of (12.27) we recall, from Subsection 9.2.1, the group (9.33)

$$S_e \ltimes (S_2)^e \tag{12.28}$$

with the multiplication law defined by equation (9.34).

Let us equip the graded vector space (12.24) with a left action of the group (12.28).

For this purpose, we declare that elements $\tau \in S_e$ and

$$\sigma_j = (\mathsf{id}, \dots, \mathsf{id}, \underbrace{(12)}_{j\text{-th spot}}, \mathsf{id}, \dots \mathsf{id}) \in (S_2)^e$$

act on generators (12.21) as

$$\tau(\rho_i) = \rho_{\tau(i)}\,, \qquad \tau(\rho'_i) = \rho'_{\tau(i)}\,,$$

and

$$\sigma_j(\rho_i) = \begin{cases} \rho_i & \text{if } i \neq j \\ \rho'_i & \text{if } i = j\,, \end{cases} \qquad \sigma_j(\rho'_i) = \begin{cases} \rho'_i & \text{if } i \neq j \\ \rho_i & \text{if } i = j\,, \end{cases}$$

respectively.

Next, we extend the action of elements $\{\sigma_j\}_{1\le j\le e}$ to the space (12.24) by incorporating appropriate sign factor which appear if odd variables $\rho_1, \dots, \rho_e$ "move around". Finally, we declare that elements of S_e act by automorphisms (of the commutative algebra) $S(V_e)$ and then extend the action of S_e to the space (12.24) by the formula:

$$\tau\big(X_1 \otimes X_2 \otimes \cdots \otimes X_n\big) = \tau(X_1) \otimes \tau(X_2) \otimes \cdots \otimes \tau(X_n)\,.$$

For example, the transposition $(23) \in S_3$ sends the vector $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ to the vector

$$\rho'_1\rho_3 \otimes \rho'_3\rho_1 \otimes 1 \otimes \rho_2\rho'_2$$

and the element σ_1 sends the vector $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ to the vector

$$-\rho_1\rho_2 \otimes \rho'_2\rho'_1 \otimes 1 \otimes \rho_3\rho'_3\,.$$

The sign factor in the action of σ_1 appeared because the variables ρ_1 and ρ_2 changed their order.

Due to Exercise 12.8 below the kernel of Θ (12.27) is spanned by vectors of the form

$$(X_1 \otimes X_2 \otimes \cdots \otimes X_n) - g(X_1 \otimes X_2 \otimes \cdots \otimes X_n)\,, \qquad g \in S_e \ltimes (S_2)^e\,. \tag{12.29}$$

Hence, we conclude that

PROPOSITION 12.7. *The map* Θ (12.27) *induces an isomorphism of graded vector spaces*

$$\mathsf{Gra}(n) \cong \bigoplus_{e\ge 0} \Big(T_n\big(S(V_e)\big)\Big)_{S_e \ltimes (S_2)^e}\,,$$

where $\Big(T_n\big(S(V_e)\big)\Big)_{S_e \ltimes (S_2)^e}$ *denotes the space of coinvariants in* (12.24). $\square$

EXERCISE 12.8. Prove that the kernel of the map Θ (12.27) is spanned by vectors of the form (12.29). *Hint: First, prove that, if a monomial* (12.25) *corresponds to a graph with multiple edges, then this monomial belongs to the span of vectors of the form* (12.29). *Second, consider linear combinations of monomials* (12.25) *each of which does not belong to the kernel of* Θ.

EXAMPLE 12.9. We mentioned above that

$$\Theta(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3) = 0\,.$$

For the monomial $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ we have

$$\sigma_1(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3) = -\rho_1\rho_2 \otimes \rho'_2\rho'_1 \otimes 1 \otimes \rho_3\rho'_3$$

$$\sigma_2(\rho_1\rho_2 \otimes \rho'_2\rho'_1 \otimes 1 \otimes \rho_3\rho'_3) = \rho_1\rho'_2 \otimes \rho_2\rho'_1 \otimes 1 \otimes \rho_3\rho'_3$$

and

$$\varsigma_{12}(\rho_1\rho'_2 \otimes \rho_2\rho'_1 \otimes 1 \otimes \rho_3\rho'_3) = \rho_2\rho'_1 \otimes \rho_1\rho'_2 \otimes 1 \otimes \rho_3\rho'_3\,,$$

where ς_{12} is the transposition (12) in S_3.

Hence,

$$\varsigma_{12}\sigma_2\sigma_1(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3) = -(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3)\,.$$

Thus

$$\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3 = \frac{1}{2}\big(\, \rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3 - \varsigma_{12}\sigma_2\sigma_1(\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3)\,\big)\,.$$

In other words, the monomial $\rho'_1\rho_2 \otimes \rho'_2\rho_1 \otimes 1 \otimes \rho_3\rho'_3$ belongs to the subspace spanned by vectors of the form (12.29).

12.4. An auxiliary cochain complex $\Lambda^{-2}\mathsf{Ger}\big(S(V_e)\big)$. Let us consider the free $\Lambda^{-2}\mathsf{Ger}$-algebra generated by $S(V_e)$

$$\Lambda^{-2}\mathsf{Ger}(S(V_e))\,. \tag{12.30}$$

Using the reduced comultiplication:

$$\widetilde{\Delta}(X) = \Delta(X) - 1 \otimes X - X \otimes 1 \tag{12.31}$$

on $S(V_e)$ we introduce on (12.30) the degree 1 derivation δ defined by the formula

$$\delta(X) = -\sum_i \{X'_i, X''_i\} \tag{12.32}$$

where $X \in S(V_e)$ and X'_i, X''_i are tensor factors in

$$\widetilde{\Delta}(X) = \sum_i X'_i \otimes X''_i\,.$$

For example, since

$$\widetilde{\Delta}(v) = 0 \qquad \forall\, v \in V_e \subset S(V_e)$$

we have

$$\delta(v) = 0 \qquad \forall\, v \in V_e \subset S(V_e)\,.$$

The Jacobi identity implies that

$$\delta^2 = 0\,.$$

Thus δ is a differential on (12.30).

It is clear that the free $\Lambda^{-1}\mathsf{Lie}$-algebra $\Lambda^{-1}\mathsf{Lie}(S(V_e))$ is a subcomplex of (12.30). Furthermore,

$$\Lambda^{-2}\mathsf{Ger}(S(V_e)) \cong \mathbf{s}^{-2}S\big(\mathbf{s}^2\Lambda^{-1}\mathsf{Lie}(S(V_e))\big) \tag{12.33}$$

as cochain complexes.

On the other hand, Theorem B.1 from Appendix[23] B implies that for every cocycle $c \in \Lambda^{-1}\mathsf{Lie}(S(V_e))$ there exists a vector $c_1 \in \Lambda^{-1}\mathsf{Lie}(S(V_e))$ and a vector $v \in V_e$ such that

$$c = v + \delta(c_1)\,.$$

Furthermore, each non-zero vector $v \in V_e$ is a non-trivial cocycle in $\Lambda^{-1}\mathsf{Lie}(S(V_e))$.

Therefore, due to Künneth's theorem,

$$H^{\bullet}\big(\,\Lambda^{-2}\mathsf{Ger}(S(V_e)), \delta\,\big) \cong \mathbf{s}^{-2}S\big(\mathbf{s}^2 V_e\big) \tag{12.34}$$

and the space

$$H^{\bullet}\big(\,\Lambda^{-2}\mathsf{Ger}(S(V_e)), \delta\,\big)$$

[23] Since formulas (12.32) and (B.5) for the differentials differ only by the overall sign factor, Theorem B.1 can be applied in this case.

is spanned by the cohomology classes of the vectors

$$b_1 \dots b_n \otimes (v_1 \otimes \dots \otimes v_n)\,, \tag{12.35}$$

where $v_i \in V_e$ and $b_1 \dots b_n$ is the generator of $\Lambda^{-2}\mathsf{Com}(n) \subset \Lambda^{-2}\mathsf{Ger}(n)$.

Thus we arrive at the following statement.

PROPOSITION 12.10. *For any cocycle*

$$c \in \Lambda^{-2}\mathsf{Ger}(S(V_e))$$

there exists a vector $c_1 \in \Lambda^{-2}\mathsf{Ger}(S(V_e))$ *such that the difference*

$$c - \delta(c_1)$$

belongs to the linear span of (12.35)*. Furthermore, a vector*

$$Y \in \mathbf{s}^{-2} S\big(\mathbf{s}^2 V_e\big)$$

is δ-exact if and only if $Y = 0$. □

12.4.1. *A equivalent description of* (12.30) *in terms of invariants.* Let $\mathcal{G}'(V_e)$ denote the following graded vector space

$$\mathcal{G}'(V_e) := \bigoplus_n \Big(\Lambda^{-2}\mathsf{Ger}(n) \otimes \big(S(V_e)\big)^{\otimes\, n}\Big)^{S_n}. \tag{12.36}$$

Since our base field has characteristic zero, this graded vector space is isomorphic to

$$\Lambda^{-2}\mathsf{Ger}(S(V_e)) = \bigoplus_n \Big(\Lambda^{-2}\mathsf{Ger}(n) \otimes \big(S(V_e)\big)^{\otimes\, n}\Big)_{S_n}. \tag{12.37}$$

For example, one may define an isomorphism $\mathfrak{I}$ from (12.37) to (12.36) by the formula:

$$\mathfrak{I}(w; X_1 \otimes \dots \otimes X_n) = \sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} (\sigma(w); X_{\sigma^{-1}(1)} \otimes \dots \otimes X_{\sigma^{-1}(n)})\,, \tag{12.38}$$

where $w \in \Lambda^{-2}\mathsf{Ger}(n)$, $X_i \in S(V_e)$, the sign factor $(-1)^{\varepsilon(\sigma)}$ comes from the usual Koszul rule, and $(w; X_1 \otimes \dots \otimes X_n)$ represents a vector in

$$\Big(\Lambda^{-2}\mathsf{Ger}(n) \otimes \big(S(V_e)\big)^{\otimes\, n}\Big)_{S_n}.$$

Let

$$\sum_t (w_t; X^t_1 \otimes \dots \otimes X^t_n)$$

be a vector in

$$\Big(\Lambda^{-2}\mathsf{Ger}(n) \otimes \big(S(V_e)\big)^{\otimes\, n}\Big)^{S_n}$$

and let δ' be a degree 1 operation on $\mathcal{G}'(V_e)$ given by the equation:

$$\delta'\left(\sum_t (w_t; X^t_1 \otimes \dots \otimes X^t_n)\right) = \tag{12.39}$$

$$\sum_t \sum_{\sigma \in \mathrm{Sh}_{n,1}} \sigma(\{w_t, b_{n+1}\}; X^t_1 \otimes \dots \otimes X^t_n \otimes 1)$$

$$-\sum_t \sum_{\tau \in \mathrm{Sh}_{2,n-1}} (-1)^{|w_t|} \tau(w_t \circ_1 \{b_1, b_2\}; \Delta X^t_1 \otimes X^t_2 \otimes \dots \otimes X^t_n)\,.$$

A direct but tedious computation shows that

$$\mathfrak{I} \circ \delta = 2\delta' \circ \mathfrak{I}\,. \tag{12.40}$$

In other words, δ' (12.39) is a differential on (12.36) and the cohomology of the cochain complex

$$\big(\mathcal{G}'(V_e), \delta'\big) \tag{12.41}$$

is isomorphic to the cohomology of (12.30) with the differential (12.32).

For our purpose, we need to switch to yet another cochain complex $\mathcal{G}(V_e)$ isomorphic to (12.41). This new cochain complex is obtained from $\mathcal{G}'(V_e)$ by exchanging the order of the tensor factors. Namely,

$$\mathcal{G}(V_e) := \bigoplus_n \Big(\big(S(V_e)\big)^{\otimes\, n} \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n}. \tag{12.42}$$

The differential $\tilde{\delta}$ induced on (12.42) by the natural isomorphism between (12.36) and (12.42) is given by the formula:

$$\tilde{\delta}\left(\sum_t (X_1^t \otimes \dots \otimes X_n^t; w_t)\right) = \tag{12.43}$$

$$\sum_t \sum_{\sigma \in \mathrm{Sh}_{n,1}} (-1)^{|X_1^t| + \dots + |X_n^t|} \sigma(X_1^t \otimes \dots \otimes X_n^t \otimes 1; \{w_t, b_{n+1}\})$$

$$- \sum_t \sum_{\tau \in \mathrm{Sh}_{2,n-1}} (-1)^{|w_t| + |X_1^t| + \dots + |X_n^t|} \tau(\Delta X_1^t \otimes X_2^t \otimes \dots \otimes X_n^t; w_t \circ_1 \{b_1, b_2\})\,.$$

Thus Proposition 12.10 implies the following statement.

Corollary 12.11. *For any cocycle*

$$c \in \mathcal{G}(V_e)$$

there exists a vector $c_1 \in \mathcal{G}(V_e)$ such that the difference

$$c - \tilde{\delta}(c_1)$$

belongs to the linear span of vectors of the form

$$\sum_{\sigma \in S_n} (v_1 \otimes \dots \otimes v_n; b_1 \dots b_n), \tag{12.44}$$

where $v_1, v_2, \dots, v_n \in V_e$ and $b_1 \dots b_n$ is the generator of $\Lambda^{-2}\mathsf{Com}(n) \subset \Lambda^{-2}\mathsf{Ger}(n)$. Furthermore, a linear combination Y of vectors of the form (12.44) *is $\tilde{\delta}$-exact if and only if $Y = 0$.* $\square$

12.5. The associated graded complex GrConv$^{\oplus}$($\mathsf{Ger}^{\vee}$, Gra). Let us recall that $\mathfrak{b}(w)$ denotes the total number of Lie brackets in a monomial $w \in \Lambda^{-2}\mathsf{Ger}(n)$.

Let

$$\sum_i v_i \otimes w_i$$

be a vector in

$$\Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}$$

such that the number $k_{\mathfrak{b}} = \mathfrak{b}(w_i)$ is the same for every monomial w_i.

It is obvious that for every monomial w'_j in

$$\partial(v \otimes w) = \sum_j v'_j \otimes w'_j$$

we have $\mathfrak{b}(w'_j) = k_{\mathfrak{b}}$ or $\mathfrak{b}(w'_j) = k_{\mathfrak{b}} + 1$.

This observation allows us to introduce an ascending filtration

$$\cdots \subset \mathcal{F}^{m-1}_{\mathfrak{b}} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \subset \mathcal{F}^{m}_{\mathfrak{b}} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \subset \dots \tag{12.45}$$

where $\mathcal{F}^{m}_{\mathfrak{b}} \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ is spanned by homogeneous vectors

$$\gamma = \sum_i v_i \otimes w_i \in \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$$

in which each monomial w_i satisfies the inequality

$$\mathfrak{b}(w_i) - |\gamma| \le m\,.$$

It is clear that the differential $\partial^{\mathrm{Gr}_{\mathfrak{b}}}$ on the associated graded complex

$$\mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \tag{12.46}$$

is obtained from the differential ∂ (12.4) by keeping only the terms which raise the number of Lie brackets in the second tensor factors. Namely,

$$\partial^{\mathrm{Gr}_{\mathfrak{b}}} = \big[\, \Gamma_{\bullet\bullet} \otimes \{b_1, b_2\},\ \big]\,. \tag{12.47}$$

Our goal is to give a convenient description of the associated graded complex (12.46) using the map Θ (12.27) introduced in Subsection 12.3 and the cochain complex (12.42) introduced in Subsection 12.4.1.

First, we observe that, as a graded vector space,

$$\mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \cong \bigoplus_{n \ge 1} \Big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n}. \tag{12.48}$$

Thus, due to Proposition 12.7, the map Θ (12.27) induces an isomorphism of graded vector spaces

(12.49)

$$\bigoplus_{n \ge 1} \bigoplus_{e \ge 0} \bigg(\Big(T_n\big(S(V_e)\big) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)_{S_e \ltimes (S_2)^e} \bigg)^{S_n} \cong \mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$$

Since the action of the group $S_e \ltimes (S_2)^e$ commutes with the action of S_n we conclude that Θ induces an isomorphism of graded vector spaces:

(12.50)

$$\Theta' \ : \ \bigoplus_{e \ge 0} \left(\bigoplus_{n \ge 1} \Big(T_n\big(S(V_e)\big) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \right)_{S_e \ltimes (S_2)^e} \to\ \mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra})\,.$$

On the other hand,

$$\bigoplus_{n \ge 1} \Big(T_n\big(S(V_e)\big) \otimes \Lambda^{-2}\mathsf{Ger}(n) \Big)^{S_n} \tag{12.51}$$

is a subspace in the cochain complex $\mathcal{G}(V_e)$ (12.42).

We claim that

PROPOSITION 12.12. *The subspace* (12.51) *is a direct summand in the cochain complex* $\mathcal{G}(V_e)$ (12.42) *with the differential* $\tilde{\delta}$ (12.43). *Furthermore, the isomorphism* (12.50) *is compatible with the differentials.*

PROOF. Let us recall, from Subsection 12.3, that V_e is the graded vector space of finite linear combinations of variables from the set (12.21).

The subspace (12.51) is spanned by vectors of the form

$$\sum_{\sigma \in S_n} \sigma(X_1 \otimes X_2 \otimes \dots \otimes X_n \,;\, w) \tag{12.52}$$

where $w \in \Lambda^{-2}\mathsf{Ger}(n)$ and

$$X_1 \otimes X_2 \otimes \dots \otimes X_n \tag{12.53}$$

is a monomial in $\big(S(V_e)\big)^{\otimes n}$ satisfying

PROPERTY 12.13. Each variable from the set (12.21) appears in (12.53) exactly once.

It is clear that this subspace is closed with respect to $\tilde{\delta}$ (12.43). Moreover, the cochain complex $\mathcal{G}(V_e)$ splits into the direct sum of (12.51) and the subcomplex spanned by vectors of the form (12.52) for which (12.53) does <u>not</u> satisfy Property 12.13.

To prove equation

$$\partial^{\mathrm{Gr}_{\mathfrak{b}}} \circ \Theta' = \Theta' \circ \tilde{\delta} \tag{12.54}$$

we consider a monomial (12.53) satisfying Property 12.13 and a vector $w \in \Lambda^{-2}\mathsf{Ger}(n)$. We denote by Γ the graph in gra_n which corresponds to the monomial (12.53).

Going through the construction of the map Θ it is easy to verify that

$$(\Theta \otimes 1) \circ \tilde{\delta} \left(\sum_{\sigma \in S_n} (-1)^{\varepsilon(\sigma)} X_{\sigma^{-1}(1)} \otimes X_{\sigma^{-1}(2)} \otimes \dots \otimes X_{\sigma^{-1}(n)} \otimes \sigma(w) \right) = \tag{12.55}$$

$$\sum_{\lambda \in \mathrm{Sh}_{n,1}} (-1)^{|\Gamma|} \sum_{\sigma \in S_n} \lambda(\Gamma_{\bullet\bullet} \circ_1 \sigma(\Gamma)) \otimes \lambda\big(\{\sigma(w), b_{n+1}\}\big) -$$

$$- \sum_{\tau \in \mathrm{Sh}_{2,n-1}} (-1)^{|\Gamma| + |w|} \sum_{\sigma \in S_n} \tau\big(\sigma(\Gamma) \circ_1 \Gamma_{\bullet\bullet}\big) \otimes \tau\big(\sigma(w) \circ_1 \{b_1, b_2\}\big)\,.$$

Since the right hand side of (12.55) equals

$$\left[\Gamma_{\bullet\bullet} \otimes \{b_1, b_2\}\ ,\ \sum_{\sigma \in S_n} \sigma(\Gamma) \otimes \sigma(w) \right]$$

equation (12.54) follows. □

Combining Corollary 12.11 with Proposition 12.12 we deduce

COROLLARY 12.14. *For the associated graded complex* (12.46) *we have*

$$H^\bullet\big(\mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^\vee, \mathsf{Gra})\big) = \begin{cases} \mathbb{K} & \text{if } \bullet = 1 \\ \mathbf{0} & \text{otherwise}\,. \end{cases} \tag{12.56}$$

Furthermore,

$$H^1\big(\mathrm{Gr}_{\mathfrak{b}}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^\vee, \mathsf{Gra})\big)$$

is spanned by the cohomology class of the vector represented by $\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2$.

PROOF. Since the cochain complex (12.51) is a direct summand in $\mathcal{G}(V_e)$ and the cohomology commutes with taking coinvariants, Corollary 12.11 implies that the cohomology of the cochain complex

$$\left(\bigoplus_{n\ge 1}\Big(T_n\big(S(V_e)\big)\otimes\Lambda^{-2}\mathsf{Ger}(n)\Big)^{S_n}\right)_{S_e\ltimes(S_2)^e} \tag{12.57}$$

is spanned by the classes of vectors of the form

$$\sum_{\sigma\in S_{2e}}\sigma(\rho_1\otimes\rho_1'\otimes\rho_2\otimes\rho_2'\otimes\cdots\otimes\rho_e\otimes\rho_e'\ ;\ b_1\dots b_{2e}), \tag{12.58}$$

$b_1\dots b_{2e}$ is the generator of $\Lambda^{-2}\mathsf{Com}(2e)\subset\Lambda^{-2}\mathsf{Ger}(2e)$.

Since variables $\rho_1,\dots,\rho_e$ are odd, it is not hard to see that (12.58) represents the zero vector in the coinvariants (12.57) whenever $e>1$.

On the other hand, the map Θ' (12.50) sends the vector

$$(\rho_1\otimes\rho_1';b_1b_2)+(\rho_1'\otimes\rho_1;b_1b_2) \tag{12.59}$$

to the non-trivial cocycle

$$2\Gamma_{\bullet\!-\!\bullet}\otimes b_1b_2\,.$$

Hence, the corollary follows from Proposition 12.12. $\square$

12.6. Proof of Theorem 12.1. Let us denote by $\mathcal{H}$ the subcomplex of $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee},\mathsf{Gra})$

$$\mathcal{H}=\mathbb{K}\langle\,\Gamma_{\bullet\!-\!\bullet}\otimes b_1b_2\,\rangle \tag{12.60}$$

spanned by the single cocycle $\Gamma_{\bullet\!-\!\bullet}\otimes b_1b_2$.

By construction, the cochain complex $\mathcal{H}$ carries the zero differential. Moreover, restricting (12.45) on $\mathcal{H}$ we get the "silly" filtration

$$\mathcal{F}^m\mathcal{H}^k=\begin{cases}\mathcal{H}^k & \text{if } m\ge -k\,,\\ \mathbf{0} & \text{otherwise}\end{cases} \tag{12.61}$$

with

$$\mathrm{Gr}\,\mathcal{H}\cong\mathcal{H}\,. \tag{12.62}$$

Corollary 12.14 implies that the embedding

$$\mathcal{H}\hookrightarrow\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee},\mathsf{Gra}) \tag{12.63}$$

induces a quasi-isomorphism on the level of associated graded complexes.

Since the filtrations on $\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee},\mathsf{Gra})$ and $\mathcal{H}$ are bounded from the left and cocomplete, Lemma A.3 from Appendix A implies that the embedding (12.63) is quasi-isomorphism of cochains complexes.

Combining this fact with Proposition 12.4 and Corollary 12.5 we conclude that the embedding

$$\mathcal{H}\hookrightarrow\mathrm{Conv}(\mathsf{Ger}^{\vee},\mathsf{Gra}) \tag{12.64}$$

is also a quasi-isomorphism of cochain complexes.

Since $\mathcal{H}$ is spanned by the cocycle $\Gamma_{\bullet\!-\!\bullet}\otimes b_1b_2$, Theorem 12.1 is proved. $\square$

13. Deformation complex of Ger versus Kontsevich's graph complex

This section is the culmination of our text. Using the results proved above, we establish here a link between the (extended) deformation complex $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$ (11.6) of the operad Ger and full graph complex fGC (See Definition 8.1.)

First, recall that, due to decompositions (8.16) and (11.10), the cohomology of the cochain complex fGC (resp. $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})$) can be expressed in terms of the cohomology of its "connected part" $\mathsf{fGC}_{\mathrm{conn}}$ (resp. $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$). Namely,

$$H^\bullet(\mathsf{fGC}) \cong \mathbf{s}^{-2}\widehat{S}\big(\mathbf{s}^2\, H^\bullet(\mathsf{fGC}_{\mathrm{conn}})\big)\,, \tag{13.1}$$

and

$$H^\bullet\big(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})\big) \cong \mathbf{s}^{-2}\widehat{S}\Big(\mathbf{s}^2\, H^\bullet\big(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}\big)\Big)\,. \tag{13.2}$$

Let us denote by $\mathfrak{R}$ the natural map of graded vector spaces

$$\mathfrak{R} \;:\; \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}} \;\to\; \mathsf{fGC}_{\mathrm{conn}} = \mathrm{Conv}(\Lambda^2\mathsf{coCom}, \mathsf{Gra})_{\mathrm{conn}} \tag{13.3}$$

given by the formula

$$\mathfrak{R}(f) = f\Big|_{\Lambda^2\mathsf{coCom}}\,.$$

It is not hard to see that $\mathfrak{R}$ is a map of cochain complexes. We observe that the map of dg Lie algebras ι_* (12.2)

$$\iota_* : \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger}) \to \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})\,.$$

satisfies the following property

$$\mathfrak{R}(\iota_*(X)) = 0\,, \qquad \forall\; X \in \Xi_{\mathrm{conn}}\,,$$

where Ξ_{conn} is defined in (11.20).

Therefore, restricting ι_* to the subcomplex Ξ_{conn} we get a map of cochain complexes

$$\psi := \iota_*\Big|_{\Xi_{\mathrm{conn}}} \;:\; \Xi_{\mathrm{conn}} \;\to\; \ker\mathfrak{R}\,. \tag{13.4}$$

We claim that

PROPOSITION 13.1. *The map ψ (13.4) is a quasi-isomorphism of cochain complexes.*

Let us postpone the proof of this proposition to Subsection 13.1 and deduce a link between the cohomology of $\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Ger})_{\mathrm{conn}}$ and the cohomology of $\mathsf{fGC}_{\mathrm{conn}}$.

Recall that, due to Corollary 12.2, $H^\bullet\big(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}\big)$ is spanned by the cohomology class of the vector $\Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2$.

Therefore, if we set

$$\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})^+_{\mathrm{conn}} = \mathbb{K} \oplus \mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}} \tag{13.5}$$

and extend the differential ∂ to $\mathrm{Conv}^+(\mathsf{Ger}^\vee, \mathsf{Gra})_{\mathrm{conn}}$ by declaring that for $1 \in \mathbb{K}$

$$\partial(1) = \Gamma_{\bullet\!-\!\bullet} \otimes b_1 b_2\,, \tag{13.6}$$

then we get an acyclic cochain complex

$$\big(\mathrm{Conv}(\mathsf{Ger}^\vee, \mathsf{Gra})^+_{\mathrm{conn}}, \partial\big)\,.$$

Similarly, we "add" to the graph complex $\mathsf{fGC}_{\mathrm{conn}}$ a one-dimensional vector space

$$\mathsf{fGC}^+_{\mathrm{conn}} = \mathbb{K} \oplus \mathsf{fGC}_{\mathrm{conn}} \tag{13.7}$$

and extend the differential by declaring that for $1 \in \mathbb{K}$

$$\partial(1) = \Gamma_{\bullet\!-\!\bullet}\,.$$

Due to Exercise 8.4 from Section 8 we have

$$\partial \Gamma_{\bullet} = \Gamma_{\bullet\!-\!\bullet}\,,$$

where $\Gamma_{\bullet}$ is the graph with the single vertex and no edges. Therefore,

$$H^{\bullet}\big(\mathsf{fGC}^+_{\mathrm{conn}}\big) \cong H^{\bullet}\big(\mathsf{fGC}_{\mathrm{conn}}\big) \oplus \mathbb{K}\langle \phi \rangle\,, \tag{13.8}$$

where ϕ is the cohomology class represented by the cocycle

$$\Gamma_{\bullet} \,-\, 1 \in \mathsf{fGC}^+_{\mathrm{conn}}\,.$$

The map $\mathfrak{R}$ (13.3) extends in the obvious way to the morphism of cochain complexes:

$$\mathfrak{R}^+ \ :\ \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^+_{\mathrm{conn}} \ \to\ \mathsf{fGC}^+_{\mathrm{conn}}\,. \tag{13.9}$$

Furthermore,

$$\ker(\mathfrak{R}^+) = \ker(\mathfrak{R})\,.$$

Thus we arrive at the diagram

(13.10)

$$\begin{tikzcd}
 & \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}} & & & \\
 & \Xi_{\mathrm{conn}} \arrow[u, hook, "\mathrm{emb}_{\Xi}"] \arrow[d, "\psi"] & & & \\
\mathbf{0} \arrow[r] & \ker(\mathfrak{R}) \arrow[r] & \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^+_{\mathrm{conn}} \arrow[r] & \mathsf{fGC}^+_{\mathrm{conn}} \arrow[r] & \mathbf{0}
\end{tikzcd}$$

The bottom row of this diagram is an exact sequence of cochain complexes. The top vertical arrow emb_{Ξ} is a quasi-isomorphism due to Theorem 11.9. The vertical arrow ψ is also a quasi-isomorphism due to Proposition 13.1. Finally the cochain complex $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^+_{\mathrm{conn}}$ in the middle of the exact sequence is acyclic.

Using diagram (13.10), we can now prove the main theorem of these notes.

THEOREM 13.2 (T. Willwacher, [**42**]). *If* $\mathsf{fGC}_{\mathrm{conn}}$ *is the "connected part" of the full graph complex* fGC (8.3) *and* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}$ *is the "connected part" of the extended deformation complex* $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})$ (11.6) *of the operad* Ger *then*

$$H^{\bullet+1}\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}\big) \cong H^{\bullet}\big(\mathsf{fGC}_{\mathrm{conn}}\big) \ \oplus\ \mathbb{K}\,. \tag{13.11}$$

PROOF. Since the cochain complex $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^+_{\mathrm{conn}}$ in (13.10) is acyclic, the connecting homomorphism induces an isomorphism

$$H^{\bullet}\big(\mathsf{fGC}^+_{\mathrm{conn}}\big) \cong H^{\bullet+1}\big(\ker(\mathfrak{R})\big)\,.$$

On the other hand,

$$H^{\bullet}\big(\ker(\mathfrak{R})\big) \cong H^{\bullet}\big(\Xi_{\mathrm{conn}}\big) \cong H^{\bullet}\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}\big)$$

because both ψ and $\mathrm{emb}_{\Xi_{\mathrm{conn}}}$ are quasi-isomorphisms.

Therefore,

$$H^{\bullet+1}\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}\big) \cong H^{\bullet}\big(\mathsf{fGC}^{+}_{\mathrm{conn}}\big)\,. \tag{13.12}$$

Thus, using the isomorphism (13.8), we arrive at the desired result (13.11). □

REMARK 13.3. The above proof gives us a concrete isomorphism from

$$H^{\bullet}\big(\mathsf{fGC}_{\mathrm{conn}}\big) \ \oplus\ \mathbb{K} \tag{13.13}$$

to

$$H^{\bullet+1}\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}\big)\,. \tag{13.14}$$

Chasing through diagram (13.10), it is not hard to see that the vector $1 \in \mathbb{K}$ in the second summand of (13.13) is sent, via this isomorphism, to the class represented by the cocycle

$$a_1 a_2 \otimes \{b_1, b_2\}$$

or the cocycle

$$-\{a_1, a_2\} \otimes b_1 b_2$$

in $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})_{\mathrm{conn}}$.

REMARK 13.4. According to [**39**], the Lie algebra $\mathfrak{grt}$ of the Grothendieck-Teichmueller group GRT embeds into $H^0\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})\big)$. Since $\mathfrak{grt}$ is infinite dimensional [**8**], the spaces $H^0\big(\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Ger})\big)$ and $H^0(\mathsf{fGC})$ are also infinite dimensional.

13.1. Proof of Proposition 13.1. Let us prove that the map

$$\psi\Big|_{\Xi^{\oplus}_{\mathrm{conn}}} \ :\ \Xi^{\oplus}_{\mathrm{conn}} \to \ker(\mathfrak{R}) \cap \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^{\oplus}_{\mathrm{conn}} \tag{13.15}$$

is a quasi-isomorphism of cochain complexes.

For this purpose we apply the general construction of Section 6.7 to the case when $\mathcal{O} = \mathsf{Gra}$.

Following Section 6.7, the cochain complex $\mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^{\oplus}$ carries the ascending filtration

$$\dots \subset \mathcal{F}^{m-1}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \subset \mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra}) \subset \dots\,, \tag{13.16}$$

where $\mathcal{F}^{m}\,\mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra})$ consists of sums

$$\sum_i v_i \otimes w_i \in \bigoplus_n \big(\mathsf{Gra}(n) \otimes \Lambda^{-2}\mathsf{Ger}(n)\big)^{S_n}$$

which satisfy

$$\mathfrak{L}_1(w_i) - |\, v_i \otimes w_i \,| \le m\,, \qquad \forall\ i\,.$$

Furthermore, due to Proposition 6.17, the formula

$$\Upsilon_{\mathsf{Gra}}\left(\sum_i v_i \otimes w_i\right) := \sum_{\sigma \in \mathrm{Sh}_{r,n}} \sum_i \sigma(v_i) \otimes \sigma(b_1 \dots b_r\, w_i(b_{r+1}, \dots, b_{r+n})) \tag{13.17}$$

$$\sum_i v_i \otimes w_i \in \Big(\mathbf{s}^{2r}\mathsf{Gra}(r+n)^{S_r} \otimes \Lambda^{-2}\mathsf{Ger}^{\heartsuit}(n)\Big)^{S_n}$$

defines an isomorphism of cochain complexes

$$\Upsilon_{\mathsf{Gra}} \ : \ \bigoplus_{n \ge 0} \Big(\mathrm{Tw}^{\oplus} \mathsf{Gra}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n} \ \to \ \mathrm{Gr}\, \mathrm{Conv}^{\oplus}(\mathsf{Ger}^{\vee}, \mathsf{Gra})\,, \tag{13.18}$$

where the differential on

$$\bigoplus_{n \ge 0} \Big(\mathrm{Tw}^{\oplus} \mathsf{Gra}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n}$$

comes from the differential ∂^{Tw} on $\mathrm{Tw}^{\oplus}\mathsf{Gra}(n)$.

Let us restrict the filtration (13.16) to the subcomplex

$$\ker(\mathfrak{R}) \cap \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^{\oplus}_{\mathrm{conn}}$$

and recall that the n-th space

$$\mathsf{fgraphs}(n) := \mathsf{fGraphs}(n) \cap \mathrm{Tw}^{\oplus} \mathsf{Gra}(n) \tag{13.19}$$

of the dg operad $\mathsf{fgraphs}$ is spanned by vectors of the form

$$\sum_{\sigma \in S_r} \sigma(\Gamma)\,,$$

where the graph $\Gamma \in \mathsf{gra}_{r+n}$ has no connected components which involve exclusively neutral vertices (i.e. vertices with labels $\le r$).

It is not hard to see that the restriction of Υ_{Gra} to

$$\bigoplus_{n \ge 2} \Big(\mathsf{fgraphs}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n}_{\mathrm{conn}}$$

gives us an isomorphism

$$\Upsilon' : \bigoplus_{n \ge 2} \Big(\mathsf{fgraphs}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n}_{\mathrm{conn}} \ \to \ \mathrm{Gr}\, \Big(\ker(\mathfrak{R}) \cap \mathrm{Conv}(\mathsf{Ger}^{\vee}, \mathsf{Gra})^{\oplus}_{\mathrm{conn}} \Big) \tag{13.20}$$

of cochain complexes.

On the other hand, Corollary 9.25 implies that the natural embedding

$$\Xi^{\oplus} \hookrightarrow \bigoplus_{n \ge 2} \Big(\mathsf{fgraphs}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n} \tag{13.21}$$

is a quasi-isomorphism of cochain complexes.

Therefore, since the cone of the embedding

$$\Xi^{\oplus}_{\mathrm{conn}} \hookrightarrow \bigoplus_{n \ge 2} \Big(\mathsf{fgraphs}(n) \otimes \Lambda^{-2} \mathsf{Ger}^{\heartsuit}(n) \Big)^{S_n}_{\mathrm{conn}} \tag{13.22}$$

is a direct summand in the cone of the embedding (13.21), the map (13.22) is also a quasi-isomorphism.

This observation allows us to conclude that the map (13.15) induces a quasi-isomorphism on the level of associated graded complexes.

Since the filtration (13.16) is locally bounded and cocomplete, Lemma A.3 implies that (13.15) is indeed a quasi-isomorphism of cochain complexes.

Thus, using the Euler characteristic trick, we conclude that the map ψ (13.4) is also a quasi-isomorphism of cochain complexes.

Proposition 13.1 is proved. □

Appendix A. Lemma on a quasi-isomorphism of filtered complexes

Let us recall that a cone $\mathrm{Cone}(f)$ of a morphism of cochain complexes $f : C \to K$ is the cochain complex

$$C \oplus \mathbf{s}K$$

with the differential

$$\partial^{\mathrm{Cone}}(v_1 + \mathbf{s}v_2) = \partial(v_1) + \mathbf{s}f(v_1) - \mathbf{s}\partial(v_2)\,,$$

where we denote by ∂ the differentials on both complexes C and K.

Let us also recall a claim which follows easily from Lemma 3 in [**12**, Section III.3.2]:

Claim A.1. *A morphism $f : C \to K$ of cochain complexes is a quasi-isomorphism if and only if the cochain complex* $\mathrm{Cone}(f)$ *is acyclic.* □

Let C be a cochain complex equipped with an ascending filtration:

$$\cdots \subset \mathcal{F}^{m-1}C \subset \mathcal{F}^{m}C \subset \mathcal{F}^{m+1}C \subset \dots\,.$$

We say that the filtration on C is *cocomplete* if

$$C = \bigcup_m \mathcal{F}^m C\,. \tag{A.1}$$

Furthermore, we say that the filtration on C is *locally bounded from the left* if for every degree d there exists an integers m_d such that

$$\mathcal{F}^{m_d} C^d = \mathbf{0}\,. \tag{A.2}$$

Let us denote by $\mathrm{Gr}(C)$ the associated graded cochain complex

$$\mathrm{Gr}(C) := \bigoplus_m \mathcal{F}^m C \big/ \mathcal{F}^{m-1} C\,. \tag{A.3}$$

We will need the following claim.

Claim A.2. *Let C be a cochain complex equipped with a cocomplete ascending filtration which is locally bounded from the left. If* $\mathrm{Gr}(C)$ *is acyclic then so is C.*

Proof. Let v be cocycle in C of degree d. Our goal is to show that there exists a vector $w \in C^{d-1}$ such that

$$v = \partial w\,.$$

Since the filtration on C is cocomplete there exists an integer m such that

$$v \in \mathcal{F}^m C^d\,.$$

Therefore v represents a cocycle in the quotient

$$\mathcal{F}^m C^d \big/ \mathcal{F}^{m-1} C^d\,.$$

On the other hand, $\mathrm{Gr}(C)$ is acyclic. Hence there exists a vector $w_m \in \mathcal{F}^m C^{d-1}$ such that

$$v - \partial(w_m) \in \mathcal{F}^{m-1} C^d\,. \tag{A.4}$$

The latter implies that the vector $v - \partial(w_m)$ represents a cocycle in the quotient

$$\mathcal{F}^{m-1} C^d \big/ \mathcal{F}^{m-2} C^d\,.$$

Hence, there exists a vector $w_{m-1} \in \mathcal{F}^{m-1} C^{d-1}$ such that

$$v - \partial(w_m) - \partial(w_{m-1}) \in \mathcal{F}^{m-2} C^d\,. \tag{A.5}$$

Continuing this process, we conclude that there exists a sequence of vectors

$$w_k \in \mathcal{F}^k C^{d-1}, \qquad k \le m$$

such that for every $k < m$ we have

$$v - \partial(w_m + w_{m-1} + \dots + w_k) \in \mathcal{F}^{k-1} C^d \,. \tag{A.6}$$

Since the filtration on C is locally bounded from the left there exists an integer $k_d < m$ such that $\mathcal{F}^{k_d - 1} C^d = \mathbf{0}$ and we get

$$v - \partial(w_m + w_{m-1} + \dots + w_{k_d}) = 0 \,.$$

The desired statement is proved. □

We are now ready to prove the following generalization of Claim A.2.

LEMMA A.3. *Let C and K be cochain complexes equipped with cocomplete ascending filtrations which are locally bounded from the left. Let $f : C \to K$ be a morphism of cochain complexes compatible with the filtrations. If the induced map of cochain complexes*

$$\mathrm{Gr}(f) : \mathrm{Gr}(C) \to \mathrm{Gr}(K)$$

is a quasi-isomorphism then so is f.

PROOF. Let us introduce the obvious ascending filtration on the cone of f

$$\dots \subset \mathcal{F}^{m-1}\mathrm{Cone}(f) \subset \mathcal{F}^m \mathrm{Cone}(f) \subset \mathcal{F}^{m+1}\mathrm{Cone}(f) \subset \dots \,,$$

$$\mathcal{F}^m \mathrm{Cone}(f) = \mathcal{F}^m C \,\oplus\, \mathbf{s}\mathcal{F}^m K \,. \tag{A.7}$$

The differential ∂^{Cone} is compatible with the filtration (A.7) because f is compatible with the filtrations on C and K.

It is obvious that the filtration (A.7) is cocomplete and locally bounded from the left. Furthermore, it is not hard to see that

$$\mathrm{Gr}(\mathrm{Cone}(f)) = \mathrm{Cone}(\mathrm{Gr}(f)) \,.$$

Therefore, Claim A.1 implies that $\mathrm{Gr}(\mathrm{Cone}(f))$ is acyclic.

Combining this observation with Claim A.2 we conclude that $\mathrm{Cone}(f)$ is also acyclic. Therefore, applying Claim A.1 once again, we deduce the statement of the lemma. □

REMARK A.4. Lemma A.3 is often used in the literature under the folklore name "standard spectral sequence argument". Unfortunately, a clean proof of this fact based on the use of a spectral sequence is very cumbersome.

Appendix B. Harrison complex of the cocommutative coalgebra $S(V)$

Let V be a finite dimensional graded vector space. We consider the symmetric algebra

$$S(V) \tag{B.1}$$

as the cocommutative coalgebra with the standard comultiplication:

$$\Delta(v_1 \dots v_n) = 1 \otimes (v_1 \dots v_n) +$$

$$\sum_{p=1}^{n-1} \sum_{\sigma \in \mathrm{Sh}_{p,n-p}} (-1)^{\varepsilon(\sigma, v_1, \dots, v_n)} \, v_{\sigma(1)} \dots v_{\sigma(p)} \otimes v_{\sigma(p+1)} \dots v_{\sigma(n)} + (v_1 \dots v_n) \otimes 1 \,, \tag{B.2}$$

where $v_1, \dots, v_n$ are homogeneous vectors in V and the sign factor $(-1)^{\varepsilon(\sigma, v_1, \dots, v_n)}$ is determined by the standard Koszul rule.

We denote by $\widetilde{\Delta}$ the reduced comultiplication which is define by the formula

$$\widetilde{\Delta}(X) = \Delta(X) - X \otimes 1 - 1 \otimes X \tag{B.3}$$

For example, $\widetilde{\Delta}(1) = -1 \otimes 1$ and $\widetilde{\Delta}(v) = 0$ for all $v \in V$.

Let us consider the free $\Lambda^{-1}\mathsf{Lie}$-algebra

$$\Lambda^{-1}\mathsf{Lie}(S(V)) \tag{B.4}$$

generated by $S(V)$.

Let us denote by X'_i and X''_i the tensor factors of

$$\widetilde{\Delta}(X) = \sum_i X'_i \otimes X''_i$$

for a vector $X \in S(V)$ and introduce the degree 1 derivation δ of the free $\Lambda^{-1}\mathsf{Lie}$-algebra (B.4) by setting

$$\delta(X) = \sum_i \{X'_i, X''_i\}. \tag{B.5}$$

Due to the Jacobi identity

$$\delta^2 = 0.$$

Hence δ is a differential on (B.4) and we call

$$\big(\Lambda^{-1}\mathsf{Lie}(S(V)), \delta\big) \tag{B.6}$$

the *Harrison complex* of $S(V)$.

It is easy to see that each non-zero vector $v \in V \subset \Lambda^{-1}\mathsf{Lie}(S(V))$ is a non-trivial cocycle in (B.6).

The following theorem and its various versions[24] are often referred to as "well-known".

THEOREM B.1. *For the Harrison complex* (B.6) *we have*

$$H^{\bullet}\big(\Lambda^{-1}\mathsf{Lie}(S(V)), \delta\big) \cong V.$$

More precisely, for every cocycle c in (B.6) *there exists a vector $v \in V \subset \Lambda^{-1}\mathsf{Lie}(S(V))$ and a vector c_1 in* (B.6) *such that*

$$c = v + \delta(c_1)$$

Furthermore, a vector $v \in V \subset \Lambda^{-1}\mathsf{Lie}(S(V))$ is an exact cocycle in (B.6) *if and only if $v = 0$.*

PROOF. To prove this theorem we embed the suspension

$$\mathbf{s}\,\Lambda^{-1}\mathsf{Lie}(S(V)) = \mathsf{Lie}(\mathbf{s}\,S(V)) \tag{B.7}$$

of (B.6) into the tensor algebra

$$T(\mathbf{s}\,S(V)) \tag{B.8}$$

generated by $\mathbf{s}\,S(V)$.

The differential δ on (B.7) can be extended to (B.8) in the obvious way:

$$\delta(\mathbf{s}X) = 2\,\mathbf{s} \otimes \mathbf{s}\big(\widetilde{\Delta}(X)\big). \tag{B.9}$$

[24] For a version of Theorem B.1 we refer the reader to [**29**, Section 3.5]. Another version of this theorem can also be deduced from statements in [**35**, Appendix B].

To compute the cohomology of $\big(T(\mathbf{s}\, S(V)), \delta\big)$ we consider the restricted dual complex

$$(T(\mathbf{s}^{-1}\, S(V')), \delta')\,, \tag{B.10}$$

where V' is the linear dual of V.

Since $T(\mathbf{s}\, S(V))$ is a free associative algebra, it is convenient to view (B.10) as the cofree coassociative coalgebra with the comultiplication given by deconcatenation. Furthermore, since δ is a derivation of (B.8), δ' is coderivation. Therefore, δ' is uniquely determined by its composition $p \circ \delta'$ with the projection

$$p : T(\mathbf{s}^{-1}\, S(V')) \to \mathbf{s}^{-1}\, S(V')\,.$$

It is easy to see that

$$p \circ \delta'(\mathbf{s}^{-1}\, X_1 \otimes \cdots \otimes \mathbf{s}^{-1}\, X_n) = \begin{cases} (-1)^{|X_1|-1}\, 2\, \mathbf{s}^{-1}\, \mu(X_1, X_2) & \text{if } n = 2 \\ 0 \qquad \text{otherwise}\,. \end{cases} \tag{B.11}$$

Here $X_1, \dots, X_n$ are homogeneous vectors in $S(V')$ and the map

$$\mu : S(V') \otimes S(V') \to S(V')$$

is defined by the formula

$$\mu(X_1, X_2) = X_1 X_2 - \varepsilon(X_1) X_2 - X_1 \varepsilon(X_2)\,, \tag{B.12}$$

where ε is the augmentation $\varepsilon : S(V') \to \mathbb{K}$ of $S(V')$.

Using (B.11), it is not hard to see that (B.10) is the Hochschild chain complex with the reversed grading and with rescaled differential

$$C_{-\bullet}(S(V'), \mathbb{K})\,.$$

Hence, due to the Hochschild-Kostant-Rosenberg theorem [**21**], we have

$$H^{\bullet}(T(\mathbf{s}^{-1}\, S(V')), \delta') \cong S(\mathbf{s}^{-1}\, V')\,. \tag{B.13}$$

If we view $S(\mathbf{s}^{-1}\, V')$ as the subspace of $T(\mathbf{s}^{-1}\, V')$ which is, in turn, a subspace of (B.10), then the Hochschild-Kostant-Rosenberg theorem can be restated as follows. For every cocycle c in (B.10) there exists a vector $X \in S(\mathbf{s}^{-1}\, V')$ and a vector c_1 in (B.10) such that

$$c = X + \delta'(c_1)\,.$$

Every vector $X \in S(\mathbf{s}^{-1}\, V')$ is a cocycle in (B.10) and $X \in S(\mathbf{s}^{-1}\, V')$ is an exact cocycle if and only if $X = 0$.

Let us now go back to the cochain complex (B.8) with the differential (B.9). Let us consider $S(\mathbf{s}\, V)$ as the subspace of

$$T(\mathbf{s}\, V) \subset T(\mathbf{s}\, S(V))\,.$$

It is clear that every vector in $S(\mathbf{s}\, V)$ is a cocycle in (B.8).

Dualizing the above statement about cocycles in (B.10) we deduce the following.

CLAIM B.2. *For every cocycle $c \in T(\mathbf{s}\, S(V))$ there exists a vector $X \in S(\mathbf{s}\, V)$ and a vector $c_1 \in T(\mathbf{s}\, S(V))$ such that*

$$c = X + \delta(c_1)\,.$$

Furthermore, a vector $X \in S(\mathbf{s}\, V)$ is a trivial cocycle in (B.8) *if and only if $X = 0$.*
□

Let us now observe that, due to the PBW theorem, we have the isomorphism of graded vector spaces

$$T(\mathbf{s}\, S(V)) \cong S\big(\mathsf{Lie}(\mathbf{s}\, S(V))\big) \tag{B.14}$$

Moreover, the differential δ is compatible with this isomorphism. In other words, the cochain complex (B.8) is isomorphic to the symmetric algebra of the cochain complex (B.7).

Since the cochain complex $S\big(\mathsf{Lie}(\mathbf{s}\, S(V))\big)$ splits into the direct sum

$$S\big(\mathsf{Lie}(\mathbf{s}\, S(V))\big) = \mathbb{K} \;\oplus\; \mathsf{Lie}(\mathbf{s}\, S(V)) \;\oplus\; \bigoplus_{m \ge 2} S^m\big(\mathsf{Lie}(\mathbf{s}\, S(V))\big)$$

the statement of the theorem follows easily from Claim B.2. □

Appendix C. Filtered dg Lie algebras. The Goldman-Millson theorem

In this section we prove a version of the Goldman-Millson theorem [**19**] which is often used in applications.

We consider a Lie algebra $\mathcal{L}$ in the category $\mathsf{Ch}_{\mathbb{K}}$ equipped with a descending filtration

$$\mathcal{L} = \mathcal{F}_1\mathcal{L} \supset \mathcal{F}_2\mathcal{L} \supset \mathcal{F}_3\mathcal{L} \supset \dots \tag{C.1}$$

which is compatible with the Lie bracket (and the differential).

We assume that $\mathcal{L}$ is complete with respect to this filtration. Namely,

$$\mathcal{L} = \lim_{k} \mathcal{L} \,\big/\, \mathcal{F}_k\mathcal{L}\,. \tag{C.2}$$

We call such Lie algebras *filtered.*

Condition (C.2) and equality $\mathcal{L} = \mathcal{F}_1\mathcal{L}$ guarantee that the subalgebra $\mathcal{L}^0$ of degree zero elements in $\mathcal{L}$ is a pro-nilpotent Lie algebra (in the category of $\mathbb{K}$-vector spaces). Hence, $\mathcal{L}^0$ can exponentiated to a pro-unipotent group which we denote by

$$\exp(\mathcal{L}^0)\,. \tag{C.3}$$

We recall that a *Maurer-Cartan element* of $\mathcal{L}$ is a degree 1 vector $\alpha \in \mathcal{L}$ satisfying the equation

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0\,, \tag{C.4}$$

where ∂ denotes the differential on $\mathcal{L}$.

For a vector $\xi \in \mathcal{L}^0$ and a Maurer-Cartan element α we consider the new degree 1 vector $\widetilde{\alpha} \in \mathcal{L}$ which is given by the formula

$$\widetilde{\alpha} = \exp(\mathrm{ad}_\xi)\,\alpha - \frac{\exp(\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}\partial\xi\,, \tag{C.5}$$

where the expressions

$$\exp(\mathrm{ad}_\xi) \qquad \text{and} \qquad \frac{\exp(\mathrm{ad}_\xi) - 1}{\mathrm{ad}_\xi}$$

are defined in the obvious way using the Taylor expansions of the functions

$$e^x \qquad \text{and} \qquad \frac{e^x - 1}{x}$$

around the point $x = 0$, respectively.

Conditions (C.2) and $\mathcal{L} = \mathcal{F}_1\mathcal{L}$ guarantee that the right hand side of equation (C.5) is defined.

It is known (see, e.g. [**3**, Appendix B] or [**19**]) that, for every Maurer-Cartan element α and for every degree zero vector $\xi \in \mathcal{L}$, the vector $\widetilde{\alpha}$ in (C.5) is also a Maurer-Cartan element. Furthermore, formula (C.5) defines an action of the group (C.3) on the set of Maurer-Cartan elements of $\mathcal{L}$.

The transformation groupoid $\mathrm{MC}(\mathcal{L})$ corresponding to this action is called the *Deligne groupoid* of the Lie algebra $\mathcal{L}$. This groupoid and its higher versions were studied extensively by E. Getzler in [**14**] and [**15**].

REMARK C.1. The transformation groupoid $\mathrm{MC}(\mathcal{L})$ may be defined without imposing the assumption $\mathcal{L} = \mathcal{F}_1\mathcal{L}$. In this more general case, the group (C.3) should be replaced by

$$\exp(\mathcal{F}_1\mathcal{L}^0)\,.$$

Let

$$\varphi : \mathcal{L} \to \widetilde{\mathcal{L}}$$

be a homomorphism of two filtered dg Lie algebras.

It is obvious that for every Maurer-Cartan element $\alpha \in \mathcal{L}$ the vector $\varphi(\alpha)$ is a Maurer-Cartan element of $\widetilde{\mathcal{L}}$. Moreover the assignment

$$\alpha \to \varphi(\alpha)$$

extends to the functor

$$\varphi_* : \mathrm{MC}(\mathcal{L}) \to \mathrm{MC}(\widetilde{\mathcal{L}}) \tag{C.6}$$

between the corresponding Deligne groupoids.

The following statement is a version of the famous Goldman-Millson theorem [**19**].

THEOREM C.2. *Let $\varphi : \mathcal{L} \to \widetilde{\mathcal{L}}$ be a quasi-isomorphism of filtered dg Lie algebras. If the restriction*

$$\varphi\Big|_{\mathcal{F}_m\mathcal{L}} \ : \ \mathcal{F}_m\mathcal{L} \to \mathcal{F}_m\widetilde{\mathcal{L}}$$

is a quasi-isomorphism for all m then the functor (C.6) *induces a bijection*

$$\varphi_* : \pi_0\big(\mathrm{MC}(\mathcal{L})\big) \to \pi_0\big(\mathrm{MC}(\widetilde{\mathcal{L}})\big) \tag{C.7}$$

from the isomorphism classes of Maurer-Cartan elements in $\mathcal{L}$ to the isomorphism classes of Maurer-Cartan elements in $\widetilde{\mathcal{L}}$.

PROOF. Using the conditions of the theorem and Exercise C.3 given below, it is not hard to see that φ induces a quasi-isomorphism

$$\mathrm{Gr}(\varphi) : \mathcal{F}_m\mathcal{L}/\mathcal{F}_{m+1}\mathcal{L} \to \mathcal{F}_m\widetilde{\mathcal{L}}/\mathcal{F}_{m+1}\widetilde{\mathcal{L}}$$

for all m.

In order to prove that the map (C.7) is surjective we need to show that for every Maurer-Cartan element $\beta \in \widetilde{\mathcal{L}}$ there exists a vector $\xi \in \widetilde{\mathcal{L}}^0$ and a Maurer-Cartan element $\alpha \in \mathcal{L}$ such that

$$\exp(\xi)(\beta) = \varphi(\alpha)\,. \tag{C.8}$$

The Maurer-Cartan equation $\partial\beta + [\beta,\beta]/2 = 0$ implies that β represents a cocycle in

$$\mathcal{F}_1\widetilde{\mathcal{L}}/\mathcal{F}_2\widetilde{\mathcal{L}}\,.$$

Hence there exists $\alpha_1 \in \mathcal{F}_1\mathcal{L}^1$ and $\xi_1 \in \mathcal{F}_1\widetilde{\mathcal{L}}^0$ such that

$$\partial\alpha_1 \in \mathcal{F}_2\mathcal{L} \tag{C.9}$$

and

$$\beta - \partial\xi_1 - \varphi(\alpha_1) \in \mathcal{F}_2\widetilde{\mathcal{L}}\,. \tag{C.10}$$

Let us denote by β_1 the Maurer-Cartan element

$$\beta_1 = \exp(\xi_1)(\beta)\,.$$

Inclusion (C.10) implies that

$$\beta_1 - \varphi(\alpha_1) \in \mathcal{F}_2\widetilde{\mathcal{L}}\,. \tag{C.11}$$

We showed that there exists a vector $\xi_1 \in \mathcal{F}_1\widetilde{\mathcal{L}}^0$ and a vector $\alpha_1 \in \mathcal{F}_1\mathcal{L}^1$ such that for

$$\beta_1 = \exp(\xi_1)(\beta)$$

we have inclusion (C.11) and the inclusion

$$\partial\alpha_1 + \frac{1}{2}[\alpha_1, \alpha_1] \in \mathcal{F}_2\mathcal{L}\,, \tag{C.12}$$

which follows from (C.9). Inclusions (C.11) and (C.12) form the base of our induction.

Now we assume that there exist vectors

$$\xi_k \in \mathcal{F}_k\widetilde{\mathcal{L}}^0\,, \qquad 1 \le k \le m$$

and $\alpha_m \in \mathcal{F}_1\mathcal{L}$ such that

$$\partial\alpha_m + \frac{1}{2}[\alpha_m, \alpha_m] \in \mathcal{F}_{m+1}\mathcal{L}\,, \tag{C.13}$$

and

$$\beta_m - \varphi(\alpha_m) \in \mathcal{F}_{m+1}\widetilde{\mathcal{L}}\,, \tag{C.14}$$

where

$$\beta_m = \exp(\xi_m)\ldots\exp(\xi_1)(\beta)\,. \tag{C.15}$$

Let us consider the vector

$$\big(\partial\varphi(\alpha_m) + \frac{1}{2}[\varphi(\alpha_m), \varphi(\alpha_m)]\big) - \partial(\varphi(\alpha_m) - \beta_m) \tag{C.16}$$

in $\mathcal{F}_{m+1}\widetilde{\mathcal{L}}^2$.

Using the Maurer-Cartan equation for β_m we can rewrite (C.16) as

$$\big(\partial\varphi(\alpha_m) + \frac{1}{2}[\varphi(\alpha_m), \varphi(\alpha_m)]\big) - \partial(\varphi(\alpha_m) - \beta_m) = \frac{1}{2}\big([\varphi(\alpha_m), \varphi(\alpha_m)] - [\beta_m, \beta_m]\big) =$$
$$\frac{1}{2}\big([\varphi(\alpha_m), \varphi(\alpha_m)] - [\varphi(\alpha_m), \beta_m] + [\varphi(\alpha_m), \beta_m] - [\beta_m, \beta_m]\big) =$$
$$\frac{1}{2}\big([\varphi(\alpha_m), \varphi(\alpha_m) - \beta_m] + [\varphi(\alpha_m) - \beta_m, \beta_m]\big)\,.$$

Thus (C.14) implies that vector (C.16) belongs to $\mathcal{F}_{m+2}\widetilde{\mathcal{L}}^2$.

On the other hand, applying the differential ∂ to the vector

$$\big(\partial\varphi(\alpha_m) + \frac{1}{2}[\varphi(\alpha_m), \varphi(\alpha_m)]\big)$$

and using (C.13) together with the Jacobi identity we conclude that

$$\partial\left(\partial\varphi(\alpha_m) + \frac{1}{2}[\varphi(\alpha_m), \varphi(\alpha_m)]\right) \in \mathcal{F}_{m+2}\widetilde{\mathcal{L}}^3 \,.$$

Combining this observation with the fact that vector (C.16) belongs to $\mathcal{F}_{m+2}\widetilde{\mathcal{L}}^2$ we deduce that

$$\varphi\big(\partial\alpha_m + \frac{1}{2}[\alpha_m, \alpha_m]\big)$$

represents an exact cocycle in

$$\mathcal{F}_{m+1}\widetilde{\mathcal{L}}/\mathcal{F}_{m+2}\widetilde{\mathcal{L}} \,.$$

Therefore, there exists a vector $\gamma_{m+1} \in \mathcal{F}_{m+1}\mathcal{L}^1$ such that

$$\partial\gamma_{m+1} + \partial\alpha_m + \frac{1}{2}[\alpha_m, \alpha_m] \in \mathcal{F}_{m+2}\mathcal{L} \,. \tag{C.17}$$

Let us denote by α'_{m+1} the vector

$$\alpha'_{m+1} = \alpha_m + \gamma_{m+1} \,.$$

Combining (C.17) with the fact that vector (C.16) belongs to $\mathcal{F}_{m+2}\widetilde{\mathcal{L}}^2$ we conclude that

$$\partial(\beta_m - \varphi(\alpha'_{m+1})) \in \mathcal{F}_{m+2}\widetilde{\mathcal{L}} \,.$$

In other words, $\beta_m - \varphi(\alpha'_{m+1})$ represents a cocycle in

$$\mathcal{F}_{m+1}\widetilde{\mathcal{L}}/\mathcal{F}_{m+2}\widetilde{\mathcal{L}} \,.$$

Therefore, there exists a vector $\xi_{m+1} \in \mathcal{F}_{m+1}\widetilde{\mathcal{L}}^0$ and a vector $\gamma'_{m+1} \in \mathcal{F}_{m+1}\mathcal{L}^1$ such that

$$\partial\gamma'_{m+1} \in \mathcal{F}_{m+2}\mathcal{L}^2 \tag{C.18}$$

and

$$\beta_m - \partial\xi_{m+1} - \varphi(\alpha_{m+1}) - \varphi(\gamma'_{m+1}) \in \mathcal{F}_{m+2}\widetilde{\mathcal{L}} \,. \tag{C.19}$$

We set

$$\alpha_{m+1} = \alpha'_{m+1} + \gamma'_{m+1}$$

and

$$\beta_{m+1} = \exp(\xi_{m+1})(\beta_m) \,.$$

Combining (C.17) together with (C.18) and (C.19) we see that α_{m+1}, β_{m+1} and ξ_{m+1} satisfy the inductive assumption for m replaced by $m+1$.

Thus, we conclude that, there exist sequences of vectors

$$\alpha_m \in \mathcal{F}_1\mathcal{L}^1 \,, \qquad \alpha_{m+1} - \alpha_m \in \mathcal{F}_{m+1}\mathcal{L}^1 \,, \qquad m \ge 1$$

and

$$\xi_m \in \mathcal{F}_m\widetilde{\mathcal{L}}^0 \,, \qquad m \ge 1$$

such that inclusions (C.13) and (C.14) hold for all m.

Since the filtrations on $\mathcal{L}$ and $\widetilde{\mathcal{L}}$ are complete the sequence $\{\alpha_m\}_{m\ge1}$ converges to a vector $\alpha \in \mathcal{L}^1$ and the sequence

$$\Big\{ \mathrm{CH}\Big(\xi_m, \dots, \mathrm{CH}\big(\xi_3, \mathrm{CH}(\xi_2, \xi_1)\big) \dots\Big) \Big\}_{m\ge1}$$

converges to a vector $\xi \in \widetilde{\mathcal{L}}^0$ such that

$$\partial\alpha + \frac{1}{2}[\alpha, \alpha] = 0$$

and

$$\exp(\xi)(\beta) = \varphi(\alpha)\,.$$

We proved that the map (C.7) is surjective.

Due to Exercise C.4 below the map (C.7) is also injective. Thus the theorem is proved. $\square$

EXERCISE C.3. If the rows in the commutative diagram of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

are exact and any 2 vertical maps are quasi-isomorphisms, then show that the third vertical map is also a quasi-isomorphism. *Hint: Consider the 5-lemma (Sec. II.5 in* [**12**]*).*

EXERCISE C.4. Prove that the map (C.7) is injective.

Appendix D. Solutions to selected exercises

Solution of Exercise 5.1. We need only to consider generators of $\mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)$ i.e. $(\mathbf{q}_n, \mathbf{s}X)$, where $\mathbf{q}_n$ is the standard n-corolla, and $X \in \mathcal{C}_\circ(n)$.

By definition,

$$F(\partial^{\mathrm{Cobar}}(\mathbf{q}_n, \mathbf{s}X)) = \partial^{\mathcal{O}} F((\mathbf{q}_n, \mathbf{s}X)) \tag{D.1}$$

if and only if

$$\alpha_F(\partial^{\mathcal{C}} X) + \partial^{\mathcal{O}} \alpha_F(X) - F(\partial''(\mathbf{q}_n, \mathbf{s}X)) = 0, \tag{D.2}$$

where $\alpha_F \in \mathrm{Conv}(\mathcal{C}, \mathcal{O})$ is the degree 1 map $\alpha_F(X) = F((\mathbf{q}_n, \mathbf{s}X))$, and

$$\partial''(\mathbf{q}_n, \mathbf{s}X) = -\sum_{z \in \pi_0(\mathsf{Tree}_2(n))} (\mathbf{s} \otimes \mathbf{s})(\mathbf{t}_z; \Delta_{\mathbf{t}_z}(X)). \tag{D.3}$$

By definition of the differential on $\mathrm{Conv}(\mathcal{C}, \mathcal{O})$, Eq. (D.2) holds if and only if

$$\big(\partial \alpha_F\big)(X) - F(\partial''(\mathbf{q}_n, \mathbf{s}X)) = 0, \tag{D.4}$$

Next, expanding the right-hand side of Eq. (D.3) gives:

$$\partial''(\mathbf{q}_n, \mathbf{s}X) = -\sum_{z \in \pi_0(\mathsf{Tree}_2(n))} \sum_{\alpha} (-1)^{|X^1_\alpha|} (\mathbf{t}_z; \mathbf{s}X^1_\alpha \otimes \mathbf{s}X^2_\alpha),$$

where X^1_α and X^2_α are tensor factors in

$$\Delta_{\mathbf{t}_z}(X) = \sum_\alpha X^1_\alpha \otimes X^2_\alpha\,.$$

Let p_z be the number of edges terminating at the second nodal vertex of $\mathbf{t}_z$ and let

$$\tilde{\mu}_{\mathbf{t}_z} : \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)(n - p_z + 1) \otimes \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)(p_z) \to \mathbb{OP}(\mathbf{s}\,\mathcal{C}_\circ)(n)$$

be the multiplication map for the tree $\mathbf{t}_z$. By definition of multiplication for the free operad, we have

$$(\mathbf{t}_z; \mathbf{s}X^1_\alpha \otimes \mathbf{s}X^2_\alpha) = \tilde{\mu}_{\mathbf{t}_z}\big((\mathbf{q}_{n-p_z+1}, \mathbf{s}X^1_\alpha) \otimes (\mathbf{q}_{p_z}, \mathbf{s}X^2_\alpha)\big)$$

Since F is a map of operads, we have the following equalities:

$$\begin{aligned}
F(\partial''(\mathbf{q}_n, \mathbf{s}X)) &= -\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\sum_{\alpha}(-1)^{|X^1_\alpha|}F\big(\tilde{\mu}_{\mathbf{t}_z}\big((\mathbf{q}_{n-p_z+1}, \mathbf{s}X^1_\alpha)\otimes(\mathbf{q}_{p_z}, \mathbf{s}X^2_\alpha)\big)\big)\\
&= -\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\sum_{\alpha}(-1)^{|X^1_\alpha|}\mu_{\mathbf{t}_z}\big(F(\mathbf{q}_{n-p_z+1}, \mathbf{s}X^1_\alpha)\otimes F(\mathbf{q}_{p_z}, \mathbf{s}X^2_\alpha)\big)\\
&= -\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\sum_{\alpha}(-1)^{|X^1_\alpha|}\mu_{\mathbf{t}_z}\big(\alpha_F(X^1_\alpha)\otimes\alpha_F(X^2_\alpha)\big)\\
&= -\sum_{z\in\pi_0(\mathsf{Tree}_2(n))}\mu_{\mathbf{t}_z}\big(\alpha_F\otimes\alpha_F\circ\Delta_{\mathbf{t}_z}(X)\big)\\
&= -\alpha_F\bullet\alpha_F(X)\\
&= -\frac{1}{2}[\alpha_F, \alpha_F](X).
\end{aligned}$$

By substituting this last equality into Eq. (D.4), we see Eq. (D.1) holds if and only if the Maurer-Cartan equation

$$\partial\alpha_F + \frac{1}{2}[\alpha_F, \alpha_F] = 0$$

holds for α_F. ◀

Solution of Exercise 5.7. Assume the Maurer-Cartan elements α_F and $\alpha_{\widetilde{F}}$ corresponding to the maps $F, \widetilde{F}\colon \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}$ are isomorphic as objects of the Deligne groupoid. By definition (see Eq. (C.5)) this implies that there exists a degree 0 element $\xi \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$ such that

$$\alpha_{\widetilde{F}} = \exp(\mathrm{ad}_\xi)\,\alpha_F - \frac{\exp(\mathrm{ad}_\xi)-1}{\mathrm{ad}_\xi}\partial\xi\ .$$

Define $\alpha(t) \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})[[t]]$ to be:

$$\alpha(t) = \exp(-t\mathrm{ad}_\xi)\,\alpha_F - \frac{\exp(-t\mathrm{ad}_\xi)-1}{\mathrm{ad}_\xi}\partial\xi\ .$$

Since α_F and ξ are elements of $\mathcal{F}_1\mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})$, and the bracket and differential are compatible with the filtration, we conclude that

$$\alpha(t) \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O})\{t\}.$$

Note $\alpha(0) = \alpha_F$ and $\alpha(1) = \alpha_{\widetilde{F}}$. Differentiation of $\alpha(t)$ gives:

$$\begin{aligned}
\frac{d\alpha(t)}{dt} &= -\mathrm{ad}_\xi\big(\exp(-t\mathrm{ad}_\xi)\,\alpha_F\big) + \exp(-t\mathrm{ad}_\xi)\partial\xi\\
&= -\mathrm{ad}_\xi\big(\exp(-t\mathrm{ad}_\xi)\,\alpha_F\big) + \exp(-t\mathrm{ad}_\xi)\partial\xi - \partial\xi + \partial\xi\\
&= -\mathrm{ad}_\xi\big(\exp(-t\mathrm{ad}_\xi)\,\alpha_F\big) - \frac{-\mathrm{ad}_\xi}{\mathrm{ad}_\xi}\Big(\exp(-t\mathrm{ad}_\xi)\partial\xi - \partial\xi\Big) + \partial\xi\\
&= -\mathrm{ad}_\xi\Big(\exp(-t\mathrm{ad}_\xi)\,\alpha_F - \frac{\exp(-t\mathrm{ad}_\xi)-1}{\mathrm{ad}_\xi}\partial\xi\Big) + \partial\xi\\
&= \partial\xi - [\xi, \alpha(t)].
\end{aligned}$$

Thus, applying Prop. C.1 of [**4**], we conclude that

$$\partial\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)] = 0$$

for all t.

Hence, equations (5.10), (5.11), and (5.12), which are described in the "only if" part of the proof, imply that

$$\alpha_H = \alpha(t) + \xi dt \in \mathrm{Conv}(\mathcal{C}_\circ, \mathcal{O}^I)$$

is a Maurer-Cartan element that corresponds to a homotopy $H\colon \mathrm{Cobar}(\mathcal{C}) \to \mathcal{O}^I$ between F and $\widetilde{F}$. ◀

Solution of Exercise 6.15. The space

$$\mathbf{s}^{2r}\big(\mathsf{Ger}(r+n)\big)^{S_r}$$

is spanned by vectors of the form

$$\mathrm{Av}(w) = \sum_{\sigma \in S_r} \sigma(w) \tag{D.5}$$

where w is a monomial in $\mathbf{s}^{2r}\mathsf{Ger}(r+n)$.

It is clear that

$$f^{-1}\big(\mathrm{Av}(w)\big) = w(\underbrace{a, a, \dots, a}_{r \text{ times}}, a_1, \dots, a_n)\,.$$

So our goal is to show that

$$\partial^{\mathrm{Tw}}\big(\mathrm{Av}(w)\big) = \tag{D.6}$$

$$\sum_{\sigma \in S_{r+1}} \sum_{i=1}^{r} \frac{(-1)^{e_i}}{2}\, w(a_{\sigma(1)}, \dots, a_{\sigma(i-1)}, \{a_{\sigma(i)}, a_{\sigma(i+1)}\}, a_{\sigma(i+2)}, \dots, a_{\sigma(r+1)},$$
$$a_{r+2}, \dots, a_{r+1+n})\,,$$

where the sign factor $(-1)^{e_i}$ comes from swapping the odd operator $\{a_{\sigma(i)}, \ \}$ with the corresponding number of brackets.

Following the definition of ∂^{Tw} (6.37) we get

$$\partial^{\mathrm{Tw}}\big(\mathrm{Av}(w)\big) = \sum_{\tau \in \mathrm{Sh}_{1,r}} \sum_{\sigma \in S_{2,\dots,r+1}} \tau\big(\{a_1, w(a_{\sigma(2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n})\}\big)$$

$$- \sum_{i=1}^{n} \sum_{\sigma \in S_r} \sum_{\tau' \in \mathrm{Sh}_{r,1}} (-1)^{e_{r+i}} \tau'\big(w(a_{\sigma(1)}, \dots, a_{\sigma(r)}, a_{r+2}, \dots, a_{r+i}, \{a_{r+1}, a_{r+i+1}\},$$
$$a_{r+i+2}, \dots, a_{r+1+n})\big)$$

$$-(-1)^{|w|} \sum_{\tau \in \mathrm{Sh}_{2,r-1}} \tau\big(w \circ_1 \{a_1, a_2\}\big) =$$

$$\sum_{\sigma \in S_{r+1}} \{a_{\sigma(1)}, w(a_{\sigma(2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n})\} \tag{D.7}$$

$$- \sum_{i=1}^{n} \sum_{\sigma \in S_{r+1}} (-1)^{e_{r+i}} w(a_{\sigma(1)}, \dots, a_{\sigma(r)}, a_{r+2}, \dots, a_{r+i}, \{a_{\sigma(r+1)}, a_{r+i+1}\}, a_{r+i+2}, \dots, a_{r+1+n})$$

$$- \sum_{\tau \in \mathrm{Sh}_{2,r-1}} \sum_{\sigma \in S_{3,\dots,r+1}} \sum_{i=1}^{r} (-1)^{e_i} \tau \circ \sigma \big(w(a_3, \dots, a_{i+1}, \{a_1, a_2\}, a_{i+2}, \dots, a_{r+1+n}) \big) ,$$

where we used the obvious identity

(D.8)
$$\mathrm{Av}(w) = \sum_{\sigma \in S_{2,\dots,r}} \sum_{i=1}^{r} w(a_{\sigma(2)}, \dots, a_{\sigma(i)}, a_1, a_{\sigma(i+1)}, \dots, a_{\sigma(r)}, a_{r+1}, \dots, a_{r+n}) .$$

Using the defining identities of Gerstenhaber algebra we simplify (D.7) further

(D.9)
$$\partial^{\mathrm{Tw}} \big(\mathrm{Av}(w) \big) =$$

$$\sum_{\sigma \in S_{r+1}} \sum_{i=2}^{r+1} (-1)^{e_i} w(a_{\sigma(2)}, \dots, a_{\sigma(i-1)}, \{a_{\sigma(1)}, a_{\sigma(i)}\}, a_{\sigma(i+1)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n})\}$$

$$- \sum_{\sigma \in S_{r+1}}^{\sigma(1)<\sigma(2)} \sum_{i=1}^{r} (-1)^{e_i} w(a_{\sigma(3)}, \dots, a_{\sigma(i+1)}, \{a_{\sigma(1)}, a_{\sigma(2)}\}, a_{\sigma(i+2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n}) =$$

$$\sum_{\sigma \in S_{r+1}} \sum_{i=2}^{r+1} (-1)^{e_i} w(a_{\sigma(2)}, \dots, a_{\sigma(i-1)}, \{a_{\sigma(1)}, a_{\sigma(i)}\}, a_{\sigma(i+1)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n})\}$$

$$- \sum_{\sigma \in S_{r+1}} \sum_{i=1}^{r} \frac{(-1)^{e_i}}{2} w(a_{\sigma(3)}, \dots, a_{\sigma(i+1)}, \{a_{\sigma(1)}, a_{\sigma(2)}\}, a_{\sigma(i+2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n}) =$$

$$\sum_{\sigma \in S_{r+1}} \sum_{i=1}^{r} \frac{(-1)^{e_i}}{2} w(a_{\sigma(3)}, \dots, a_{\sigma(i+1)}, \{a_{\sigma(1)}, a_{\sigma(2)}\}, a_{\sigma(i+2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n}) =$$

$$\sum_{\sigma \in S_{r+1}} \sum_{i=1}^{r} \frac{(-1)^{e_i}}{2} \, w(a_{\sigma(1)}, \dots, a_{\sigma(i-1)}, \{a_{\sigma(i)}, a_{\sigma(i+1)}\}, a_{\sigma(i+2)}, \dots, a_{\sigma(r+1)}, a_{r+2}, \dots, a_{r+1+n}) .$$

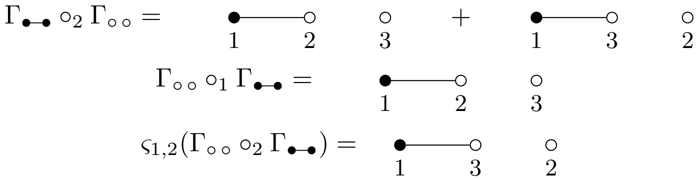
Thus equation (D.6) indeed holds and the desired statement follows. ◄

Solution of Exercise 9.26. According to the formula for ∂^{Tw} given in Eq. (9.6) we have

(D.10)
$$\partial^{\mathrm{Tw}} \Gamma_{\circ\,\circ} = \mathrm{Av}_1 \big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma_{\circ\,\circ} \big) - \mathrm{Av}_1 \big(\Gamma_{\circ\,\circ} \circ_1 \Gamma_{\bullet\!-\!\bullet} + \varsigma_{1,2} (\Gamma_{\circ\,\circ} \circ_2 \Gamma_{\bullet\!-\!\bullet}) \big),$$

where $\varsigma_{1,2}$ is the cycle $(12) \in S_3$. Recall that, in the right hand side of (D.10), both graphs $\Gamma_{\bullet\!-\!\bullet}$ and $\Gamma_{\circ\,\circ}$ are viewed as vectors in $\mathsf{Gra}(2)$, while the final result of the computation is treated as a vector in $\mathrm{Tw}\mathsf{Gra}(2)$. In particular, the colors of vertices play a role only for the final result of the computation. (See also Remark 9.3.)

Expanding the terms on the right hand side gives the following equalities:

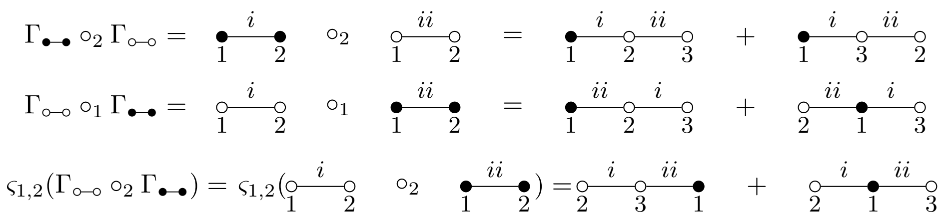


Hence, all terms cancel on the right hand side of Eq. (D.10), and therefore $\partial^{\mathrm{Tw}}\Gamma_{\circ\,\circ} = 0$.

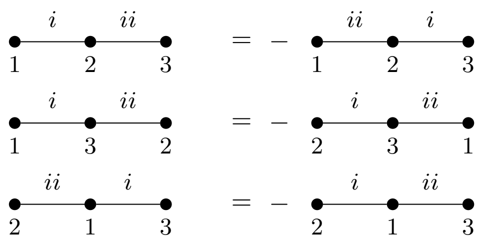
Next, applying the differential ∂^{Tw} to $\Gamma_{\circ\!-\!\circ} \in \mathrm{Tw}\mathsf{Gra}(2)$, we get

$$\partial^{\mathrm{Tw}}\Gamma_{\circ\!-\!\circ} = \mathrm{Av}_1\big(\Gamma_{\bullet\!-\!\bullet} \circ_2 \Gamma_{\circ\!-\!\circ}\big) + \mathrm{Av}_1\big(\Gamma_{\circ\!-\!\circ} \circ_1 \Gamma_{\bullet\!-\!\bullet} + \varsigma_{1,2}(\Gamma_{\circ\!-\!\circ} \circ_2 \Gamma_{\bullet\!-\!\bullet})\big). \tag{D.11}$$

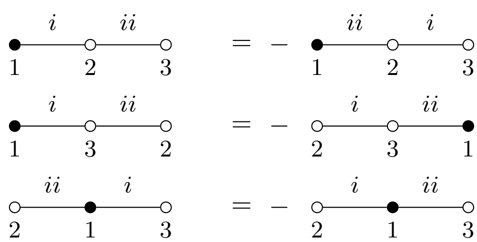
We expand the terms on the right hand side, being mindful of the ordering on edges, and Remark 9.3.



By definition of the operad Gra, we have the following equalities in $\mathsf{Gra}(1+2)$:



Thus, in $\mathrm{Tw}\mathsf{Gra}(2)$, we have:



Hence, all terms on the right hand side of Eq. (D.11) cancel, and therefore $\partial^{\mathrm{Tw}}\Gamma_{\circ\!-\!\circ} = 0$. ◀

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Geometric quantization; a crash course

Eugene Lerman

Contents

Early in 2011 Sam Evens acting on behalf of the organizers of the summer school on quantization at Notre Dame asked me to give a short series of lectures on geometric quantization. These lectures were meant to prepare a group of graduate mathematics students for talks at the conference on quantization which were to follow the summer school. I was told to assume that the students had attended an introductory course on manifolds. But I was not to assume any prior knowledge of symplectic geometry. The notes that follow resulted from this request. They are a mostly faithful record of four one-hour lectures (except lecture 4). It should be said that there exist many books on geometric quantization starting with Souriau [**14**], Sniatycki [**12**], Simms and Woodhouse [**13**] , Guillemin and Sternberg [**7**], Wallach [**15**] and Woodhouse [**16**] and continuing with Bates and Weinstein [**3**] and Ginzburg, Guillemin and Karshon [**10**]. There are also a number of one hundred page surveys on geometric quantization such as the ones by Ali and Englis [**1**] and by Echeverria-Enriquez *et al.* [**5**]. Clearly I could not have squeezed a semester or more worth of mathematics into four lectures. Since I had to pick and choose, I decided to convey the flavor of the subject by proceeding as follows. In the first lecture I tried to explain how to formulate the Newton's law of motion in terms of symplectic geometry. This naturally require an introduction of the notions of symplectic manifolds, Hamiltonian vector fields and Poisson brackets. In lecture 2 I described prequantization. Lecture 3 dealt with polarizations. I have mostly limited myself to real polarizations. In the original version I tried to explain half-forms. Here I stick with densities. In lecture 4 I came back to prequantization and

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tried to explain why it is more natural to prequantize a differential cocycle rather than just a two form. In other words prequantization is taken up from a more functorial point of view — differential cohomology and stacks. For reasons of space and time the treatment is not very detailed. The notes also contain two appendices: the first one recalls bits and pieces of category theory; the second discusses densities.

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1. An outline of the notes

The goal of this mini course is — starting with a classical system (which is modeled as a symplectic manifold together with a function called the Hamiltonian) — to produce a quantum system, that is, a collection of (skew)adjoined operators on a Hilbert space.

Here is a more detailed plan of the lectures (the possibly unfamiliar terms are to be defined later in the course):

- We go from Newton's law of motion to a symplectic formulation of classical mechanics, while cutting quite a few corners along the way.
- Next we have a crash course on symplectic geometry. The two key points are:
  - A function h on a symplectic manifold (M, ω) uniquely defines a vector field Ξ_h on the manifold M.
  - There is a Poisson bracket, that is, an $\mathbb{R}$-bilinear map
  $$C^\infty(M) \times C^\infty(M) \to C^\infty(M), \quad (f, g) \mapsto \{f, g\},$$
  which has a number of properties. In particular, the bracket $\{f, g\}$ makes $C^\infty(M)$ into a Lie algebra.
- Next we'll discuss prequantization: Given a symplectic manifold (M, ω) and a corresponding Poisson bracket $\{\cdot, \cdot\}$ we want to find/construct a complex line bundle $\pi : \mathbb{L} \to M$ with a Hermitian inner product $< \cdot, \cdot >$ and a connection ∇ on L such that:
  - the connection ∇ preserves the inner product $< \cdot, \cdot >$
  - $curv(\nabla) = (2\pi\sqrt{-1})\omega$

  Given such a bundle we get a prequantum Hilbert Space $\mathcal{H}_0$, which consists of L^2 sections of $\mathbb{L} \to M$. We'll observe:
  - Each function $f \in C^\infty(M)$ defines an operator $Q_f : \mathcal{H}_0 \to \mathcal{H}_0$,
  $$Q_f(s) = (2\pi\sqrt{-1})fs + \nabla_{\Xi_f} s$$
  where Ξ_f is the Hamiltonian vector field generated by the function f.
  - The map
  $$C^\infty(M) \xrightarrow{Q} \{\text{skew Hermitian Operators on } \mathcal{H}_0\}$$
  given by $f \mapsto Q_f$ is a map of Lie algebras.
- There is problem with prequantization: quantum mechanics tells us that the space $\mathcal{H}_0$ is too big. Here is an example.

EXAMPLE 1.1. Consider a particle in $\mathbb{R}^3$. The corresponding classical phase space is $M = T^*\mathbb{R}^3$. The associated line bundle $\mathbb{L} = T^*\mathbb{R}^3 \times \mathbb{C}$ is trivial, and the prequantum Hilbert space is $\mathcal{H}_0 = L^2(T^*\mathbb{R}^3, \mathbb{C})$. Physics tells us that what we should have as our quantum phase space the vector space $L^2(\mathbb{R}^3, \mathbb{C})$.

One then uses polarizations to cut the number of variables in half. The introduction of polarizations leads to a number of technical problems. In other words, this is where our trouble really begins.

In the first appendix to the paper we remind the reader what categories, functors, natural transformations and equivalences of categories are. In the second appendix we discuss densities.

2. From Newton's law of motion to geometric mechanics in one hour

Consider a single particle of mass m moving on a line $\mathbb{R}$ subject to a force $F(q,t)$. Newton's law of motion in this case says: the trajectory $q(t)$ of the particles solves the second order ordinary differential equation (ODE):

$$m\frac{d^2q}{dt^2} = F(q(t), q'(t), t). \tag{2.1}$$

We now make two simplifying assumptions: (1) the force F only depends on position q and (2) the force F is conservative — that is, $F(q) = -V'(q)$ for some $V \in C^\infty(M)$. Then (2.1) becomes:

$$m\frac{d^2q}{dt^2} = -V'(q(t)). \tag{2.2}$$

The standard way to deal with equation (2.2) is to introduce a new variable p ("momentum") so that $p = m\dfrac{dq}{dt}$ and convert (2.2) into a system of first order ODEs. That is, if $p = m\dfrac{dq}{dt}$ then $m\dfrac{d^2q}{dt^2} = \dfrac{dp}{dt}$. Thus every solution of

$$\begin{cases} \dfrac{dq}{dt} = \dfrac{1}{m}p \\ \dfrac{dp}{dt} = -V'(q) \end{cases} \tag{2.3}$$

solves (2.2). On the other hand, a solution of (2.3) is an integral curve of a vector field $\Xi(p,q)$, where

$$\Xi(p,q) = \frac{1}{m}p\ \frac{\partial}{\partial q} - V'(q)\ \frac{\partial}{\partial p}.$$

Note: the energy $H(q,p) = \dfrac{1}{2m}p^2 + V(q)$ is conserved, i.e., it is constant along solutions of (2.3). In fact the function H completely determines the vector field Ξ in the following sense. Consider the two-form $\omega = dp \wedge dq$ on $\mathbb{R}^2$. It is easy to see that

$$\iota(\Xi)\omega = -dH.$$

So H determines Ξ. Notice $\omega = dp \wedge dq$ is nondegenerate, so for any 1-form α the equation $\omega(X,\cdot) = \alpha(\cdot)$ has a unique solution. To summarize: Newton's equations and $m\dfrac{d^2q}{dt} = -V'(q)$ are equivalent to integrating the vector field Ξ_H defined by $\iota(\Xi_H)\omega = -dH$. We now generalize this observation.

DEFINITION 2.1. A symplectic form ω on a manifold M is a closed nondegenerate 2-form. The pair (M, ω) is called a symplectic manifold.

REMARK 2.2. $d\omega = 0$ will give us an important property of the Poisson bracket: the Jacobi identity.

Standard examples of symplectic manifolds.

EXAMPLE 2.3. $(\mathbb{R}^2, dp \wedge dq)$

EXAMPLE 2.4. (Σ, ω) where Σ is an orientable surface and ω is an area form (nowhere zero form) on Σ. Note that since the surface Σ is two dimensional, $d\omega$ is automatically 0.

EXAMPLE 2.5. Let Q be any manifold and set $M = T^*Q$. If α denotes the tautological 1-form then $\omega = d\alpha$ is a symplectic form on M. Here are some details. "Recall" that there are two ways of defining the tautological 1-form (also called the Liouville form) α.

(1) In local coordinates α is defined as follows. If $(q_1, \dots q_n)$ is a coordinate chart on Q and $(q_1, \dots, q_n, p_1, \dots, p_n)$ the corresponding coordinates on the cotangent bundle T^*M, then

$$\alpha = \sum p_i dq_i.$$

It is not hard to check that $d\alpha = \omega = \sum dp_i \wedge dq_i$ is non-degenerate. It is closed by construction, hence it's symplectic. It is not obvious that α (and hence ω) are globally defined forms.

(2) Alternatively, we have the projection $\pi : T^*Q \to Q$ and $d\pi : T_{(q,p)}(T^*Q) \to T_qQ$. So given $q \in Q$, $p \in T_q^*Q$ and $v \in T_{(q,p)}(T^*Q)$ define

$$\alpha_{(q,p)}(v) = p(d\pi(v)).$$

It is a standard exercise to check that the two constructions agree. In the first construction of α it is clear that α is smooth and $d\alpha$ is nondegenerate. In the second construction it is clear that α is globally defined.

DEFINITION 2.6. The Hamiltonian vector field Ξ_f of a function f on a symplectic manifold (M, ω) is the unique vector field defined by $\omega(\Xi_f, \cdot) = -df$.

Warning: the opposite sign convention is also frequently used in literature: $\omega(\Xi_f, \cdot) = df$.

REMARK 2.7. The function f is always constant along the integral curves of its Hamiltonian vector field Ξ_f.

PROOF. Let γ be an integral curve of the vector field Ξ_f. Then

$$\frac{d}{dt} f(\gamma(t)) = \Xi_f(f) = df(\Xi_f) = -\omega(\Xi_f, \cdot)(\Xi_f) = -\omega(\Xi_f, \Xi_f).$$

Since ω is skew-symmetric, $\omega(\Xi_f, \Xi_f) = 0$. Hence $\dfrac{d}{dt} f(\gamma(t)) = 0$, i.e., $f(\gamma(t))$ is a constant function of t, which is what we wanted to prove. $\square$

Poisson Bracket.

DEFINITION 2.8. The Poisson bracket $\{\cdot,\cdot\}$ on a symplectic manifold (M,ω) is a map

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

defined by

$$\{f,g\} = \Xi_f(g).$$

REMARK 2.9. The Poisson bracket has a number of useful properties which we list below. Proofs may be found in any symplectic geometry book.

(1) For any three function $f,g,h \in C^\infty(M)$ we have

$$\{f, g\cdot h\} = \Xi_f(g\cdot h) = \Xi_f(g)\cdot h + g\cdot \Xi_f(h) = \{f,g\}\cdot h + g\cdot\{f,h\}.$$

(2) For any pair of functions f,g we have $\{f,g\} = \Xi_f(g) = dg(\Xi_f) = -\omega(\Xi_g,\Xi_f) = \omega(\Xi_f,\Xi_g) = -\{g,f\}$. In particular $\{f,f\} = 0$.

(3) One can show that the equation $d\omega = 0$ implies that

$$\{f,\{g,h\}\} = \{\{f,g\},h\} + \{g,\{f,g\}\},$$

which is the Jacobi identity. In other words the pair $(C^\infty(M),\{\cdot,\cdot\})$ is a Lie algebra.

(4) It is not hard to show that $\{f,g\} = 0$ if and only if g is constant along integral curves of f. Indeed, let γ be an integral curve of the vector field Ξ_f. Then

$$\frac{d}{dt} g(\gamma(t)) = \Xi_f(g)\,(\gamma(t)) = \{f,g\}(\gamma(t)).$$

Therefore if the function g is constant along γ then the bracket $\{f,g\}$ is zero along γ. The converse is true as well. This generalizes the fact that a function f is constant along the integral curves of its Hamiltonian vector field Ξ_f.

(5) One can show that $\iota([\Xi_f,\Xi_g])\omega = -d\{f,g\}$. Hence if the Poisson bracket $\{f,g\}$ of two functions f,g is 0 then flows of their Hamiltonian vector fields commute. Here is a better interpretation of the same fact: The map

$$C^\infty(M) \to \text{ vector fields on } M, \quad f \mapsto \Xi_f$$

is a map of Lie algebras: $\Xi_{\{f,g\}} = [\Xi_f,\Xi_g]$.

REMARK 2.10. It is not hard to show that is (M,ω) is a symplectic manifold, then its dimension is necessarily even. This only involves linear algebra.

Say $\dim M = 2n$. Then one can show further that the $2n$-form $\omega^n := \overbrace{\omega\wedge\cdots\wedge\omega}^{n}$ (n-fold wedge product) is nowhere zero, hence defines an orientation of M. In particular, this allows us to integrate any compactly supported function $f \in C^\infty(M)$ over M by integrating the form $f\omega^n$. The space $L^2(M,\omega)$ is then defined as the completion of the space $C_c^\infty(M,\mathbb{C})$ of compactly supported functions with respect to the L^2 norm

$$\|f\| := \left(\int_M |f|^2\omega^n\right)^{1/2}.$$

It is a Hilbert space with the Hermitian inner product

$$\langle\langle f,g\rangle\rangle := \int_M \bar{f} g\omega^n$$

(in the convention I prefer, the Hermitian inner products are complex-linear in the second variable).

We end the section with an easy lemma that we will need later.

LEMMA 2.11. *Let (M, ω) be a symplectic manifold and $f : M \to \mathbb{R}$ a smooth function. Then the Lie derivative $\mathcal{L}_{\Xi_f}\omega$ of the symplectic form with respect to the Hamiltonian vector field of the function f is zero:*

$$\mathcal{L}_{\Xi_f}\omega = 0.$$

PROOF. This is an application of Cartan's formula: for a differential form σ its Lie derivative $\mathcal{L}_X\sigma$ with respect to a vector field X is given by $\mathcal{L}_X\sigma = \iota(X)d\sigma + d\iota(X)\sigma$, where, as above, $\iota(X)\sigma$ denotes the contraction of X and σ, etc. Thus

$$\mathcal{L}_{\Xi_f}\omega = \iota(\Xi_f)d\omega + d\iota(\Xi_f)\omega.$$

The first summand above is 0 since $d\omega = 0$. By definition of Ξ_f, $\iota(\Xi_f)\omega = -df$. Since $d(df) = 0$, the second summand is zero as well. □

3. Prequantization

The goal of this section is to describe geometric prequantization. This is a procedure for turning a classical mechanical system mathematically formalized as a symplectic manifold (M, ω) together with its Poisson algebra of smooth functions $C^\infty(M)$ ("classical observables") into a quantum system formalized as a Hilbert space $\mathcal{H}_0$ together with the Lie algebra of (densely defined) skew-Hermitian operators $\{Q_f | f \in C^\infty(M)\}$. Moreover the map

$$Q : C^\infty(M) \to \mathrm{End}(\mathcal{H}_0), \quad f \mapsto Q_f,$$

should (and would) be a map of Lie algebras:

$$Q_{\{f,g\}} = [Q_f, Q_g]$$

for all functions $f, g \in C^\infty(M)$. We start by recalling some notation.

NOTATION 3.1. We denote the space of sections of a vector bundle $E \to M$ over a manifold M by $\Gamma(E)$. Thus the space of vector fields on a manifold M is denoted by $\Gamma(TM)$.

3.1. Connections.

We start by recalling a number of standard definitions and facts. By a fact I mean a theorem the proof of which will take us too far afield. Such proofs may be found in any number of textbooks.

DEFINITION 3.2. A connection ∇ on a (complex) vector bundle $E \xrightarrow{\pi} M$ is a $\mathbb{C}$-bilinear map

$$\Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X s$$

such that

(1) $\nabla_{fX} s = f\nabla_X s$ for all functions $f \in C^\infty(M, \mathbb{C})$ and all vector fields $X \in \Gamma(TM)$ (i.e., ∇ is $C^\infty(M, \mathbb{C})$ linear in the first variable) and
(2) $\nabla_X(fs) = X(f)s + f\nabla_X s$ for all functions $f \in C^\infty(M, \mathbb{C})$ and all vector fields $X \in \Gamma(TM)$ (i.e., ∇ is a derivation in second slot).

If the vector bundle E carries a fiber-wise Hermitian inner product $\langle \cdot, \cdot \rangle$ we can talk about the connections respecting this structure. More precisely

DEFINITION 3.3. A connection ∇ on a vector bundle $E \xrightarrow{\pi} M$ with a fiber-wise inner product $\langle\cdot,\cdot\rangle$ is Hermitian if

$$X(\langle s, s'\rangle) = \langle \nabla_X s, s'\rangle + \langle s, \nabla_X s'\rangle$$

for all vector fields X on M and all section $s, s' \in \Gamma(E)$.

Next recall that given any complex vector bundle $E \xrightarrow{\pi} M$ we can consider the bundle $\mathrm{End}(E) \to M$ of endomorphisms with a fiber $\mathrm{End}(E)_x$ at a point $x \in M$ given by

$$\mathrm{End}(E)_x = \{A : E_x \to E_x \mid A \text{ is } \mathbb{C} \text{ linear}\}.$$

We also have the subbundle of skew Hermitian maps $\mathrm{End}(E, \langle\cdot,\cdot\rangle) \subset \mathrm{End}(E)$ with typical fiber

$$\mathrm{End}(E, \langle\cdot,\cdot\rangle)_x = \{A : E_x \to E_x \mid A \text{ is } \mathbb{C} \text{ linear and } \langle Av, w\rangle + \langle w, Av\rangle = 0 \ \forall v, w \in E_x\}.$$

We have the following *Fact* :

FACT 3.4. The space of Hermitian connections on a vector bundle $(E, \langle\cdot,\cdot\rangle)$ is non empty. In fact it is an infinite dimensional affine space: the difference of two connections is a $\mathrm{End}(E, \langle\cdot,\cdot\rangle)$ valued 1-form.

DEFINITION 3.5. Let $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ be a connection on a vector bundle $E \to M$. The curvature R^∇ of the connection is a section of $\Lambda^2(T^*M) \otimes \mathrm{End}(E)$ (i.e., an $\mathrm{End}(E)$ valued 2-form). It is defined by

$$R^\nabla(X,Y)s = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s$$

for all vector fields X and Y and all sections $s \in \Gamma(E)$.

If ∇ is a Hermitian connection then its curvature R^∇ is a 2-form with values in $\mathrm{End}(E, \langle\cdot,\cdot\rangle)$. Furthermore if $E \xrightarrow{\pi} M$ is a complex line bundle then

$$\mathrm{End}(E, \langle\cdot,\cdot\rangle) \simeq M \times \sqrt{-1}\mathbb{R}$$

hence

$$\frac{1}{\sqrt{-1}} R^\nabla \in \Omega^2(M, \mathbb{R}).$$

That is $\frac{1}{\sqrt{-1}} R^\nabla$ is an ordinary real valued 2-form.

Fix a manifold M and consider the collection $\mathcal{D}(M)$ of all triples $(L, \langle\cdot,\cdot\rangle, \nabla)$, where $L \to M$ is a complex line bundle, $\langle\cdot,\cdot\rangle$ is a Hermitian inner product on L and ∇ is a Hermitian connection on $(L, \langle\cdot,\cdot\rangle)$. Then curvature defines a map

$$\mathcal{D}(M) \to \Omega^2(M), \quad (L, \langle\cdot,\cdot\rangle, \nabla) \mapsto \frac{1}{\sqrt{-1}} R^\nabla \tag{3.1}$$

from the collection $\mathcal{D}(M)$ to the set of (real-valued) 2-forms $\Omega^2(M)$.

To define geometric prequantization one needs to invert this map. That is, given a symplectic manifold (M, ω) one would like to find a Hermitian line bundle with a Hermitian connection ∇ so that

$$\frac{1}{\sqrt{-1}} R^\nabla = \omega.$$

However there are two problems: (1) the map (3.1) is not 1-1 and (2) not all symplectic forms are in the image of the map. The first problem has to do with the fact that taking curvature of a connection is very much like taking the exterior derivative of a 1-form. So recovering connection from its curvature is also like recovering a 1-form from its exterior derivative.

The second problem is topological. It has to do with the fact that (isomorphism classes of) complex line bundles are parametrized by degree 2 integral cohomology classes, that is, elements of $H^2(M,\mathbb{Z})$. Moreover the cohomology class $c_1(M)$ of a line bundle $E \to M$ and the de Rham class $[\frac{1}{2\pi\sqrt{-1}}R^\nabla]$ defined by the curvature R^∇ of a Hermitian connection ∇ on E are closely related: $[\frac{1}{2\pi\sqrt{-1}}R\nabla]$ is the image of $c_1(E)$ under the natural map

$$\iota : H^2(M,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \otimes \mathbb{R} \simeq H^2_{dR}(M).$$

Consequently the integral of the 2-form $\frac{1}{2\pi\sqrt{-1}}R^\nabla$ over any smooth integral 2-cycle in M has to be an integer. Hence the only symplectic forms that can be prequantized (that is, can occur as curvatures) are the forms whose integration over integral 2-cycles give integers. And if a symplectic form ω is integral (that is, the de Rham class $[\omega]$ lies in the image of the map ι above), the lift of $[\omega]$ to $H^2(M,\mathbb{Z})$ need not be unique. A solution to these problems (independently due to Kostant and to Souriau) can be stated as follows:

THEOREM 3.6. *Suppose the de Rham cohomology class of a closed 2-form σ on a manifold M lies in the image of $\iota : H^2(M,\mathbb{Z}) \to H^2_{dR}(M)$. Then there exists a Hermitian line bundle $E \to M$ with a Hermitian connection ∇ such that $\frac{1}{2\pi\sqrt{-1}}R^\nabla = \sigma$.*

There is another solution to this problem that I find more satisfactory and to which Theorem 3.6 is a corollary. It has the additional merit of allowing one to prequantize orbifolds as well. It involves thinking of $\mathcal{D}(M)$ not just as a set but as a collection of objects in a category and upgrading the map (3.1) to a functor. The target of this functor is a category of differential cocycles: the objects of this category involve integral cocycles *and* differential forms. The functor will turn out to be an equivalence of categories. So it can be easily inverted (up to homotopy). We will take this up in the last section of the notes. In the mean time we proceed with prequantization.

DEFINITION 3.7. Suppose $(E \to M, \langle\cdot,\cdot\rangle, \nabla)$ is a Hermitian line bundle with connection such that $\omega := \frac{1}{2\pi\sqrt{-1}}R^\nabla$ is symplectic. The prequantization is a linear map

$$Q : C^\infty(M) \to \mathrm{Hom}(\Gamma(E),\Gamma(E)), \quad f \mapsto Q_f$$

where the operator Q_f is defined by

$$Q_f(s) = \nabla_{\Xi_f} s - 2\pi\sqrt{-1} f \cdot s$$

for all functions $f \in C^\infty(M)$ and all section $s \in \Gamma(E)$. Here, as before, Ξ_f denotes the Hamiltonian vector field of f with respect to the symplectic form ω.

REMARK 3.8. Our definition of Q differs from a more traditional one by $\sqrt{-1}$. The physicists like to identify the Lie algebra of the unitary group with Hermitian matrices (and operators).

We next prove:

LEMMA 3.9. *The prequantization map $Q : C^\infty(M) \to \mathrm{Hom}(\Gamma(E),\Gamma(E))$ is a map of Lie algebras:*

$$[Q_f, Q_g]s = Q_{\{f,g\}}s$$

for all sections $s \in \Gamma(E)$ and all functions $f, g \in C^\infty(M)$.

PROOF. Since $\omega := \frac{1}{2\pi\sqrt{-1}} R^\nabla$, we have, by definition of curvature that

$$([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})s = 2\pi\sqrt{-1}\omega(X,Y)\cdot s$$

for all vector fields X, Y on M and all sections $s \in \Gamma(E)$. Hence

$$[\nabla_{\Xi_f}, \nabla_{\Xi_g}]s = \nabla_{[\Xi_f,\Xi_g]}s + 2\pi\sqrt{-1}\omega(\Xi_f, \Xi_g)\cdot s$$

for all $f, g \in C^\infty(M)$. Since $\omega(\Xi_f, \Xi_g) = \{f, g\}$ and since $[\Xi_f, \Xi_g] = \Xi_{\{f,g\}}$ we get

$$[\nabla_{\Xi_f}, \nabla_{\Xi_g}]s = \nabla_{\Xi_{\{f,g\}}}s + 2\pi\sqrt{-1}\{f,g\}s. \tag{3.2}$$

Next observe that

$$\begin{aligned} Q_f(Q_g s) &= Q_f(\nabla_{\Xi_g}s - 2\pi\sqrt{-1}g\cdot s) \\ &= \nabla_{\Xi_f}(\nabla_{\Xi_g}s - 2\pi\sqrt{-1}g\cdot s) - 2\pi\sqrt{-1}f\cdot(\nabla_{\Xi_g}s - 2\pi\sqrt{-1}g\cdot s) \\ &= \nabla_{\Xi_f}(\nabla_{\Xi_g}s) - 2\pi\sqrt{-1}\Xi_f(g)\cdot s - 2\pi\sqrt{-1}g\nabla_{\Xi_f}s - 2\pi\sqrt{-1}f\nabla_{\Xi_g}s - 4\pi^2 fg\cdot s \end{aligned}$$

Similarly,

$$Q_g(Q_f s) = \nabla_{\Xi_g}(\nabla_{\Xi_f}s) - 2\pi\sqrt{-1}\Xi_g(f)\cdot s - 2\pi\sqrt{-1}f\nabla_{\Xi_g}s - 2\pi\sqrt{-1}g\nabla_{\Xi_f}s - 4\pi^2 g\cdot f\cdot s.$$

Hence

$$\begin{aligned} [Q_f, Q_g]s &= Q_f(Q_g s) - Q_g(Q_f s) \\ &= [\nabla_{\Xi_f}, \nabla_{\Xi_g}]s - 2\pi\sqrt{-1}(\{f,g\} - \{g,f\})s \\ &= [\nabla_{\Xi_f}, \nabla_{\Xi_g}]s - 4\pi\sqrt{-1}\{f,g\}s \\ &= \nabla_{\Xi_{\{f,g\}}}s + 2\pi\sqrt{-1}\{f,g\}s - 4\pi\sqrt{-1}\{f,g\}s \quad \text{by (3.2)} \\ &= \nabla_{\Xi_{\{f,g\}}}s - 2\pi\sqrt{-1}\{f,g\}s = Q_{\{f,g\}}s. \end{aligned}$$

□

DEFINITION 3.10. Let (M, ω) be a symplectic manifold of dimension $2m$ and $(E \to M, \langle\cdot,\cdot,\rangle)$ be a Hermitian line bundle as before. A section $s \in \Gamma(E)$ is square integrable if the integral $\int_M \langle s, s\rangle\, \omega^m$ converges.

Clearly any section s with compact support is square integrable. Moreover, for any two compactly supported sections s, s' of $E \to M$ the function $\langle s, s'\rangle$ is compactly supported, hence the integral $\int_M \langle s, s'\rangle\, \omega^m$ converges.

NOTATION 3.11. We denote the space of compactly supported sections of the bundle $E \to M$ by $\Gamma_c(E)$:

$$\Gamma_c(E) := \{s \in \Gamma(E) \mid \operatorname{supp}(s) \text{ is compact }\}.$$

The space $\Gamma_c(E)$ of compactly supported sections carries a natural Hermitian inner product defined by

$$\langle\langle s, s'\rangle\rangle = \int_M \langle s, s'\rangle\, \omega^m$$

for all $s, s' \in \Gamma_c(E)$.

DEFINITION 3.12. The prequantum Hilbert space associated with a prequantum line bundle $(E \to M, \langle\cdot,\cdot,\rangle)$ is the completion of the inner product space $(\Gamma_c(E), \langle\langle\cdot,\cdot\rangle\rangle)$ with the respect to the corresponding L^2 norm:

$$\mathcal{H}_0 := \text{ the completion of } \Gamma_c(E).$$

LEMMA 3.13. *The prequantization map* $Q : C^\infty(M) \to \mathrm{Hom}(\Gamma_c(E), \Gamma_c(E))$ *is skew-Hermitian:*

$$\langle\langle Q_f s, s' \rangle\rangle + \langle\langle s, Q_f s' \rangle\rangle = 0$$

for all compactly supported sections s, s' *of* $E \to M$.

PROOF. Observe that

$$\langle 2\pi\sqrt{-1} f s, s' \rangle + \langle s, 2\pi\sqrt{-1} f s' \rangle = 0$$

for all functions $f \in C^\infty(M, \mathbb{R})$ and all square-integrable sections s, s'. Next note that since the connection ∇ is Hermitian we have

$$\int_M \langle \nabla_{\Xi_f} s, s' \rangle \omega^m + \int_M \langle s, \nabla_{\Xi_f} s' \rangle \omega^m = \int_M \Xi_f \langle s, s' \rangle \omega^m.$$

Since the Lie derivative $\mathcal{L}_{\Xi_f}\omega$ of the symplectic form with respect to any Hamiltonian vector field Ξ_f zero (see Lemma 2.11), we have $\mathcal{L}_{\Xi_f}\omega^m = 0$ as well. Hence

$$\mathcal{L}_{\Xi_f}(\langle s, s' \rangle \omega^m) = \mathcal{L}_{\Xi_f}(\langle s, s' \rangle)\omega^m + \langle s, s' \rangle \mathcal{L}_{\Xi_f}(\omega^m) = \Xi_f \langle s, s' \rangle \omega^m + 0.$$

On the other hand by Cartan's magic formula, for any top degree form μ we have

$$\mathcal{L}_{\Xi_f}\mu = \iota(\Xi_f) d\mu + d(\iota(\Xi_f)\mu) = d(\iota(\Xi_f)\mu),$$

since $d\mu = 0$. We conclude that

$$\begin{aligned}
\int_M \langle \nabla_{\Xi_f} s, s' \rangle \omega^m + \int_M \langle s, \nabla_{\Xi_f} s' \rangle \omega^m &= \int_M \Xi_f \langle s, s' \rangle \omega^m \\
&= \int_M \mathcal{L}_{\Xi_f}\left(\langle s, s' \rangle \omega^m\right) \\
&= \int_M d\left(\imath(\Xi_f)\langle s, s' \rangle \omega^m\right) = 0,
\end{aligned}$$

where the last equality holds by Stokes' theorem. The result follows. $\square$

REMARK 3.14. Note that the operators Q_f are not bounded in the L^2 norm since they involve differentiation. So they do not extend to bounded operators on the completion $\mathcal{H}_0$. However, they are elliptic operators, and consequently extend to closed densely defined operators on $\mathcal{H}_0$. Not surprisingly their domain of definition consists of square integrable sections with square integrable (distributional) first derivatives.

4. Polarizations

Recall that geometric prequantization associates to an integral symplectic manifold (M, ω) a Hilbert space $\mathcal{H}_0$ and to each real-valued function f on M a skew-Hermitian operator $Q_f : \mathcal{H}_0 \to \mathcal{H}_0$. Unfortunately this is not correct physics.

EXAMPLE 4.1. Suppose our classical configuration space is $\mathbb{R}$, the real line. This is the example with which we started these lectures. The corresponding classical phase space is the cotangent bundle $M = T^*\mathbb{R}$ with the canonical symplectic form $\omega = dp \wedge dq$. The corresponding prequantum line bundle E is trivial: $E = T^*\mathbb{R} \times \mathbb{C} \to T^*\mathbb{R}$. Hence the prequantum Hilbert space is $\mathcal{H}_0$ is the space $L^2(T^*\mathbb{R}, \mathbb{C})$ of complex valued square integrable functions. Quantum mechanics tells us that the correct Hilbert space consists of square-integrable functions of one variable $L^2(\mathbb{R}, \mathbb{C})$, not of functions of two variables.

A standard solution to this problem is to introduce a polarization. To define polarizations we start with linear algebra.

Definition 4.2. A Lagrangian subspace L of a symplectic vector space (V, ω) is a subspace satisfying two conditions:

(1) L is isotropic: $\omega(v, v') = 0$ for all vectors $v, v' \in L$;
(2) L is maximally isotropic: for any isotropic subspace L' of V containing L we must have $L = L'$.

Remark 4.3. A standard argument shows that if $L \subset (V, \omega)$ is Lagrangian then

$$\dim L = \frac{1}{2} \dim V.$$

Example 4.4. If $(V, \omega) = (\mathbb{R}^2, \omega = dp \wedge\, dq)$ then any line L in V is Lagrangian.

The analogous definition for submanifolds is as follows:

Definition 4.5. An immersed submanifold L of a symplectic manifold (M, ω) is Lagrangian if $T_xL \subset (T_xM, \omega_x)$ is a Lagrangian subspace for each point $x \in L$.

Definition 4.6. A (real) polarization on a symplectic manifold (M, ω) is a subbundle $\mathcal{F} \subset TM$ of its tangent bundle such that

(1) $\mathcal{F}$ is Lagrangian: $\mathcal{F}_x \subset (T_xM, \omega_x)$ is a Lagrangian subspace for all points $x \in M$.
(2) $\mathcal{F}$ is integrable (or involutive): for all local sections X, Y of $\mathcal{F} \to M$, the Lie bracket $[X, Y]$ is again a local section of $\mathcal{F}$. This conditions is often abbreviated as $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$.

Remark 4.7. If $\mathcal{F} \subset TM$ is an integrable distribution, then by the Frobenius theorem there exists a foliation $\mathcal{L}_\mathcal{F}$ of M tangent to the distribution $\mathcal{F}$. If in addition $\mathcal{F}$ is Lagrangian then the leaves of $\mathcal{L}_\mathcal{F}$ are immersed Lagrangian submanifolds of M.

Example 4.8. Suppose the symplectic manifold M is a cotangent bundle of some manifold N: $M = T^*N$ with its standard symplectic form. Then M has a polarization $\mathcal{F}$ given by the kernel of the differential of $\pi : T^*N \to N$:

$$\mathcal{F} = \ker(d\pi) : T(T^*N) \to TN.$$

In local coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n) : T^*U \to \mathbb{R}^n \times \mathbb{R}^n$ ($U \subset N$ open) on T^*N this polarization is given by

$$\mathcal{F} = \left\{ \sum_{i=1}^n a_i \frac{\partial}{\partial p_i} \mid a_i \in C^\infty(U) \right\}.$$

The polarization $\mathcal{F}$ is called the vertical polarization of T^*N.

Example 4.9. Consider the punctured plane $M := \mathbb{R}^2 \smallsetminus \{0\}$ with the symplectic form $dp \wedge dq$. The collection of circles $\big\{ C_r := \{(p, q) \in M \mid p^2 + q^2 = r^2\} \big\}_{r>0}$ forms a Lagrangian foliation of M. The tangent lines to the circle define a polarization $\mathcal{F}$ of M.

Remark 4.10. Real polarizations need not exist. Here is an example. It is not hard to show that any real line bundle over the two-sphere S^2 has to be trivial, hence has to have a nowhere zero section. Thus if S^2 has a polarization then it has a nowhere zero vector field, which contradicts a well-known theorem.

Since real polarization need not exist for interesting classical systems, one generalizes the notion of a real polarization to that of a complex polarization. A complex polarization on a symplectic manifold (M, ω) is a complex Lagrangian involutive subbundle of the complexified tangent bundle $TM \otimes \mathbb{C}$. We will not discuss them further. There is a well-developed theory of complex polarizations that an interested reader may consult.

Finally there are examples due to Mark Gotay [**6**] of symplectic manifolds that admit no polarizations whatsoever, real or complex. One can reconcile oneself to this fact by thinking that not all classical mechanical systems have quantum counterparts.

DEFINITION 4.11. Let $\mathcal{F}$ be a polarization on (M, ω) and $(\mathbb{L} \xrightarrow{\pi} M, \langle\cdot,\cdot\rangle, \nabla)$ a prequantum line bundle. A section $s \in \Gamma(\mathbb{L})$ is covariantly constant along $\mathcal{F}$ if $\nabla_X s = 0$ for all sections $X \in \Gamma(\mathcal{F})$.

NOTATION 4.12. We denote the space of sections of the prequantum line bundle $\mathbb{L} \to M$ covariantly constant along a polarization $\mathcal{F}$ by $\Gamma_{\mathcal{F}}(\mathbb{L})$. Thus

$$\Gamma_{\mathcal{F}}(\mathbb{L}) := \{s \in \Gamma(\mathbb{L}) \mid \nabla_X s = 0 \text{ for all } X \in \Gamma(\mathcal{F})\}.$$

REMARK 4.13. It is common to refer to the space $\Gamma_{\mathcal{F}}(\mathbb{L})$ as the space of polarized sections.

EXAMPLE 4.14. If $M = T^*N$, $\mathbb{L} = T^*N \times \mathbb{C}$ and $\mathcal{F} \subset TM$ is the vertical polarization, then the space $\Gamma_{\mathcal{F}}(\mathbb{L}) \subset C^\infty(T^*N, \mathbb{C})$ consists of functions constant along the fibers of $\pi : T^*N \to N$. Thus $\Gamma_{\mathcal{F}}(\mathbb{L}) = \pi^* C^\infty(N, \mathbb{C})$.

EXAMPLE 4.15. Consider again the punctured plane $M = \mathbb{R}^2 \smallsetminus \{0\}$ with the polarization $\mathcal{F}$ defined by circles. We may take the trivial bundle $\mathbb{L} = M \times \mathbb{C} \to M$ as the prequantum line bundle. Its space of sections is simply the space of complex valued functions on M. The Hermitian inner product on $\mathbb{L}$ is defined by the standard Hermitian inner product on $\mathbb{C}$.

For any real-valued 1-form α on M the map $\nabla : \Gamma(TM) \times C^\infty(M, \mathbb{C}) \to C^\infty(M, \mathbb{C})$ defined by

$$\nabla_X f = Xf + \sqrt{-1}\alpha(X) f$$

is a Hermitian connection. Now consider the real-valued 1-form α on on M given in polar coordinates (r, θ) by the equation $\alpha = r^2 d\theta$. A section $f \in C^\infty(M, \mathbb{C})$ of $\mathbb{L}$ is covariantly constant along the polarization defined by the circles (cf. Example 4.9) if and only if

$$\frac{\partial f}{\partial \theta} = -\sqrt{-1} r^2 f. \tag{4.1}$$

A function f solves the above equation if and only if it is of the form

$$f(r, \theta) = g(r) e^{-\sqrt{-1} r^2 \theta}$$

for some function $g(r)$. Such an f is a well-defined function on the punctured plane only if $r^2 \in \mathbb{Z}$. Thus (4.1) has no nonzero smooth solutions.

Let us go back to the general situation: a prequantum line bundle $(\mathbb{L}, \langle\cdot,\cdot\rangle, \nabla)$ over a symplectic manifold (M, ω) and a real polarization $\mathcal{F} \subset TM$. We now make several assumptions:

(1) The space of leaves $N := M/\mathcal{F}$ is a Hausdorff manifold and the quotient map $\pi : M \to M/\mathcal{F} \equiv N$ is a submersion.
(2) The space of polarized sections $\Gamma_{\mathcal{F}}(\mathbb{L})$ is nonzero.

REMARK 4.16. If we assume that the quotient map $\pi : M \to M/\mathcal{F}$ is *proper* then it places very severe restrictions on what the connected components of the leaves of $\mathcal{F}$ can be: they have to be compact tori. See Duistermaat [**4**], for example. In particular the fact that compact leaves of the polarization of a punctured plane turned out to be circles (i.e., one dimensional tori) should come as no surprise (q.v. Example 4.15). More generally the leaves of a Lagrangian fibration are open subsets of quotients of the form V/Γ where V is a finite dimensional real vector spaces and $\Gamma \subset V$ a discrete subgroup. That is, $V/\Gamma \simeq (\mathbb{R}^k/\mathbb{Z}^k) \times \mathbb{R}^l$ for some k, l with $k + l = \dim V$.

The issue with existence of nonzero parallel sections reduces to the holonomy of the connection being trivial along the leaves. Since the curvature of the connection vanishes identically on each leaf, the obstruction to the existence of nonzero parallel (polarized) sections lies in the representations of the fundamental groups of the leaves. This is why it is not uncommon for the fundamental groups of the leaves to be assumed away. Fortunately there are examples of fibrations with simply connected leaves that are slightly more general than the cotangent bundles. They are the so-called "twisted cotangent bundles" and amount to the following. Let Q be a manifold with an integral closed two-form τ (which may be degenerate) and let $M = T^*Q$. It is not hard to check that the two-form $\omega = \pi^*\tau + \omega_{T^*Q}$ is symplectic. Here $\pi : T^*Q \to Q$ is the canonical projection and ω_{T^*Q} is the canonical symplectic form on T^*Q (q.v. Example 2.5). The Lagrangian foliation of (T^*Q, ω) is provided by the fibers of π, which are contractible.

DEFINITION 4.17. Given a polarization $\mathcal{F} \subset TM$, the space of polarization preserving functions is the space $C^\infty_{\mathcal{F}}(M)$ defined by

$$C^\infty_{\mathcal{F}}(M) := \{f \in C^\infty(M) \mid [\Xi_f, X] \in \Gamma(\mathcal{F}) \text{ for all } X \in \Gamma(\mathcal{F})\},$$

where, as before, Ξ_f denotes the Hamiltonian vector field of the function f.

REMARK 4.18. It is not hard to show that if f is a polarization preserving function then the flow of its Hamiltonian vector field Ξ_f preserves the leaves of the foliation defined by the distribution $\mathcal{F}$.

REMARK 4.19. If $\Xi_f \in \Gamma(\mathcal{F})$ then, since $\mathcal{F}$ is involutive, $f \in C^\infty_{\mathcal{F}}(M)$. Using again the fact that $\mathcal{F}$ is Lagrangian, it is not hard to show that if $f = \pi^*h$ for some $h \in C^\infty(M/\mathcal{F})$ then $\Xi_f \in \Gamma(\mathcal{F})$. In particular the space $C^\infty_{\mathcal{F}}(M)$ is non-trivial.

LEMMA 4.20. *The subspace $C^\infty_{\mathcal{F}}(M)$ of $C^\infty(M)$ is closed under the Poisson bracket.*

PROOF. Suppose $f, g \in C^\infty_{\mathcal{F}}(M)$ and $X \in \Gamma(\mathcal{F})$. Then

$$\Xi_{\{f,g\}} = [\Xi_f, \Xi_g].$$

Hence

$$[X, \Xi_{\{f,g\}}] = [X, [\Xi_f, \Xi_g]] = [[X, \Xi_f], \Xi_g] + [\Xi_f, [X, \Xi_g]]$$

where the second equality hold by the Jacobi identity. Since $[X, \Xi_f], [X, \Xi_g] \in \Gamma(\mathcal{F})$ by assumption, and since $\Gamma(\mathcal{F})$ is a subspace of $\Gamma(TM)$ that is closed under brackets, we have $[X, \Xi_{\{f,g\}}] \in \Gamma(\mathcal{F})$. ☐

LEMMA 4.21. *If $f \in C^\infty_{\mathcal{F}}(M)$ is a polarization-preserving function then the operator Q_f defined by the prequantization map $Q : C^\infty(M) \to \mathrm{End}(\Gamma(\mathbb{L}))$ preserves the space $\Gamma_{\mathcal{F}}(\mathbb{L})$ of $\mathcal{F}$ polarized sections, the sections covariantly constant along $\mathcal{F}$.*

Hence we get a map of Lie algebras

$$Q : C^\infty_{\mathcal{F}}(M) \to \mathrm{End}(\Gamma_{\mathcal{F}}(\mathbb{L})).$$

PROOF. Note first that for any vector field $X \in \Gamma(TM)$ and any function $f \in C^\infty(M)$

$$X(f) = df(X) = (\iota(\Xi_f)\omega)(X) = \omega(X, \Xi_f).$$

To prove the lemma we need to show that

$$\nabla_X(Q_f s) = 0$$

for all vector fields $X \in \Gamma(\mathcal{F})$, all functions $f \in C^\infty_{\mathcal{F}}(M)$ and all polarized sections $s \in \Gamma_{\mathcal{F}}(\mathbb{L})$. Now

$$\nabla_X(\nabla_{\Xi_f} s - 2\pi\sqrt{-1} f s) = \nabla_X(\nabla_{\Xi_f} s) - 2\pi\sqrt{-1}X(f)\, s - 2\pi\sqrt{-1} f \nabla_X s.$$

Note that by assumption $2\pi\sqrt{-1} f\nabla_X s = 0$. By definition of curvature,

$$\nabla_X(\nabla_{\Xi_f} s) = \nabla_{\Xi_f}(\nabla_X s) + \nabla_{[X,\Xi_f]} s + R^\nabla(X, \Xi_f)s.$$

By assumption on X, f and s, the first two terms are 0. Also, by definition of the connection

$$R^\nabla = 2\pi\sqrt{-1}\omega.$$

We conclude that

$$\nabla_X(\nabla_{\Xi_f} s) = 0 + 0 + 2\pi\sqrt{-1}\omega(X, \Xi_f)s.$$

Putting it all together we see that

$$\nabla_X(Q_f s) = 2\pi\sqrt{-1}\omega(X,\Xi_f)s - 2\pi\sqrt{-1}X(f)\, s = 2\pi\sqrt{-1}(\omega(X,\Xi_f) - \omega(X,\Xi_f))s = 0.$$

$\square$

We would now like to define an inner product on the space $\Gamma_{\mathcal{F}}(\mathbb{L})$ of polarized sections. If the fibers of $\pi : M \to M/\mathcal{F}$ are compact, then as before we can define an inner product on a subspace of $\Gamma_{\mathcal{F}}(\mathbb{L})$ consisting of square integrable sections — c.f. Definition 3.10 and the subsequent discussion. The completion of this space would give us the desired Hilbert space. However, as we have seen in Example 4.15, the space of (smooth) polarized section can be 0. This is not just an accident of the particular example, but is fairly typical, since fibers of proper Lagrangian fibrations are tori. The solution to this problem — the lack of nonzero smooth polarized sections — is to consider distributional polarized sections. We will not say anything further on this topic in these notes. A curious reader may consult the discussion of distributional sections in [**12**].

If the leaves of the polarization on a symplectic manifold (M, ω) are not compact (as is the case of the vertical polarization on a cotangent bundle T^*N) then none of the polarized sections are square integrable with respect to the symplectic volume form ω^m ($m = \frac{1}{2}\dim M$). On the other hand, the Hermitian inner product $\langle s, s'\rangle$ of two polarized sections $s, s' \in \Gamma_{\mathcal{F}}(\mathbb{L})$ is constant along the fibers of the submersion $\pi : M \to M/\mathcal{F}$, hence descends to a function on the leaf space $M/\mathcal{F}$. Thus it is tempting to push the function $\langle s, s'\rangle$ down to $M/\mathcal{F}$ and integrate it over the leaf space. The problem is that the leaf space $M/\mathcal{F}$ has no preferred measure or volume. For instance suppose (M, ω) is a 2-dimensional symplectic vector space

(V, ω_V). Here we think of ω_V as a constant coefficient differential form. Then any line $\ell \subset V$ defines a polarization whose space of leaves is the quotient vector space V/ℓ. While the vector space V/ℓ is one dimensional in this example, and thus isomorphic to the real line $\mathbb{R}$, there is no preferred identification of V/ℓ with $\mathbb{R}$ and no preferred measure on V/ℓ.

Let us recap where we are. We have an integral symplectic manifold (M, ω), a prequantum line bundle $\mathbb{L} \to M$ with connection ∇ whose curvature is $2\pi\sqrt{-1}\omega$, a Lagrangian foliation $\mathcal{F}$ of M so that the space of leaves $M/\mathcal{F}$ is a Hausdorff manifold, the quotient map $\pi : M \to M/\mathcal{F}$ is a fibration and the holonomy representation of the fundamental groups of the leaves with respect to the connection ∇ are trivial. In this case the prequantum line bundle $\mathbb{L} \to M$ descends to a Hermitian line bundle $\mathbb{L}/\mathcal{F} \to M/\mathcal{F}$. We now consider the new complex line bundle $\mathbb{L}/\mathcal{F} \otimes |T(M/\mathcal{F})|^{1/2}$, the bundle $\mathbb{L}/\mathcal{F}$ twisted by the bundle of half densities on $M/\mathcal{F}$ (q.v. Definition B.11). We have an isomorphism

$$\Gamma(\mathbb{L}/\mathcal{F} \otimes |T(M/\mathcal{F})|^{1/2}) \simeq \Gamma(\mathbb{L}/\mathcal{F}) \otimes_{C^\infty(M/\mathcal{F})} \Gamma(|T(M/\mathcal{F})|^{1/2}) \tag{4.2}$$

of $C^\infty(M/\mathcal{F})$ modules.

It makes sense to talk about the sections of $\mathbb{L}/\mathcal{F} \otimes |T(M/\mathcal{F})|^{1/2}$ being square integrable, and it makes sense to define a sesquilinear pairing of two square integrable sections. This is done as follows. As we observed in Remark B.14, the bundle $|T(M/\mathcal{F})|^{1/2}$ of half densities is trivial, that is, it has a nowhere zero global section. Hence any section of $\mathbb{L}/\mathcal{F} \otimes |T(M/\mathcal{F})|^{1/2}$ is of the form $s \otimes \mu$ where $s \in \Gamma(\mathbb{L}/\mathcal{F}) \simeq \Gamma_{\mathcal{F}}(\mathbb{L})$ and μ is a 1/2 density on $M/\mathcal{F}$. Now given a polarized section $s \in \Gamma_{\mathcal{F}}(\mathbb{L})$ and a 1/2 density μ on $M/\mathcal{F}$ we can form a 1 density

$$\langle s, s\rangle \bar{\mu}\mu = ||s||^2|\mu|^2$$

on $M/\mathcal{F}$. Moreover the map

$$(\mathbb{L}/\mathcal{F}) \otimes_{C^\infty(M/\mathcal{F})} \Gamma(|T(M/\mathcal{F})|^{1/2}) \to |M/\mathcal{F}|^1, \quad s \otimes \mu \mapsto ||s||^2|\mu|^2$$

is well-defined. Here, as in Appendix B, $|M/\mathcal{F}|^1$ denotes the space of 1-densities on the manifold $M/\mathcal{F}$, which is the space of sections of the bundle $|T(M/\mathcal{F})|^1$ of 1-densities. It is not hard to show that the space of compactly supported polarized sections of the twisted prequantum line bundle forms a vector space with a Hermitian inner product given by

$$\langle\langle s_1 \otimes \mu_1, s_2 \otimes \mu_2\rangle\rangle := \int_{M/\mathcal{F}} \langle s_1, s_2\rangle \bar{\mu}_1\mu_2.$$

The completion of this complex vector space with respect to $\langle\langle\cdot,\cdot\rangle\rangle$ is the intrinsic quantum space associate with the data $(\mathbb{L} \to M, \nabla, \mathcal{F}, \langle\cdot,\cdot\rangle)$. A bit more effort gives a representation of the Lie algebra $C^\infty_{\mathcal{F}}(M)$ on this quantum space. See [**13**],[**12**] or [**16**].

5. Prequantization of differential cocycles

Let $L \xrightarrow{\pi} M$ be a complex line bundle, $\langle\cdot,\cdot\rangle$ a Hermitian inner product and ∇ a Hermitian connection. In Lecture 1 we were trying to find a section of the curvature map

$$curv : (L \xrightarrow{\pi} M, \langle\cdot,\cdot\rangle, \nabla) \longmapsto \frac{1}{\sqrt{-1}} R^\nabla \in \Omega^2(M), \tag{5.1}$$

which is neither 1-1 nor onto. Recall that a Hermitian line bundle with connection over a manifold M defines a principal S^1 bundle with a connection 1-form and conversely. Since 1-forms obviously pull back, it will be convenient for us to replace the problem of finding the section of (5.1) by the problem of finding a section of

$$curv : (S^1 \to P \to M, A \in \Omega^1(P,\mathbb{R})^{S^1}) \to F_A \in \Omega^2(M,\mathbb{R}), \tag{5.2}$$

where F_A denotes the curvature of the connection 1-form A. Note that here we think of the circle S^1 as $\mathbb{R}/\mathbb{Z}$ and not as the group $U(1)$ of unit complex numbers. Hence now our connection 1-forms are $\mathbb{R}$-valued.

For a given manifold M, the collection of all principal S^1-bundles with connection 1-forms over M forms a category, which we will denote by $DBS^1(M)$.[1] More precisely the objects of the category $DBS^1(M)$ are pairs (P, A), where P is a principal S^1 bundle over M and $A \in \Omega^1(P)$ is a connection 1-form on P. Given two objects (P, A), (P', A') of $DBS^1(M)$ the set $\mathrm{Hom}((P, A), (P', A'))$ of morphisms between them is defined by

$$\begin{aligned}\mathrm{Hom}((P,A),(P',A')) = \{\phi : P \to P' \mid \phi \text{ is } S^1 \text{ equivariant},\\ \phi \text{ induces identity on } M,\ \phi^*A' = A\}.\end{aligned}$$

Notice that all morphisms are invertible, so the category $DBS^1(M)$ is a groupoid by definition (see Appendix A).

Our solution of non-invertability of the curvature map proceeds along the following lines. We will construct a category $\mathcal{DC}(M)$ of differential cocycles, so that:

(1) The objects of $\mathcal{DC}(M)$ involve differential forms.
(2) There is an equivalence of categories $DBS^1(M) \xrightarrow{DCh} \mathcal{DC}(M)$.

Since equivalences of categories are invertible (up to natural isomorphisms) this will achieve our objective.[2] The construction of the category $\mathcal{DC}(M)$ was carried out as a toy example in a paper by Hopkins and Singer [**9**], which is where we copy the definition from. In constructing the functor DCh we will follow [**11**]. The construction of $\mathcal{DC}$ requires several steps.

Step 1: Categories from cochain complexes. Let $A^\bullet = \{A^\bullet \xrightarrow{d} A^{\bullet+1}\}$ be a cochain complex of abelian groups. For example, $A^\bullet = \Omega^\bullet(M)$, the complex of differential forms on a manifold M. For each index $n \geq 0$ there is a category $\mathcal{H}^n(A^\bullet)$ with the set $\{z \in A^n | dz = 0\}$ of cocycles of degree n being its set of objects. The set of morphisms $\mathrm{Hom}(z, z')$ for two cocycles z, z' is defined by

$$\begin{aligned}\mathrm{Hom}(z, z') = \{(z, [b]) \in \ker(d : A^n \to A^{n+1}) \times A^{n-1}/dA^{n-2} \mid z' = z + db\}\\ \simeq \{[b] \in A^{n-1}/dA^{n-2} \mid z' = z + db\}\end{aligned}$$

The composition of morphisms is addition $+$:

$$(z', [b']) \circ (z, [b]) = (z, [b + b'']).$$

The category $\mathcal{H}^n(A^\bullet)$ is a groupoid with the set $\pi_0(\mathcal{H}^n(A^\bullet)$ of equivalence classes of objects being the cohomology group $H^n(A^\bullet)$. The category $\mathcal{H}^n(A^\bullet)$ may also be

[1]This is not a standard notation. The $BS^1(M)$ is supposed to remind the reader of the classifying space BS^1, maps into which classify principal S^1 bundles. The D stands for "differential," i.e., the connection.

[2]A reader who is not fluent in category theory may wish at this point to contemplate Example A.23 of two rather different looking but equivalent categories.

viewed as an action groupoid for the action of A^{n-1}/dA^{n-2} on $\ker(d : A^n \to A^{n+1})$ by way of $d : A^{n-1} \to A^n$.

Next suppose we have a contravariant functor from the category Man of manifolds and smooth maps to the category $\mathsf{CoChain}$ of cochain complexes (such a functor is often called a presheaf of cochain complexes):

$$A^\bullet : \mathsf{Man}^{op} \to \mathsf{CoChain}, \quad M \mapsto A^\bullet(M).$$

An example to keep in mind is the functor that assigns to each manifold the complex of differential forms and to a map of manifolds the pullback of differential forms. Then each smooth map $f : M \to N$ between manifolds induces a functor

$$\mathcal{H}^n(f) : \mathcal{H}^n(A^\bullet(N)) \to \mathcal{H}^n(A^\bullet(M))$$

with

$$z' \xleftarrow[(z,[b])]{} z \qquad \longmapsto \qquad (f^*z' \xleftarrow[(f^*z,[f^*b])]{} f^*z) \ .$$

Step 2: The presheaf of differential cocycles. We need to introduce more notation. Denote by $C^\bullet(M,\mathbb{Z})$ the complex of (C^∞) singular integral cochains on a manifold M and by $C^\bullet(M,\mathbb{R})$ the complex of (C^∞) singular real-valued cochains. We have maps of complexes

$$C^\bullet(M,\mathbb{Z}) \to C^\bullet(M,\mathbb{R}) \quad \text{and} \quad \Omega^\bullet(M) \to C^\bullet(M,\mathbb{R}).$$

The second map sends a differential k-form σ to a functional on the space of real k-chains: the value of this functional on a chain s is the integral $\int_s \sigma$. We would like to find a complex $DC^\bullet(M)$ that fits into the diagram

$$\begin{tikzcd} DC^\bullet(M) \arrow[r, dashed] \arrow[d, dashed] & \Omega^\bullet(M) \arrow[d] \\ C^\bullet(M,\mathbb{Z}) \arrow[r] & C^\bullet(M,\mathbb{R}) \end{tikzcd} \ .$$

We define this complex as follows:

$$DC^k(M) = \{(c,h,\omega) \in C^k(M,\mathbb{Z}) \times C^{k-1}(M,\mathbb{R}) \times \Omega^k(M) \mid \omega = 0 \ for\ k < 2\}$$

with the differential $\tilde{d}$ defined by

$$\tilde{d}(c,h,\omega) = (\delta c, \omega - c - \delta h, d\omega).$$

Note that in defining $\tilde{d}$ we suppressed the maps $C^\bullet(M,\mathbb{Z}) \to C^\bullet(M,\mathbb{R})$ and $\Omega^\bullet(M) \to C^\bullet(M,\mathbb{R})$. In particular, when we think of a differential k-form τ as a real cochain, we write its differential as $\delta\tau$. That is, for any $k+1$ chain γ we have

$$\delta\tau(\gamma) = \int_\gamma d\tau.$$

It is not hard to show that

$$\tilde{d} \circ \tilde{d} = 0.$$

Indeed,

$$\tilde{d}(\tilde{d}(c,h,\omega)) = \tilde{d}(\delta c, \omega - c - \delta h, d\omega) = (\delta^2 c, d\omega - \delta c - \delta(\omega - c - \delta h), d^2\omega) = (0,0,0).$$

We are now ready to define the category $\mathcal{DC}(M)$ of differential cocycles by setting

$$\mathcal{DC}(M) := \mathcal{H}^2(DC^\bullet(M)).$$

By construction the set of objects $\mathcal{DC}(M)_0$ of this category is

$$\mathcal{DC}(M)_0 = \{(c,h,\omega) \in C^2(M,\mathbb{Z})\times C^1(M,\mathbb{R})\times\Omega^2(M) \mid \delta c = 0,\ d\omega = 0,\ \omega = c-\delta h\}$$

and the morphisms are defined by

$$\mathrm{Hom}((c,h,\omega),(c',h',\omega'))$$
$$= \left\{ [e,k,0] \;\middle|\; e \in C^1(M,\mathbb{Z}),\ k \in C^0(M,\mathbb{R}) \text{ and } \begin{array}{c} c'-c = \delta e \\ h'-h = -\delta k - e \\ \omega - \omega' = 0 \end{array} \right\}.$$

The following theorem then holds (it is presented as a warm-up example in [**9**]):

THEOREM 5.1. *For each manifold M there exists an equivalence of categories*

$$DCh_M : DBS^1(M) \to \mathcal{DC}(M)$$

with $DCh_M(P,A) = (c(P,A), h(P,A), F_A)$ for each principal circle bundle with connection (P,A). Here as before F_A denotes the curvature of a connection A.

REMARK 5.2. The functor that assigns a prequantum line bundle to a differential cocycle is the "homotopy inverse" of the functor DCh_M.

Finally here is an outline of an argument as to why this theorem is true and how you would go about writing down the functor DCh. I will be following the presentation in [**11**]. Here are the main ideas:

(1) Do it for all manifolds at once.
(2) Restate the theorem for presheaves of categories:
    There exists a morphism $DCh_{(\cdot)} : DBS^1(\cdot) \to \mathcal{DC}(\cdot)$ of presheaves of categories with the desired properties.
    Here the $\cdot$ is a place holder for a manifold.
(3) Write down the functor DCh explicitly for the sub-presheaf of trivial bundles $DBS^1_{triv}(\cdot) \subset DBS^1(\cdot)$. This is not hard. The set of objects of the presheaf $DBS^1_{triv}(\cdot)$ on a manifold M is the set

$$DBS^1_{triv}(M)_0 = \{(M\times S^1, a+d\theta) | a \in \Omega^1(M)\},$$

The set of morphisms is

$$\mathrm{Hom}((M\times S^1, a+d\theta),(M\times S^1, a'+d\theta)) = \{f : M \to S^1 | a = a' + f^*d\theta\}.$$

For each manifold M we therefore define the functor $DCh_M : DBS^1_{triv}(M) \to \mathcal{DC}(M)$ as follows: on objects

$$DCh_M(M\times S^1, a+d\theta) = (0,a,da),$$

on morphisms

$$DCh_M(f : M \to S^1) = [\delta(\tilde{f}) - f^*d\theta, \tilde{f}, 0],$$

where $\tilde{f} : M \to \mathbb{R}$ is any lift of f (not necessarily continuous). We think of $\tilde{f}$ as a real 0-cochain:

$$\tilde{f}\left(\sum_{p\in M} n_p\right) = \sum n_p \tilde{f}(p)$$

for any zero chain $\sum\limits_{p\in M} n_p$.

The rest of the argument uses the following facts (see [**11**] for details):

(1) The functor $DCh_M : DBS^1_{triv}(M) \to \mathcal{DC}(M)$ is bijective on Hom's (that is, it is a fully faithful functor).
(2) If M is contractible (e.g. an open ball) then DCh_M is essentially surjective. Of course, for a general manifold M not all bundles over M are trivial, but they are all locally trivial.
(3) Any bundle with a connection can be glued together out of the trivial ones.
(4) Differential cochains glue like bundles. Showing this requires work.

These last two facts amount to saying that the two presheaves $DBS^1(\cdot)$ and $\mathcal{DC}(\cdot)$ are stacks over $\underline{Man}$, the category of Manifolds.

The presheaf DBS^1_{triv} is not a stack since gluing a bunch of trivial bundles together need not result in a trivial bundle. There is an operation on presheaves of categories called stackification. This is a version of sheafification for sheaves of categories. The stackification of DBS^1_{triv}, not surprisingly, is DBS^1. Therefore, by the universal property of stackifications there is a unique functor $DCh : DBS^1(\cdot) \to \mathcal{DC}(\cdot)$ making the following diagram

$$\begin{tikzcd}
DBS^1(\cdot) \arrow[dr, dashed, "\exists!"] & \\
& \mathcal{DC}(\cdot) \\
DBS^1_{triv}(\cdot) \arrow[uu, hook] \arrow[ur, "DCh"'] &
\end{tikzcd}$$

commute. The resulting functor

$$DCh : DBS^1(\cdot) \to \mathcal{DC}(\cdot)$$

is an equivalence of categories since it is an equivalence of categories locally.

Appendix A. Elements of category theory

A.1. Basic notions.

We start by recalling the basic definitions of category theory, mostly to fix our notation. This appendix may be useful to the reader with some background in category theory; the reader with little to no experience in category theory may wish to consult a textbook such as [**2**].

Definition A.1 (Category). A **category** A consists of

(1) A collection[3] A_0 of *objects*;
(2) For any two objects $a, b \in \mathsf{A}_0$, a set $\mathrm{Hom}_{\mathsf{A}}(a, b)$ of of *morphisms* (or *arrows*);
(3) For any three objects $a, b, c \in \mathsf{A}_0$, and any two arrows $f \in \mathrm{Hom}_{\mathsf{A}}(a, b)$ and $g \in \mathrm{Hom}_{\mathsf{A}}(b, c)$, a *composite* $g \circ f \in \mathrm{Hom}_{\mathsf{A}}(a, c)$, i.e., for all triples of objects $a, b, c \in \mathsf{A}_0$ there is a *composition map*

[3] A collection may be too big to be a set; we will ignore the set-theoretic issues this may lead to.

$$\circ\colon \operatorname{Hom}_{\mathsf{A}}(b,c)\times \operatorname{Hom}_{\mathsf{A}}(a,b)\to \operatorname{Hom}_{\mathsf{A}}(a,c),$$

$$\operatorname{Hom}_{\mathsf{A}}(b,c)\times \operatorname{Hom}_{\mathsf{A}}(a,b)\ni (g,f)\mapsto g\circ f\in \operatorname{Hom}_{\mathsf{A}}(a,c).$$

This composition operation is *associative* and has *units*, that is,

i. for any triple of morphisms $f \in \operatorname{Hom}_{\mathsf{A}}(a,b)$, $g \in \operatorname{Hom}_{\mathsf{A}}(b,c)$ and $h \in \operatorname{Hom}_{\mathsf{A}}(c,d)$ we have
$$h\circ(g\circ f) = (h\circ g)\circ f\,;$$
ii. for any object $a \in \mathsf{A}_0$, there exists a morphism $1_a \in \operatorname{Hom}_{\mathsf{A}}(a,a)$, called the *identity*, which is such that for any $f \in \operatorname{Hom}_{\mathsf{A}}(a,b)$ we have
$$f = f\circ 1_a = 1_b\circ f\,.$$

We denote the collection of all morphisms of a category A by A_1:

$$\mathsf{A}_1 = \bigsqcup_{a,b\in\mathsf{A}_0} \operatorname{Hom}_{\mathsf{A}}(a,b).$$

Remark A.2. The symbol "$\circ$" is customarily suppressed in writing out compositions of two morphisms. Thus

$$gf \equiv g\circ f.$$

Example A.3 (Category Set of sets). The collection Set of all sets forms a category. The objects of Set are sets, the arrows of Set are ordinary maps and the composition of arrows is the composition of maps.

Example A.4 (Category Vect of vector spaces). The collection Vect of all real vector spaces (not necessarily finite dimensional) forms a category. Its objects are vector spaces and its morphisms are linear maps. The composition of morphisms is the ordinary composition of linear maps.

Example A.5 (The category Mat of coordinate vector spaces). The objects of this category are coordinate vector spaces $0 = \mathbb{R}^0, \mathbb{R}^1, \ldots, \mathbb{R}^n \ldots$. The set of morphism from $\mathbb{R}^m$ to $\mathbb{R}^n$ is the set of all $n\times m$ matrices. The composition of morphisms is given by a matrix multiplication.

Remark A.6. For a category A there are two maps from the collection A_1 of its arrows to the collection A_0 of objects called source and target and denoted respectively by s and t. They are defined by requiring that

$$s(f) = a \quad\text{and}\quad t(f) = b \quad\text{for any } f\in \operatorname{Hom}_{\mathsf{A}}(a,b).$$

Definition A.7. A *subcategory* A of a category B is a collection of some objects A_0 and some arrows A_1 of B such that:

- For each object $a \in \mathsf{A}_0$, the identity 1_a is in A_1;
- For each arrow $f \in \mathsf{A}_1$ its source and target $s(f), t(f)$ are in A_0;
- for each pair $(f,g) \in \mathsf{A}_0 \times \mathsf{A}_0$ of composable arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ the composite $g\circ f$ is in A_1 as well.

Remark A.8. Naturally a subcategory is a category in its own right.

Example A.9. The collection FinSet of all finite sets and all maps between them is a subcategory of Set hence a category. The collection $\mathsf{FinVect}$ of real finite dimensional vector spaces and linear maps is a subcategory of Vect.

EXAMPLE A.10. A subcategory $\mathsf{FinVect}^{iso}$ is defined to have the same objects as the category of $\mathsf{FinVect}$. Its morphisms are $\mathsf{isomorphisms}$ of vector spaces. Since the composition of two linear isomorphisms is an isomorphism $\mathsf{FinVect}^{iso}$ is a subcategory of $\mathsf{FinVect}$.

Note that for any object V in $\mathsf{FinVect}^{iso}$, that is, for any finite dimensional vector space V, the set of morphisms $\mathrm{Hom}(V,V)$ in $\mathsf{FinVect}^{iso}$ is $GL(V)$, the Lie group of invertible linear maps from V to V.

Compare this to the fact that in the category $\mathsf{FinVect}$ the set of morphisms $\mathrm{Hom}(V,V)$ is $\mathrm{End}(V)$, the space of all linear maps from V to itself.

DEFINITION A.11 (isomorphism). An arrow $f \in \mathrm{Hom}_{\mathsf{A}}(a,b)$ in a category A is an *isomorphism* if there is an arrow $g \in \mathrm{Hom}_A(b,a)$ with $g \circ f = 1_a$ and $f \circ g = 1_b$. We think of f and g as inverses of each other and may write $g = f^{-1}$. Clearly $g = f^{-1}$ is also an isomorphism.

Two objects $a, b \in \mathsf{A}_0$ are *isomorphic* if there is an isomorphism $f \in \mathrm{Hom}_{\mathsf{A}}(a,b)$. We will also say that a is isomorphic to b.

DEFINITION A.12 (Groupoid). A *groupoid* is a category in which every arrow is an isomorphism.

EXAMPLE A.13. The category $\mathsf{FinVect}^{iso}$ is a groupoid.

DEFINITION A.14 (Functor). A (covariant) *functor* $F\colon \mathsf{A} \to \mathsf{B}$ from a category A to a category B is a map on the objects and arrows of A such that every object $a \in \mathsf{A}_0$ is assigned an object $Fa \in \mathsf{B}_0$, every arrow $f \in \mathrm{Hom}_{\mathsf{A}}(a,b)$ is assigned an arrow $Ff \in \mathrm{Hom}_{\mathsf{B}}(Fa, Fb)$, and such that composition and identities are preserved, namely

$$F(f \circ g) = Ff \circ Fg, \quad F1_a = 1_{Fa}.$$

A *contravariant* functor G from A to B is a map on the objects and arrows of A such that every object $a \in \mathsf{A}_0$ is assigned an object $Ga \in \mathsf{B}_0$, every arrow $f \in \mathrm{Hom}_{\mathsf{A}}(a,b)$ is assigned an arrow $Gf \in \mathrm{Hom}_{\mathsf{B}}(Gb, Ga)$ (note the order reversal), such that identities are preserved, and the composition of arrows is reversed:

$$G(f \circ g) = G(g) \circ G(f)$$

for all composable pairs of arrows f, g of A.

EXAMPLE A.15. There is a natural functor $\iota : \mathsf{Mat} \to \mathsf{FinVect}$ which is the identity on objects and maps an $n \times m$ matrix to the corresponding linear map.

EXAMPLE A.16. The functor $(-)^* : \mathsf{FinVect} \to \mathsf{FinVect}$ that takes the duals, that is, $(V \xrightarrow{A} W) \mapsto (V^* \xleftarrow{A^*} W^*)$ is a contravariant functor.

REMARK A.17. Since functors are maps, functors can be composed.

DEFINITION A.18. A functor $F\colon \mathsf{A} \to \mathsf{B}$ is

(1) *full* if $F\colon \mathrm{Hom}_{\mathsf{A}}(a,a') \to \mathrm{Hom}_{\mathsf{B}}(Fa, Fa')$ is surjective for all pairs of objects $a, a' \in \mathsf{A}_0$;
(2) *faithful* if $F\colon \mathrm{Hom}_{\mathsf{A}}(a,a') \to \mathrm{Hom}_{\mathsf{B}}(Fa, Fa')$ is injective for all pairs of objects $a, a' \in \mathsf{A}_0$
(3) *fully faithful* if $F\colon \mathrm{Hom}_{\mathsf{A}}(a,a') \to \mathrm{Hom}_{\mathsf{B}}(Fa, Fa')$ is a bijection for all pairs of objects $a, a' \in \mathsf{A}_0$;

(4) *essentially surjective* if for any object $b \in \mathsf{B}_0$ there is an object $a \in \mathsf{A}_0$ and an isomorphism $f \in \mathrm{Hom}_{\mathsf{B}}(F(a), b)$. That is, for any object b of B there is an object a of A so that b and $F(a)$ are isomorphic.

EXAMPLE A.19. The functor $\iota : \mathsf{Mat} \to \mathsf{FinVect}$ is fully faithful (since any linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is uniquely determined by what it does on the standard basis) and essentially surjective (since any real vector space of dimension n is isomorphic to $\mathbb{R}^n$).

DEFINITION A.20 (Natural Transformation). Let $F, G\colon \mathsf{A} \to \mathsf{B}$ be a pair of functors. A *natural transformation* $\tau\colon F \Rightarrow G$ is a family of $\{\tau_a\colon Fa \to Ga\}_{a\in\mathsf{A}_0}$ of morphisms in B, one for each object a of A, such that, for any $f \in \mathrm{Hom}_{\mathsf{A}}(a, a')$, the following diagram commutes:

$$\begin{tikzcd} Fa \arrow[r, "Ff"] \arrow[d, "\tau_a"'] & Fb \arrow[d, "\tau_b"] \\ Ga \arrow[r, "Fg"] & Gb \end{tikzcd}$$

If each τ_a is an isomorphism, we say that τ is a *natural isomorphism* (an older term is *natural equivalence*).

DEFINITION A.21 (Equivalence of categories). An *equivalence of categories* consists of a pair of functors

$$F\colon \mathsf{A} \to \mathsf{B}, \quad E\colon \mathsf{B} \to \mathsf{A}$$

and a pair of natural isomorphisms

$$\alpha\colon 1_{\mathsf{A}} \Rightarrow E \circ F \quad \beta\colon 1_{\mathsf{B}} \Rightarrow F \circ E.$$

In this situation the functor F is called *the pseudo-inverse* or the *homotopy inverse* of E. The categories A and B are then said to be *equivalent categories.*

PROPOSITION A.22. *A functor $F\colon \mathsf{A} \to \mathsf{B}$ is (part of) an equivalence of categories if and only if it is fully faithful and essentially surjective.*

PROOF. See [**2**, Proposition 7.25] □

EXAMPLE A.23. The categories Mat of matrices and $\mathsf{FinVect}$ of finite dimensional vector spaces are equivalent categories since the functor $\iota : \mathsf{Mat} \to \mathsf{FinVect}$ (q.v. Example A.15) is fully faithful and essentially surjective. Note that the functor ι is *not* surjective on objects.

Appendix B. Densities

In this section we borrow from a manuscript by Guillemin and Sternberg [**8**].

B.1. Densities on a vector space. Consider an n-dimensional real vector space V. Recall that a basis $\{v_1, \ldots, v_n\}$ defines a linear isomorphism $\mathsf{v} : \mathbb{R}^n \to V$ by $\mathsf{v}(x_1, \ldots, x_n) = \sum x_i v_i$. Conversely any linear isomorphism $\mathsf{v} : \mathbb{R}^n \to V$ defines a basis $\{v_1, \ldots, v_n\}$ of V by setting the i^{th} basis vector v_i to be the image $\mathsf{v}(e_i)$ of the standard i^{th} basis vector e_i of $\mathbb{R}^n$ under the isomorphism v. From now on we will not distinguish between a basis of V and a linear isomorphism from $\mathbb{R}^n$ to V.

Definition B.1 (Frame). We denote the space of bases of an n dimensional real vector space V by $\mathcal{F}r(V)$ and refer to it as the space of frames of V.

Note that there is a natural right action $\bullet$ of the Lie group $GL(n,\mathbb{R}) \equiv GL(\mathbb{R}^n)$ on the space of frames $\mathcal{F}r(V)$ of an n-dimensional vector space V by composition on the right:

$$\mathsf{v} \bullet A := \mathsf{v} \circ A$$

for all isomorphisms $\mathsf{v} : \mathbb{R}^n \to V$ and all $A \in GL(n,\mathbb{R})$. Moreover this action is free and transitive: given $\mathsf{v}, \mathsf{v}' \in \mathcal{F}r(V)$ and $A \in GL(n,\mathbb{R})$

$$\mathsf{v} \bullet A = \mathsf{v} \quad \text{if and only if } A = (\mathsf{v}^{-1} \circ \mathsf{v}').$$

Remark B.2. A space X with a free and transitive action of a group G is called a G torsor.

With these preliminaries out of the way we are ready to define α-densities on a vector space.

Definition B.3 (α-density). Let α denote a complex number. An α-density (also called a density of order α) on an n-dimensional real vector space V is a map

$$\tau : \mathcal{F}r(V) \to \mathbb{C} \quad \text{with} \quad \tau(\mathsf{v} \bullet A) = \tau(\mathsf{v})|\det(A)|^\alpha$$

for all $\mathsf{v} \in \mathcal{F}r(V)$ and all $A \in GL(n,\mathbb{R})$.

Notation B.4. Since α-densities on a fixed vector space V are complex valued functions, they form a complex vector space. We denote it by $|V|^\alpha$. In other words

$$|V|^\alpha := \{\tau : \mathcal{F}r(V) \to \mathbb{C} \mid \tau(\mathsf{v}\bullet A) = \tau(\mathsf{v})|\det(A)|^\alpha \text{ for all } \mathsf{v} \in \mathcal{F}r(V), A \in GL(n,\mathbb{R})\}.$$

Remark B.5. Alternatively one may view the space of frames $\mathcal{F}r(V)$ as a principal $GL(n,\mathbb{R})$ bundle over a point. An α-density is then a section of the associated bundle $(\mathcal{F}r(V) \times \mathbb{C})/GL(n,\mathbb{R})$, where $GL(n,\mathbb{R})$ acts on $\mathbb{C}$ by the character $A \mapsto |\det A|^\alpha$. Since $(\mathcal{F}r(V) \times \mathbb{C})/GL(n,\mathbb{R})$ is a complex line bundle over a point, it is a one dimensional complex vector space. Hence the space of sections of this bundle, i.e., the space of densities $|V|^\alpha$, has complex dimension 1. In particular we have proved:

Lemma B.6. *The space $|V|^\alpha$ of α-densities on a vector space V is a complex 1-dimensional vector space.*

Remark B.7. Any nonzero n-form $\omega \in \Lambda^n(V^*)$ on an n-dimensional real vector space V defines an α-density $|\omega|^\alpha$ by the formula

$$|\omega|^\alpha(v_1, \dots, v_n) := |\omega(v_1, \dots, v_n)|^\alpha$$

for all frames $\{v_1, \dots, v_n\} \in \mathcal{F}r(V)$.

Conversely, since the space $|V|^\alpha$ of α densities is 1-dimensional, any density $\tau \in |V|^\alpha$ is of the form $\tau = c|\omega|^\alpha$ for some nonzero n-form $\omega \in \Lambda^n(V^*)$ and a constant $c \in \mathbb{C}$.

Remark B.8. Densities pull back under linear isomorphisms. If $T : W \to V$ is an isomorphism of n-dimensional vector spaces and $\tau \in |V|^\alpha$ is a density then its pullback $T^*\tau : \mathcal{F}r(W) \to \mathbb{C}$ is defined by

$$T^*\tau\,(w_1, \dots w_n) = \tau(Tw_1, \dots Tw_n)$$

for any frame $\mathsf{w} = \{w_1, \ldots, w_n\} \in \mathcal{F}r(W)$. Note that since T is an isomorphism the tuple $(Tw_1, \ldots Tw_n)$ *is* a frame of V, so the definition of pullback does make sense.

REMARK B.9. Densities can be multiplied: if $\rho \in |V|^\alpha$ and $\tau \in |V|^\beta$ are two densities of order α and β respectively then their product defined by

$$(\rho \cdot \tau)(\mathsf{v}) = \rho(\mathsf{v})\tau(\mathsf{v})$$

is easily seen to be a density of order $\alpha+\beta$. Since the multiplication map $(\rho, \tau) \mapsto \rho\cdot\tau$ is $\mathbb{C}$ bilinear, we get a $\mathbb{C}$ linear map

$$|V|^\alpha \otimes |V|^\beta \to |V|^{\alpha+\beta}.$$

Since the map is nonzero, it is an isomorphism of vector spaces by dimension count. In particular we have a canonical isomorphism

$$|V|^{1/2} \otimes |V|^{1/2} \xrightarrow{\simeq} |V|^1.$$

REMARK B.10. It makes sense to take a complex conjugate of a density. If $\rho \in |V|^\alpha$ is an α-density then $\bar{\rho} : \mathcal{F}r(V) \to \mathbb{C}$ defined by

$$\bar{\rho}(\mathsf{v}) := \overline{\rho(\mathsf{v})}$$

for all $\mathsf{v} \in \mathcal{F}r(V)$ is easily seen to be an $\bar{\alpha}$-density.

Therefore we can define on the space $|V|^{1/2}$ of half-densities a $|V|^1$-valued Hermitian inner product by

$$(\mu, \tau) := \bar{\mu}\tau.$$

B.2. Densities on manifolds. Recall that for any real vector bundle $E \to M$ of rank k over a manifold M we have the principal $GL(k, \mathbb{R})$ bundle $\mathcal{F}r(E) \to M$, the so-called frame bundle of $E \to M$. A typical fiber $\mathcal{F}r(E)_q$ of this bundle above a point q of M consists of the frames of the fiber E_q of the bundle E:

$$\mathcal{F}r(E)_q := \mathcal{F}r(E_q).$$

Recall also that we can think of $\mathcal{F}r(E)$ as an open subset of the vector bundle $\mathrm{Hom}(M \times \mathbb{R}^k, E) \to M$; $\mathcal{F}r(E)$ consists of isomorphisms. There is a natural right $GL(k, \mathbb{R})$ action of $\mathcal{F}r(E)$ making it a principal $GL(k, \mathbb{R})$ bundle.

Next recall that given a principal G-bundle $G \to P \to M$ over a manifold M and a (complex) representation $\rho : G \to GL(W)$, we can build out of this data a complex vector bundle $P \times^\rho W \to M$ over M. This bundle is the quotient of the trivial bundle $P \times W$ by a free and proper left action of G:

$$P \times^\rho W := (P \times W)/G, \quad g \cdot (p, w) := (pg^{-1}, \rho(g)w).$$

It will be useful to recall that the space of sections of the associated bundle $\Gamma(P \times^\rho W)$ is isomorphic to the space of equivariant W-valued functions on P:

$$\Gamma(P \times^\rho W) \simeq \{\varphi : P \to W \mid \varphi(p \cdot g) = \rho(g)^{-1}\varphi(p)\}.$$

DEFINITION B.11 (densities on a manifold). We define the complex line bundle $|TM|^\alpha \to M$ of α-densities on a manifold M to be the associated bundle

$$|TM|^\alpha := \mathcal{F}r(TM) \times^{|\det|^{-\alpha}} \mathbb{C},$$

where the representation $|\det|^{-\alpha} : GL(k, \mathbb{R}) \to GL(\mathbb{C})$ is defined by

$$|\det|^{-\alpha}(A)z := |\det A|^{-\alpha} z \quad \text{for all} \quad A \in GL(k, \mathbb{R}), z \in \mathbb{C}.$$

We refer to the sections of the bundle $|TM|^\alpha \to M$ as α-densities on the manifold M. We denote the space of α-densities on a manifold M by $|M|^\alpha$:

$$|M|^\alpha := \Gamma(\mathcal{F}r(TM) \times^{|\det|^{-\alpha}} \mathbb{C}).$$

REMARK B.12. We may and will identify the space of α-densities on a manifold M with the space of equivariant complex functions on its frame bundle:

$$|M|^\alpha \simeq \{\tau : \mathcal{F}r(TM) \to \mathbb{C} \mid \tau(\mathsf{v} \bullet A) = |\det A|^\alpha \tau(\mathsf{v}) \quad \text{for all } A \in GL(m, \mathbb{R}), \mathsf{v} \in \mathcal{F}r(TM)\}.$$

REMARK B.13. Note that by design the fiber of the bundle $|TM|^\alpha \to M$ at a point q is the 1-dimensional complex vector space $|T_qM|^\alpha$ of α-densities on the tangent space T_qM.

REMARK B.14. By construction the transition maps for the complex line bundle $|TM|^\alpha \to M$ take their values in positive real numbers. It follows that $|TM|^\alpha \to M$ is a trivial bundle but not canonically. In particular its space of sections $|M|^\alpha$ is a rank 1 $C^\infty(M, \mathbb{C})$ module. Since the space of sections of the tensor product is isomorphic to the tensor product of sections (as $C^\infty(M)$ modules), it follows that for any complex line bundle $\mathbb{L} \to M$, a section of $\mathbb{L} \otimes |TM|^\alpha$ is of the form $s \otimes \tau$ for some section $s \in \Gamma(\mathbb{L})$ of $\mathbb{L} \to M$ and some α-density τ.

REMARK B.15. Since densities on a vector space pull back under a linear isomorphism, densities on a manifold pull back under local diffeomorphisms: If $F : N \to M$ is a local diffeomorphism between two manifolds and $\tau : \mathcal{F}r(TM) \to \mathbb{C}$ an α-density on M, the pullback $F^*\tau \in |N|^\alpha$ is defined as follows: for any point $q \in N$ and any frame $(v_1, \ldots, v_n)$ of T_qN

$$(F^*\tau)_q(v_1, \ldots, v_n) := \tau_{F(q)}(dF_q v_1, \ldots, dF_q v_n).$$

REMARK B.16. For any open subset U of $\mathbb{R}^m$ we have the canonical α-density $|dx_1 \wedge \cdots dx_m|^\alpha$ (q.v. Remark B.7). Here, of course, $x_1, \ldots, x_m$ are the Cartesian coordinates on $\mathbb{R}^m$. This density defines an isomorphism $U \times \mathbb{C} \to |TU|^\alpha$. Therefore, for any α-density τ on U there exists a unique function $f_\tau \in C^\infty(U)$ with

$$\tau = f_\tau |dx_1 \wedge \cdots dx_n|^\alpha.$$

Indeed,

$$f_\tau = \tau\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_m}\right).$$

Putting the preceding remarks together we note that for a diffeomorphism $F : U \to V$ of open subsets of $\mathbb{R}^m$ and an α-density $f(y)|dy_1 \wedge \ldots \wedge dy_m|^\alpha$ on V we have

$$F^*\left(f(y)|dy_1 \wedge \ldots \wedge dy_m|^\alpha\right) = f(F(x))|\det dF_x|^\alpha |dx_1 \wedge \cdots \wedge dx_n|^\alpha. \tag{B.1}$$

Formula (B.1) is a reason why 1-densities can be integrated over manifolds. The story parallels the familiar story of integration of top degree forms on oriented manifolds. The 1-densities have the advantage that the manifold doesn't have to be oriented (or even be orientable) for their integrals to make sense.

The story proceeds as follows. If $U \subset \mathbb{R}^n$ is an open set and $\tau \in |U|^1$ a 1-density, then, as remarked above,

$$\tau = f_\tau |dx_1 \wedge \ldots \wedge dx_n|$$

for a unique complex valued smooth function f_τ on U. One defines the integral $\int_U \tau$ by

$$\int_U \tau := \int_U f_\tau.$$

Or, if you prefer,

$$\int_U f \; |dx_1 \wedge \ldots \wedge dx_n| := \int_U f \; dx_1 \ldots dx_n$$

for any smooth integrable function f on the set U. This is a complex valued integral.

If M is a manifold of dimension m, $\phi : U \to \mathbb{R}^m$ a coordinate chart and τ is a 1-density on M with support in U one defines

$$\int_M \tau := \int_{\phi(U)} (\phi^{-1})^* \tau.$$

If $\psi : U \to \mathbb{R}^m$ is another coordinate chart, then

$$(\psi^{-1})^* \tau = f(y) \, |dy_1 \wedge \cdots \wedge dy_m|$$

for some $f \in C^\infty(\psi(U))$. Consider the diffeomorphism $F = \psi \circ \phi^{-1} : \phi(U) \to \psi(U)$. We have

$$(\phi^{-1})^* \tau = (\psi^{-1} \circ \psi \phi^{-1})^* \tau = F^* (\psi^{-1})^* \tau.$$

By (B.1)

$$(\phi^{-1})^* \tau = f(F(x)) |\det DF_x| |dx_1 \wedge \cdots \wedge dx_m|.$$

Therefore

$$\begin{aligned}\int_{\phi(U)} (\phi^{-1})^* \tau \;&=\; \int_{\phi(U)} f(F(x)) \, |\det DF_x| \, dx_1 \cdots dx_m \\ &= \int_{F(\phi(U)) = \psi(U)} f(y) \, dy_1 \cdots dy_m = \int_{\psi(U)} (\psi^{-1})^* \tau, .\end{aligned}$$

where the second equality holds by the change of variables formula for functions on regions of $\mathbb{R}^m$. Therefore integrals of 1-densities supported in coordinate charts are well-defined.

Next one makes sense of integrability of non-negative densities. A 1-density τ on a manifold M is non-negative if its value at any frame $\{v_1, \ldots v_m\} \subset T_q M$ is a non-negative real number. It is not hard to see that "non-negativity" is well-defined. Now choose a locally finite cover $\{U_\alpha\}$ of M by coordinate charts and choose a partition of unity $\{\rho_\alpha\}$ subordinate to the cover. We say that a non-negative 1-density is integrable if the sum

$$\sum_\alpha \int_M \rho_\alpha \tau = \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau$$

converges. We then define $\int_M \tau$ to be the sum:

$$\int_M \tau := \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau.$$

The rest of the definition of integration of 1-densities proceeds just like for functions. A given real-valued 1-density τ can be written as a difference of two (continuous) non-negative 1-densities τ_+ and τ_-:

$$\tau = \tau_+ - \tau_-.$$

We call τ integrable if $\int_M \tau_+$ and $\int_M \tau_-$ are finite. We then set $\int_M \tau = \int_M \tau_+ - \int_M \tau_-$. Finally we define a complex valued 1-density τ integrable if its real and imaginary parts are integrable; we define its integral to be the sum

$$\int_M \tau = \int_M Re(\tau) + \sqrt{-1}\int_M Im(\tau).$$

B.3. The "Intrinsic" Hilbert space. Suppose $(\mathbb{L} \to M, \langle\cdot,\cdot\rangle)$ is a Hermitian line bundle. Then given a 1-density τ on M we can define a Hermitian pairing pairing of sections of $\mathbb{L}$ by

$$\langle\langle s, s'\rangle\rangle := \int_M \langle s, s'\rangle \tau.$$

There is an associated Hilbert space of "square integrable" sections, which, of course, depends on the choice of our density τ. There is also a more intrinsic pairing of sections of a slightly different bundle. Consider the tensor product of complex line bundles $\mathbb{L} \otimes |M|^{1/2} \to M$. By Remark B.14 a section of $\mathbb{L} \otimes |M|^{1/2} \to M$ is of the form $s \otimes \mu$, where $s \in \Gamma(\mathbb{L})$ and μ is a $\frac{1}{2}$-density. Now given two sections $s_1 \otimes \mu_1$ and $s_2 \otimes \mu_2$ we can pair them to get a 1-density $\langle s_1, s_2\rangle \bar{\mu}_1 \mu_2$ (q.v. Remark B.10). Hence the Hermitian inner product

$$\Gamma(\mathbb{L} \otimes |M|^{1/2}) \times \Gamma(\mathbb{L} \otimes |M|^{1/2}) \to \mathbb{R}, \quad \langle\langle s_1 \otimes \mu_1, s_2 \otimes \mu_2\rangle\rangle := \int_M \langle s_1, s_2\rangle \bar{\mu}_1 \mu_2$$

makes sense (whenever the integral converges). It is easy to see that the integral above does converge for all sections in the space

$$L^2(\mathbb{L}\otimes|M|^{1/2}, M)\cap\Gamma(\mathbb{L}\otimes|M|^{1/2}) := \left\{ s \otimes \mu \in \Gamma(\mathbb{L} \otimes |M|^{1/2}) \;\middle|\; \int_M \langle s, s\rangle\, |\mu|^2 < \infty \right\}.$$

The completion of the space with respect to the Hermitian inner product gives us the "intrinsic Hilbert space of square-integrable sections" $L^2(\mathbb{L} \otimes |M|^{1/2}, M)$.

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Lectures on group-valued moment maps and Verlinde formulas

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1. Introduction

The theory of q-Hamiltonian G-spaces has its origins in the theory of Hamiltonian loop group actions, which in turn is motivated by moduli spaces of flat bundles over surfaces with boundary (cf. [**10, 59**]).

It is a well-known guiding principle that loop groups for compact, simply connected Lie groups G behave in many ways like compact Lie groups. The role of the co-adjoint action of G on the dual of the Lie algebra $\mathfrak{g}^*$ is played by the *gauge action* of LG on the space $L\mathfrak{g}^* = \Omega^1(S^1, \mathfrak{g})$ of connections on the trivial G-bundle over S^1, where the pairing with $L\mathfrak{g} = \Omega^0(S^1, \mathfrak{g})$ is given by integration over S^1 (using an invariant inner product on the Lie algebra $\mathfrak{g}$). See e.g. [**61**]. The action of the based loop group $L_0G \subset LG$ on $L\mathfrak{g}^*$ is *free*, with quotient $L\mathfrak{g}^*/L_0G$ the group G and quotient map the holonomy of a connection. Given an infinite-dimensional Hamiltonian LG-manifold $\mathcal{M}$, with symplectic form σ and equivariant moment map $\Psi\colon \mathcal{M} \to L\mathfrak{g}^*$, the action of L_0G on $\mathcal{M}$ is again free (by equivariance), and we obtain a space $M = \mathcal{M}/L_0G$, with an action of $G = LG/L_0G$, and with an equivariant map $\Phi\colon M \to G$ induced by Ψ:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\Psi]{} & L\mathfrak{g}^* \\ \big\downarrow{\scriptstyle /L_0G} & & \big\downarrow{\scriptstyle /L_0G} \\ M & \xrightarrow[\Phi]{} & G \end{array}$$

Under suitable properness assumptions on Ψ, the space M is finite-dimensional and compact. The original idea of q-Hamiltonian geometry is to replace the infinite-dimensional space $\mathcal{M}$ with the finite-dimensional space M, regarding Φ as a moment map. An obstacle is that the symplectic form σ on $\mathcal{M}$ is not L_0G-basic, and hence it does not descend, in general. It was observed in [**3**] that there is nevertheless a 2-form ω on M, canonically determined by σ, and satisfying a G-valued moment map condition. The 2-form ω is neither closed nor non-degenerate, in general, but both properties are replaced by precise conditions involving the moment map.

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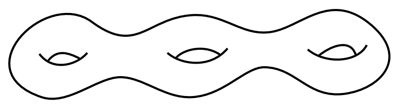
The axioms for q-Hamiltonian spaces [**3**] seemed a little strange at first, but led to useful theory with remarkably good properties. A few years later, Ping Xu [**73**] and Bursztyn-Crainic [**19**] provided a more conceptual interpretation of group-valued moment maps in terms of Dirac geometry. This led to major simplifications of the proofs (e.g. [**2, 56**]), and paved the way for applications involving non-compact groups or in complex-holomorphic settings (e.g. [**15, 20, 23**]).

These notes will give an overview of the theory of group-valued moment maps, avoiding loop groups. Particular emphasis will be on the subject of *quantization* of group-valued moment maps, and its application to Verlinde formulas for moduli space. The notes are based on lectures at the 'Summer School on Quantization' at Notre Dame University, May 31-June 4, 2011. Some additional material is included from a lecture series at the IGA workshop in Adelaide, September 2011. The audience for the summer school were postdoctoral and graduate students, with a variety of backgrounds. I made an effort to keep the lectures at a moderate pace, and to present motivation and foundational material, without going into technical details. These notes, while more detailed than the actual lectures, are written with a similar audience in mind.

2. Motivation: Moduli spaces of flat bundles

Suppose G is a compact, simply connected Lie group, and $\cdot$ an invariant inner product ('metric') on its Lie algebra $\mathfrak{g}$. Let Σ be a closed, connected, oriented 2-manifold of genus h

$\Sigma =$ 

Since G is assumed to be simply connected, any principal G-bundle over Σ is trivial. Let $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ be the infinite-dimensional affine space of connections on the trivial G-bundle over Σ. (We are treating infinite-dimensional manifolds in an informal manner; in any case we will soon pass to a finite-dimensional picture.) The group $\mathcal{G}(\Sigma) = \mathrm{Map}(\Sigma, G)$ acts on $\mathcal{A}(\Sigma)$ by *gauge transformations*,

$$g.A = \mathrm{Ad}_g(A) - g^*\theta^R.$$

(We denote by $\theta^R, \theta^L \in \Omega^1(G, \mathfrak{g})$ the right-invariant Maurer-Cartan form on G.) The *curvature*

$$\mathrm{curv}(A) = \mathrm{d}A + \tfrac{1}{2}[A, A] \in \Omega^2(\Sigma, \mathfrak{g})$$

transforms nicely under this action: $\mathrm{curv}(g.A) = \mathrm{Ad}_g \,\mathrm{curv}(A)$. In particular, the subset $\mathcal{A}_{\mathrm{flat}} = \{A \in \Omega^1(\Sigma, \mathfrak{g}) |\ \mathrm{curv}(A) = 0\}$ of *flat connections* is gauge invariant. Let

$$\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma)$$

be the *moduli space of flat connections* on the trivial G-bundle over Σ. As observed by Atiyah-Bott [**11, 10**], the space $\mathcal{M}(\Sigma)$ carries a natural symplectic structure, depending only on the choice of the metric $\cdot$ on $\mathfrak{g}$. Here the symplectic form is obtained by symplectic reduction, as follows. (We suggest the book [**21**] for background on symplectic reduction; this particular example is discussed on p. 158 of the book.) First, the affine space $\mathcal{A}(\Sigma)$ carries a symplectic form, given on tangent vectors $a, b \in T_A\Omega^1(\Sigma, \mathfrak{g}) = \Omega^1(\Sigma, \mathfrak{g})$ by

$$\omega_A(a, b) = \int_\Sigma a \cdot b.$$

The action of the gauge group $\mathcal{G}(\Sigma)$ preserves this 2-form, and is in fact *Hamiltonian*, with moment map the curvature $\mathrm{curv}\colon \mathcal{A}(\Sigma) \to \Omega^2(\Sigma, \mathfrak{g})$. That is,

$$\omega(\xi_{\mathcal{A}(\Sigma)}, \cdot) = -\mathrm{d} \int_\Sigma \mathrm{curv} \cdot \xi.$$

Here the integral on the right hand side is the function $A \mapsto \int_\Sigma \mathrm{curv}(A) \cdot \xi$, and 'd' is the exterior differential on the infinite-dimensional manifold $\mathcal{A}(\Sigma)$. The moduli space is hence recognized as a symplectic reduction

$$\mathcal{M}(\Sigma) = \mathcal{A}(\Sigma)/\!\!/\mathcal{G}(\Sigma) = \mathrm{curv}^{-1}(0)/\mathcal{G}(\Sigma).$$

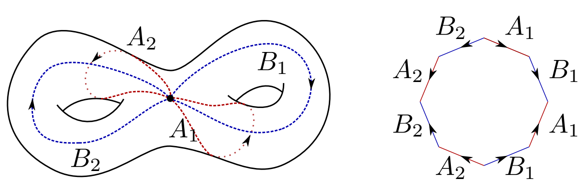
To see that $\mathcal{M}(\Sigma)$ is finite-dimensional, choose a base point x_0 on Σ, and let $\mathcal{G}(\Sigma, x_0) \subset \mathcal{G}(\Sigma)$ be the gauge transformations that are trivial at the base point. For any flat connection A on Σ, its holonomy along a based loop in Σ depends only on the homotopy class of that loop. It hence determines a group homomorphism $\kappa(A)\colon \pi_1(\Sigma; x_0) \to G$. Under the gauge action of $g \in \mathcal{G}(\Sigma)$, $\kappa(g.A) = \mathrm{Ad}_{g(x_0)}(\kappa(A))$. Conversely (using that G is simply connected), any homomorphism $\pi_1(\Sigma; x_0) \to G$ arises from a flat connection. Hence there is a canonical identification,

$$\mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma, x_0) \cong \mathrm{Hom}(\pi_1(\Sigma; x_0), G),$$

equivariant for the action of $\mathcal{G}(\Sigma)/\mathcal{G}(\Sigma, x_0) \cong G$. In particular,

$$\mathcal{M}(\Sigma) = \mathrm{Hom}(\pi_1(\Sigma; x_0), G)/G.$$

To be more explicit, we use a presentation of the fundamental group. This is done, as usual, by cutting the surface along A-cycles (winding around the handles) and B-cycles (going along the handles), as in the picture:



After cutting, the surface becomes a polygon with $4h$ sides, where h is the genus (number of handles) of the surface. Each handle gives rise to a word $A_iB_iA_i^{-1}B_i^{-1}$, and we obtain the relation $\prod_{i=1}^h A_iB_iA_i^{-1}B_i^{-1} = 1$ since the boundary of the polygon is contractible. Thus

$$\pi_1(\Sigma; x_0) = \big\langle A_1, B_1, \ldots, A_h, B_h \big| \prod_{i=1}^h A_iB_iA_i^{-1}B_i^{-1} \big\rangle$$

is a presentation of the fundamental group. Letting $a_i, b_i \in G$ be the holonomies of a connection along the paths A_i, B_i we obtain,

$$\mathrm{Hom}(\pi_1(\Sigma; x_0), G) = \Phi^{-1}(e)$$

where $\Phi\colon G^{2h} \to G$ is the map

$$\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1},$$

and finally

$$\mathcal{M}(\Sigma) = \Phi^{-1}(e)/G.$$

From this description, it is evident that $\mathcal{M}(\Sigma)$ is a compact space. If $h \geq 2$, then the subset $\Phi^{-1}(e)_{\mathrm{reg}}$ of points whose stabilizer in G equals the center $Z(G) \subset G$, is open and dense in $\Phi^{-1}(e)$. One may check that Φ has maximal rank at such points. It follows that $\Phi^{-1}(e)_{\mathrm{reg}}/G$ is a smooth symplectic manifold of dimension $(2h-2)\dim G$.

In the 1990s, the holonomy picture was used as a starting point for finite-dimensional constructions of the symplectic form on the moduli space, and an investigation of its cohomology. Important references include [**35, 37, 38, 42, 44, 45, 71**]. Jeffrey and Huebschmann [**38, 42**] developed an approach where the *logarithm* of the map Φ is viewed as a moment map, proving that $\mathcal{M}(\Sigma)$ can be written as a symplectic quotient of a finite-dimensional Hamiltonian G-space. Unfortunately, since the 'logarithm' is not globally defined, one cannot take this Hamiltonian space to be compact, and consequently many of the standard techniques of Hamiltonian geometry do not apply. One of the purposes of the theory of group-valued moment maps is to provide a more natural framework, in which the holonomy map $\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^{h} a_i b_i a_i^{-1} b_i^{-1}$ (rather than its logarithm) is directly viewed as a moment map.

3. Group-valued moment maps

Given a Lie group G, we denote by $\theta^L \in \Omega^1(G, \mathfrak{g})$ the *left-invariant Maurer-Cartan form* and by $\theta^R \in \Omega^1(G, \mathfrak{g})$ the *right-invariant Maurer-Cartan form.* In terms of a matrix representation of G, we have

$$\theta^L = g^{-1}\,\mathrm{d}g, \quad \theta^R = \mathrm{d}g g^{-1}.$$

Suppose $\mathfrak{g}$ carries an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form ('metric'), denoted by a dot '$\cdot$'. Thus $\mathrm{Ad}_g(\xi_1) \cdot \mathrm{Ad}_g(\xi_2) = \xi_1 \cdot \xi_2$ for all $\xi_1, \xi_2 \in \mathfrak{g}$. We denote by

$$\eta = \tfrac{1}{12}[\theta^L, \theta^L] \cdot \theta^L \in \Omega^3(G)$$

the *Cartan 3-form.* Since $\theta^R = \mathrm{Ad}_g\,\theta^L$ and since $\cdot$ is invariant, we may also write $\eta = \frac{1}{12}[\theta^R, \theta^R] \cdot \theta^R$. Thus η is a bi-invariant form on G, and hence it is closed: $\mathrm{d}\eta = 0$.

Definition 3.1 (Alekseev-Malkin-M [**3**]). A *q-Hamiltonian G-space* (M, ω, Φ) is a G-manifold M, with a G-invariant 2-form $\omega \in \Omega^2(M)$ and a G-equivariant map $\Phi \in C^\infty(M, G)$, called the *moment map*, satisfying

(i) $\iota(\xi_M)\omega = -\frac{1}{2}\Phi^*(\theta^L + \theta^R) \cdot \xi, \quad \xi \in \mathfrak{g}$
(ii) $\mathrm{d}\omega = -\Phi^*\eta$,
(iii) $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

Here the G-equivariance of Φ is relative to the conjugation action on G.

REMARK 3.2. In the original definition [**3**], an alternative version of condition (iii) was used, requiring

$$\text{(iii}'\text{)} \qquad \ker(\omega_m) = \{\xi_M(m) |\ \mathrm{Ad}_{\Phi(m)}\,\xi = -\xi\}.$$

However, assuming conditions (i), (ii) one may show that (iii') is equivalent to (iii). This was observed by Bursztyn-Crainic [**19**] and Xu [**73**], independently.

REMARK 3.3. In [**3**], the theory of group-valued moment maps was developed under the assumption that the metric $\cdot$ on $\mathfrak{g}$ is positive definite, which only happens if the adjoint group $G/Z(G)$ is compact. Using the more conceptual approach via Dirac geometry, initiated by [**19**], the main results all generalize to possibly non-compact groups (e.g. semi-simple Lie groups with $\cdot$ the Killing form on $\mathfrak{g}$), as well as to the holomorphic category. For details, see [**2, 56**].

Let us contrast the definition of q-Hamiltonian spaces with the usual definition of a Hamiltonian G-space. The latter is given by a G-manifold M with an invariant 2-form ω and an equivariant map $\Phi\colon M \to \mathfrak{g}^*$ satisfying the conditions,

(i) $\iota(\xi_M)\omega = -\langle \mathrm{d}\Phi, \xi\rangle$,
(ii) $\mathrm{d}\omega = 0$,
(iii) $\ker(\omega) = 0$.

REMARK 3.4. Assuming (i),(ii), the condition $\ker(\omega) = 0$ can be shown to be equivalent to the condition $\ker(\omega) \cap \ker(\mathrm{d}\Phi) = 0$.

We will now discuss the main examples and basic properties of q-Hamiltonian spaces parallel to their Hamiltonian counterparts.

3.1. Examples.

3.1.1. *Coadjoint orbits, conjugacy classes.* The first examples of Hamiltonian G-spaces are the orbits $\mathcal{O} \subset \mathfrak{g}^*$ of the co-adjoint action

$$g.\mu = (\mathrm{Ad}_{g^{-1}})^*\mu, \quad g \in G,\ \mu \in \mathfrak{g}^*.$$

(The choice of an invariant metric on $\mathfrak{g}$ identifies the coadjoint and adjoint actions; hence we will denote the coadjoint action also by $\mathrm{Ad}_g\,\mu := (\mathrm{Ad}_{g^{-1}})^*\mu$.) The moment map is the inclusion $\Phi\colon \mathcal{O} \hookrightarrow \mathfrak{g}^*$. The 2-form on the coadjoint orbit $\mathcal{O}$ is determined by the moment map condition, and is given at any point $\mu \in \mathcal{O}$ by the formula

$$\omega(\xi_{\mathcal{O}}, \xi'_{\mathcal{O}})_\mu = \langle \mu, [\xi, \xi']\rangle, \quad \xi, \xi' \in \mathcal{O}.$$

Similarly, the first examples of q-Hamiltonian G-spaces are the orbits of the conjugation action on G. The moment map for a conjugacy class is the inclusion $\Phi\colon \mathcal{C} \hookrightarrow G$, and the 2-form is uniquely determined by the moment map condition:

$$\omega(\xi_{\mathcal{C}}, \xi'_{\mathcal{C}})_a = \tfrac{1}{2}(\mathrm{Ad}_a - \mathrm{Ad}_{a^{-1}})\xi \cdot \xi'.$$

Since $\mathrm{d}\Phi$ is injective in this example, condition (iii) is automatic. Note that the 2-form ω may well-be degenerate: If elements of $\mathcal{C}$ square to central elements, the 2-form is even zero. Note also that conjugacy classes may be odd-dimensional (e.g. the conjugacy class $\mathcal{C} \cong S^1$ of $\mathrm{O}(2)$ consisting of reflections in the plane) or non-orientable (e.g. the conjugacy class $\mathcal{C} \cong \mathbb{R}P(2)$ of $\mathrm{SO}(3)$ consisting of rotations by π). On the other hand, one can show that connected q-Hamiltonian G-spaces for connected, simply connected groups G are always even-dimensional and oriented (see [**7, 4**]).

3.1.2. *Cotangent bundles, the double.* The cotangent bundle T^*G, with the cotangent lift of the $G \times G$-action on G, $(g_1, g_2).a = g_1 a g_2^{-1}$, is an example of a Hamiltonian $G \times G$-space. Using left trivialization $T^*G \cong G \times \mathfrak{g}^*$ of the cotangent bundle, the cotangent action reads $(g_1, g_2).(a, \mu) = (g_1 a g_2^{-1}, \mathrm{Ad}_{g_2} \mu)$. The two components of the moment map are $\Phi_1(a, \mu) = \mathrm{Ad}_a(\mu)$, $\Phi_2(a, \mu) = -\mu$.

Similarly, an example of a q-Hamiltonian $G \times G$-space is the *double* $D(G) \cong G \times G$, with action

$$(g_1, g_2).(a, b) = (g_1 a g_2^{-1},\ g_2 b g_1^{-1})$$

moment map components

$$\Phi_1(a, b) = ab, \quad \Phi_2(a, b) = a^{-1} b^{-1},$$

and 2-form

$$\omega = \tfrac{1}{2} a^*\theta^L \cdot b^*\theta^R + \tfrac{1}{2} a^*\theta^R \cdot b^*\theta^L$$

(here we view a, b as maps $D(G) \to G$). Replacing the variable b with $d = ba$ makes this look similar to the action on T^*G in left trivialization; for instance $\Phi_1 = \mathrm{Ad}_a(d)$, $\Phi_2 = d^{-1}$.

One can also consider T^*G with the cotangent lift of the conjugation action, with corresponding moment map $(a, \mu) \mapsto \mathrm{Ad}_a \mu - \mu$. The q-Hamiltonian analogue is the double $\mathbf{D}(G) = G \times G$ [1] with the action $g.(a, b) = (\mathrm{Ad}_g(a), \mathrm{Ad}_g(b))$, with moment map the Lie group commutator

$$\Phi(a, b) = aba^{-1}b^{-1},$$

and with the 2-form

$$\omega = \tfrac{1}{2} a^*\theta^L \cdot b^*\theta^R + \tfrac{1}{2} a^*\theta^R \cdot b^*\theta^L + \tfrac{1}{2}(ab)^*\theta^L \cdot (a^{-1}b^{-1})^*\theta^R.$$

This is a special case of the *fusion* operation to be discussed below.

3.1.3. *Linear spaces, spheres.* The space $\mathbb{C}^n = \mathbb{R}^{2n}$, with its standard symplectic form, is a Hamiltonian $\mathrm{U}(n)$-space. Similarly, the even-dimensional sphere S^{2n} is a q-Hamiltonian $\mathrm{U}(n)$-space, where the action is defined by the embedding $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n) \subset \mathrm{SO}(2n+1)$. This example was found independently in [**7**], [**40**] for $n = 2$, and generalized to higher dimensions by Hurtubise-Jeffrey-Sjamaar [**39**]. There is a similar pair of examples, due to Eshmatov [**27**], of a Hamiltonian $\mathrm{Sp}(n)$-action on the quaternionic space $\mathbb{H}^n$, and a q-Hamiltonian $\mathrm{Sp}(n)$-action on quaternionic projective space $\mathbb{H}\,\mathrm{P}(n)$.

3.1.4. *Moduli spaces of surfaces with boundary.* Assume G simply connected. Let Σ be a compact, connected surface with a single boundary component. Fix a base point $x_0 \in \partial(\Sigma)$ on the boundary, and let

$$\mathcal{M}(\Sigma) = \mathcal{A}_{\mathrm{flat}}(\Sigma)/\mathcal{G}(\Sigma; x_0)$$

be the moduli space of flat connections on Σ, up to gauge transformations that are trivial at x_0. The space $\mathcal{M}(\Sigma)$ carries a residual action of $\mathcal{G}(\Sigma)/\mathcal{G}(\Sigma; x_0) \cong G$, and the map taking the holonomy around $\partial\Sigma$ descends to a G-equivariant map $\Phi\colon \mathcal{M}(\Sigma) \to G$. A generalization of the Atiyah-Bott gauge theory construction discussed above gives 2-form ω on $\mathcal{M}(\Sigma)$, making $(\mathcal{M}(\Sigma), \omega, \Phi)$ into a q-Hamiltonian G-space. More generally, if Σ has r boundary components, fix one base point on each boundary component. Then the moduli space $\mathcal{M}(\Sigma)$ of flat connections modulo based gauge transformations is a q-Hamiltonian G^r-space. It turns out that the

[1] We use the bold face notation to indicate that we consider the double as a G-space, rather than as a $G \times G$-space.

space associated to a cylinder (2-punctured sphere) is isomorphic to $D(G)$, while the space associated to a 1-punctured torus is isomorphic to $\mathbb{D}(G)$.

3.2. Basic constructions: products. Given two Hamiltonian G-spaces, their direct product, with the diagonal G-action and with the sum of moment maps and 2-forms, is again a Hamiltonian G-space. For q-Hamiltonian spaces, the product operation uses the product of the moment maps, but it is necessary to modify the sum of the 2-forms.

PROPOSITION 3.5. [**3**] *Suppose* (M_i, ω_i, Φ_i), $i = 1, 2$ *are two q-Hamiltonian* G*-spaces. Then their* fusion product

$$(M_1 \times M_2, \omega_1 + \omega_2 + \tfrac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R, \Phi_1\Phi_2),$$

is again a q-Hamiltonian G*-space.*

Here the modification of the 2-form is required due to the following property of the 3-form η under group multiplication $\mathrm{Mult}\colon G \times G \to G$,

$$\mathrm{Mult}^*\eta = \mathrm{pr}_1^*\,\eta + \mathrm{pr}_2^*\,\eta - \tfrac{1}{2}\mathrm{d}\,\mathrm{pr}_1^*\,\theta^L \cdot \mathrm{pr}_2^*\,\theta^R,$$

where $\mathrm{pr}_1, \mathrm{pr}_2\colon G \times g \to G$ are the two projections. More generally, if $(M, \omega, (\Phi_1, \Phi_2))$ is a q-Hamiltonian $G \times G$-space, then we obtain a q-Hamiltonian G-space

$$(M_{fus}, \omega_{fus}, \Phi_{fus}),$$

where M_{fus} is M with the diagonal G-action, $\Phi_{fus} = \Phi_1\Phi_2$ and $\omega_{fus} = \omega + \frac{1}{2}\Phi_1^*\theta^L \cdot \Phi_2^*\theta^R$.

REMARK 3.6. The fusion property finds a natural proof within the framework of *Dirac structures* [**2, 19, 56**] . Here, the axioms of a q-Hamiltonian are absorbed into a morphism of Manin pairs (strong Dirac morphism) from M into G, equipped with the so-called *Cartan-Dirac structure.* Since group multiplication in G is again a morphism of Manin pairs, the fusion operation becomes simply a composition of morphisms.

As an application, the space G^{2h}, with G acting diagonally by conjugation and with moment map $\Phi(a_1, b_1, \ldots, a_h, b_h) = \prod_{i=1}^h a_i b_i a_i^{-1} b_i^{-1}$ carries the structure of a q-Hamiltonian G-space as an h-fold fusion of the double $\mathbf{D}(G)$. The following nice way of looking at the 2-form was described by Pavol Ševera in [**64**]. For any manifold X, the space $C^\infty(X, G) \times \Omega^2(X)$ has a group structure

$$(q_1, \omega_1)(q_1, \omega_2) = (q_1 q_2, \omega_1 + \omega_2 + \tfrac{1}{2}q_1^*\theta^L \cdot q_2^*\theta^R).$$

Take $X = G^{2h}$, with elements $x = (a_1, b_1, \ldots, a_h, b_h)$, and let $q_1, \ldots, q_{4h}\colon G^{2g} \to G$ be the maps

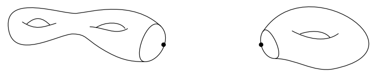
$$x \mapsto \; a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, a_2^{-1}, b_2^{-1}, \ldots, a_h^{-1}, b_h^{-1}.$$

Then $(q_1, 0) \cdots (q_{4h}, 0) = (\Phi, \omega)$ defines the q-Hamiltonian 2-form $\omega \in \Omega^2(G^{2h})$ and the moment map Φ.

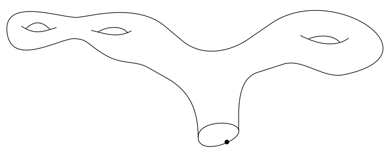
The name '*fusion*' corresponds to the fusion of surfaces, as in the following example. See [**59**] for a similar discussion for Hamiltonian loop group actions.

EXAMPLE 3.7 (Fusion of moduli spaces). Suppose G is simply connected. For any compact, oriented surface Σ with boundary component, with a marked point on each boundary component, we denote by $\mathcal{M}(\Sigma)$ the moduli space of flat connections on Σ, up to gauge transformations that are trivial at the marked points. (See

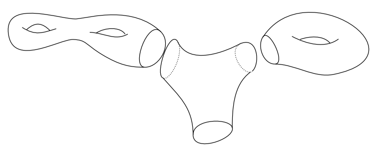
Section 3.1.4.) Suppose Σ has (at least) two boundary components. For instance, Σ could be a disjoint union of two surfaces Σ_1 and Σ_2 with one boundary component, as in the following picture.



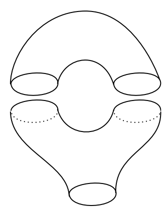
Then the fusion $\mathcal{M}(\Sigma)_{fus}$ is naturally identified with the moduli space $\mathcal{M}(\Sigma_{fus})$ of the surface Σ_{fus},



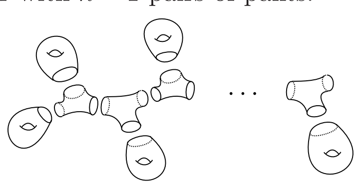
which is obtained by joining the two boundary components of Σ by a pair of pants: the two pant legs are attached to two boundaries.



For example, the moduli space of flat connections on the cylinder can be identified with the double $D(G) \cong G \times G$, a q-Hamiltonian $G \times G$ space. Fusing $D(G)$ with itself, we obtain the moduli space $\mathbf{D}(G)$ of flat connections on the punctured torus.



One can construct the punctured surface of genus h by joining h copies of the punctured torus together with $h-1$ pairs of pants.



Thus, the moduli space of flat $\mathfrak{g}$-connections on the punctured surface of genus h is

$$\mathcal{M}(\Sigma_h) := \underbrace{\mathbf{D}(G) \times \cdots \times \mathbf{D}(G)}_{h} \cong G^{2h}.$$

3.3. Reduction. Symplectic reduction of q-Hamiltonian G-spaces works just the same as for ordinary Hamiltonian spaces. Suppose (M, ω, Φ) is a q-Hamiltonian G-space, such that the group unit e is a regular value of the moment map. Then it is automatic that the G-action on $\Phi^{-1}(e)$ has discrete stabilizers. If G is compact (or more generally if the action is proper), it follows that the quotient

$$M/\!\!/G = \Phi^{-1}(e)/G$$

is an orbifold. Furthermore, the pull-back of ω to $\Phi^{-1}(e)$ is *basic*, and the resulting 2-form on $M/\!\!/G$ is *symplectic* – even though ω itself was neither closed nor non-degenerate. This is possible because ω_m is non-degenerate for all $m \in \Phi^{-1}(e)$, and since its pull-back to $\Phi^{-1}(e)$ is closed. If e is a singular value of Φ the space $M/\!\!/G$ is a singular symplectic space in the sense of Sjamaar-Lerman [**66**].

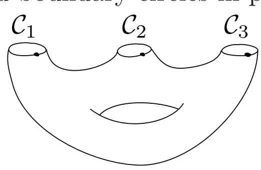
As an application of reduction, we obtain a symplectic structure on the moduli space of flat G-bundles, viewed as a symplectic quotient,

$$\mathcal{M}(\Sigma_h) = G^{2h}/\!\!/G.$$

Note that e is never a regular value of $\Phi\colon G^{2h} \to G$, since $\Phi^{-1}(e)$ contains the point $(e,\ldots,e)$ whose stabilizer is the entire group G. More generally, if $\mathcal{C}_1,\ldots,\mathcal{C}_r \subset G$ are conjugacy classes,

$$\mathcal{M}(\Sigma_h^r, \mathcal{C}_1,\ldots,\mathcal{C}_r) = (G^{2h} \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r)/\!\!/G \tag{3.1}$$

is the moduli space of flat G bundles over a surface with r boundary components, with holonomies around boundary circles in prescribed conjugacy classes $\mathcal{C}_j$.



One of the main results in [**3**] asserts that, for G compact and simply connected, the symplectic structure obtained by q-Hamiltonian reduction coincides with that coming from the Atiyah-Bott construction.

3.4. Convexity theorem. We next describe some convexity results for q-Hamiltonian spaces. Here we assume that the group G is compact and simply connected.

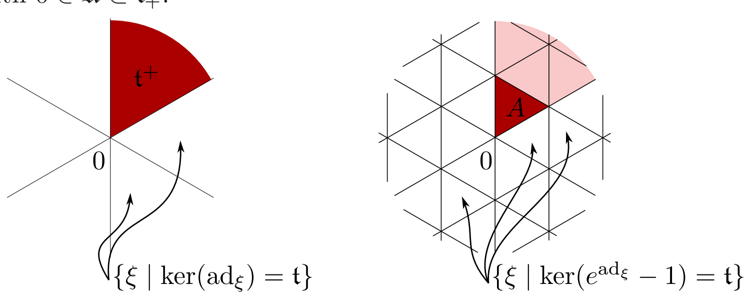
3.4.1. *Weyl chamber, alcove.* We fix a maximal torus T in G, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. An *open Weyl chamber* in $\mathfrak{t}$ is a connected component of the set

$$\{\xi \in \mathfrak{t}|\ \ker(\mathrm{ad}_\xi) = \mathfrak{t}\} = \{\xi \in \mathfrak{t}|\ G_\xi = T\},$$

while an *open alcove* is a connected component of the set

$$\{\xi \in \mathfrak{t}|\ \ker(e^{\mathrm{ad}_\xi} - 1) = \mathfrak{t}\} = \{\xi \in \mathfrak{t}|\ G_{\exp\xi} = T\},$$

here G_ζ resp. G_g are the stabilizers of $\zeta \in \mathfrak{g}$ resp. $g \in G$ under the adjoint action. Pick an open Weyl chamber, let $\mathfrak{t}_+$ be its closure, and let $\mathfrak{A}$ be the unique closed alcove with $0 \in \mathfrak{A} \subset \mathfrak{t}_+$.



Let $\mathfrak{t}_+^*$ be the image of $\mathfrak{t}_+$ under the isomorphism $\mathfrak{t} \cong \mathfrak{t}^*$ defined by the invariant metric on $\mathfrak{g}$. (This does not depend on the choice of metric.)

The fundamental Weyl chamber labels the set of coadjoint orbits $\mathcal{O} \subset \mathfrak{g}^*$, in the sense that every such orbit is of the form $G.\mu$ for a unique element $\mu \in \mathfrak{t}_+^*$. (See [**17**, Ch. IX, §2, Proposition 7].) Similarly, the fundamental Weyl alcove labels the conjugacy classes $\mathcal{C} \subset G$, in the sense that every conjugacy class is of the form $G. \exp \xi$ for a unique $\xi \in \mathfrak{A}$. (See [**17**, Ch. IX, §5, Corollary 2].)

3.4.2. *Convexity theorem.* The following result is known as the Hamiltonian *convexity theorem.*

THEOREM 3.8 (Atiyah [**9**], Guillemin-Sternberg [**32, 34**], Kirwan [**49**]). *Let (M, ω, Φ) be a compact connected Hamiltonian G-space. Then*

(1) *the fibers of Φ are connected,*
(2) *the set*
$$\Delta(M) = \{\mu \in \mathfrak{t}_+^* |\ \mu \in \Phi(M)\}$$
is a convex polytope.

Similarly, the q-Hamiltonian convexity theorem states:

THEOREM 3.9 (M-Woodward [**59**]). *Let (M, ω, Φ) be a compact connected q-Hamiltonian G-space. Then*

(1) *the fibers of Φ are connected,*
(2) *the set*
$$\Delta(M) = \{\xi \in \mathfrak{A} |\ \exp \xi \in \Phi(M)\}$$
is a convex polytope.

The result was phrased in [**59**] in terms of Hamiltonian loop group actions; the formulation in terms of q-Hamiltonian spaces follows using the equivalence theorem in [**3**].

3.4.3. *Eigenvalue problems.* The Hamiltonian convexity theorem has nice applications to eigenvalue problems. Let $\mathbb{R}_+^n$ be the set of $\lambda \in \mathbb{R}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. For a complex Hermitian $n \times n$ matrix A, let $\lambda(A) \in \mathbb{R}_+^n$ be its ordered tuple of eigenvalues. One then has:

COROLLARY 3.10 (Horn polytope). *Let $\mu, \mu' \in \mathbb{R}_+^n$ be given. Then the set $\gamma \in \mathbb{R}_+^n$ for which there exist Hermitian matrices A, A' with*
$$\lambda(A) = \mu,\ \lambda(A') = \mu', \quad \lambda(A + A') = \gamma,$$
is a convex polytope.

In short, the possible eigenvalues of a sum of Hermitian matrices with prescribed eigenvalues form a convex polytope. Corollary 3.10 follows by identifying Hermitian matrices with $\mathfrak{u}(n)^*$, the cone $\mathbb{R}_+^n$ with the positive Weyl chamber $\mathfrak{t}_+^*$, and matrices with prescribed eigenvalues with coadjoint orbits $\mathcal{O} \subset \mathfrak{u}(n)^*$. The Corollary is then an immediate consequence of the Hamiltonian convexity theorem applied to a product of coadjoint orbits $\mathcal{O} \times \mathcal{O}'$. A description of the faces of this polytope in terms of explicit eigenvalue inequalities was known as the *Horn conjecture*, this was solved by Klyachko [**50**] in 1994. For more general compact groups, the inequalities for the moment polytopes of products of coadjoint orbits in general were determined by Berenstein-Sjamaar [**13**].

The q-Hamiltonian convexity theorem has applications to *multiplicative* eigenvalue problems. The eigenvalues of any special unitary matrix $A \in \mathrm{SU}(n)$ may be written in the form $e^{2\pi\sqrt{-1}\lambda_1(A)}, \ldots, e^{2\pi\sqrt{-1}\lambda_n(A)}$, for a unique $\lambda(A) \in \mathbb{R}^n$ with

$$\sum_{i=1}^{n} \lambda_i(A) = 0, \quad \lambda_1(A) \geq \cdots \geq \lambda_n(A) \geq \lambda_1(A) - 1.$$

COROLLARY 3.11 (M-Woodward). *Given $\mu, \mu' \in \mathbb{R}^n$, the set*

$$\{\gamma \in \mathbb{R}^n |\ \exists A, A' \in \mathrm{SU}(n)\colon \lambda(A) = \mu,\ \lambda(A') = \mu', \quad \lambda(AA') = \gamma\},$$

is a convex polytope.

In short, the possible eigenvalues of a product of special unitary matrices with prescribed eigenvalues forms a convex polytope. The corollary is obtained by applying Theorem 3.9 to a fusion product of two conjugacy classes, $\mathcal{C} \times \mathcal{C}'$. The problem of determining the faces of this polytope was solved by Agnihotri-Woodward [**1**]. The moment polytope for products of conjugacy classes in a general compact simply connected Lie group was determined by Teleman-Woodward [**67**].

3.4.4. *Connectivity of the fibers.* Let us also note the following consequences of the first part of Theorem 3.9, concerning connectivity of the fibers of the moment map.

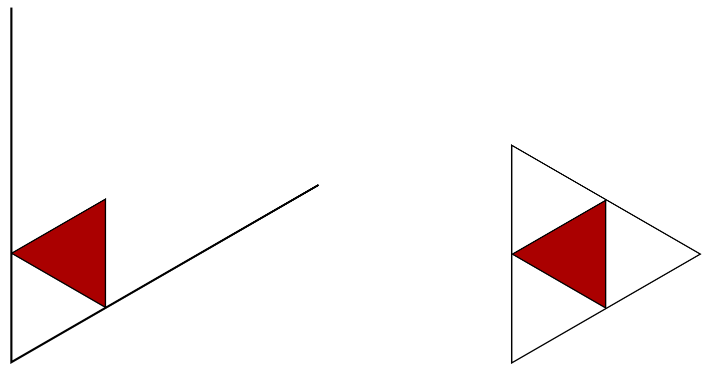
COROLLARY 3.12. *Let G be a compact, simply connected Lie group, and (M, ω, Φ) a compact connected q-Hamiltonian G-space. Then the symplectic quotient $M/\!/G$ is connected.*

In particular, the moduli spaces (3.1) are connected.

COROLLARY 3.13. *For any compact, simply connected Lie group G, the fibers of the commutator map $G \times G \to G$, $(a, b) \mapsto aba^{-1}b^{-1}$ are connected.*

This follows by applying Theorem 3.9 to the double $\mathbf{D}(G)$. Note that this result is not easy to prove 'by hand'.

3.4.5. *Multiplicity-free spaces.* An interesting class of Hamiltonian G-spaces are the *multiplicity-free spaces*. These are spaces such that the map $M/G \to \Delta(M)$ is a homeomorphism; equivalently, the symplectic quotients are 0-dimensional. In case G is a torus, Delzant [**22**] proved that multiplicity-free spaces are determined by their moment polytopes. This result was extended by Woodward [**72**] to 'reflective' multiplicity-free spaces for non-abelian groups. The classification of multiplicity-free spaces in general is more involved, and was completed only recently by F. Knop [**51**] following Losev's proof of the 'Knop conjecture'. The definition of multiplicity-free spaces carries over verbatim to the q-Hamiltonian setting. For instance, the q-Hamiltonian $\mathrm{SU}(n)$-space S^{2n} and the q-Hamiltonian $\mathrm{Sp}(n)$-space $\mathbb{H}\mathrm{P}(n)$ are multiplicity free. The following picture shows the moment polytopes for a reflective multiplicity free Hamiltonian SU(3)-space (left) and a reflective multiplicity free q-Hamiltonian SU(3)-space (right). These examples are due to Chris Woodward.



3.5. Volume forms. The *Liouville form* of a symplectic manifold (M, ω) is the volume form defined as $\Gamma = \frac{1}{n!}\omega^n$, or equivalently as the top degree part of the exponential of ω,

$$\Gamma = (\exp \omega)_{[top]}.$$

In local Darboux coordinates $q_1, p_1, \ldots, q_n, p_n$, one has $\omega = \sum_i \mathrm{d}q_i \wedge \mathrm{d}p_i$, and the Liouville form is $\mathrm{d}q_1 \wedge \mathrm{d}p_1 \cdots \wedge \mathrm{d}q_n \wedge \mathrm{d}p_n$. Given a compact Hamiltonian G-space (M, ω, Φ), one defines the *Duistermaat-Heckman measure* [**26**] to be the push-forward on $\mathfrak{g}^*$ of the associated measure, $\mathfrak{m} = \Phi_*|\Gamma|$. It has interesting properties, and may be calculated using localization techniques.

For a q-Hamiltonian G-space (M, ω, Φ), we saw that the 2-form ω may be degenerate or even zero. Assuming that G is compact and simply connected, it turns out that there is nevertheless a distinguished volume form on M. In particular, M carries a canonical orientation. The construction involves a certain differential form on G.

Proposition 3.14. *For any compact, simply connected Lie group G, the function $g \mapsto \det(\frac{\mathrm{Ad}_g + 1}{2})$ admits a smooth global square root, equal to 1 at $g = e$. Furthermore, there is a well-defined smooth differential form $\psi \in \Omega(G)$, given on the set where* $\det(\mathrm{Ad}_g + 1) \neq 0$ *by*

$$\psi = \det{}^{1/2}(\tfrac{1 + \mathrm{Ad}_g}{2})\ \exp(\tfrac{1}{4}\tfrac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\theta^L \cdot \theta^L).$$

Note that the set where $\det(\mathrm{Ad}_g + 1) \neq 0$ is open and dense in G. Note that the 2-form inside the exponential becomes singular on the subset where $\det(\mathrm{Ad}_g + 1) = 0$, but the scalar factor in front of the exponential has zeroes there. The Proposition says that the zeroes compensate the singularities, so that the form extends smoothly across the set $\det(\mathrm{Ad}_g + 1) = 0$.

Theorem 3.15. [**7**] *Suppose G is compact and simply connected. For any q-Hamiltonian G-space (M, ω, Φ), the top degree part of the form* $\exp(\omega)\Phi^*\psi$ *is a G-invariant volume form,*

$$\Gamma = (e^{\omega}\Phi^*\psi)_{[top]}.$$

In particular, M is even-dimensional and carries a canonical orientation. A conceptual explanation of the volume form is given in [**2, 56**], where the differential form ψ is identified as a *pure spinor*, and the Theorem is interpreted as the non-degeneracy of a pairing between two pure spinors. As shown in [**7**], the push-forward $\mathfrak{m} = \Phi_*|\Gamma| \in \mathcal{E}'(G)$ plays the role of a *Duistermaat-Heckman measure*, with properties similar to the Hamiltonian Duistermaat-Heckman measure. In particular, it encodes volumes of symplectic quotients, and for G compact and simply connected it can be computed by localization [**5**].

3.6. Kirwan surjectivity. There are many other aspects of the Hamiltonian theory that carry over the q-Hamiltonian setting, with suitable modifications. One result of central importance for Hamiltonian spaces is the *Kirwan surjectivity theorem*. We assume that G is compact. For any G-manifold M, let $H_G^\bullet(M)$ be its equivariant cohomology ring with coefficients in $\mathbb{R}$. It may be realized as the cohomology of the *Cartan complex* $(\Omega_G^\bullet(M), \mathrm{d}_G)$ where

$$\Omega_G^k(M) = \bigoplus_{2i+j=k} (S^i\mathfrak{g}^* \otimes \Omega^j(M))^G.$$

Viewing elements of $\Omega_G(M)$ as G-equivariant polynomial maps $\beta\colon \mathfrak{g} \to \Omega(M)$, the differential is given by

$$(\mathrm{d}_G\beta)(\xi) = \mathrm{d}\beta(\xi) - \iota(\xi_M)\beta(\xi), \quad \xi \in \mathfrak{g}.$$

EXAMPLE 3.16. (1) If (M, ω, Φ) is a Hamiltonian G-space, then $\omega^G = \omega + \Phi \in \Omega_G^2(M)$ is an example of a closed equivariant 2-form.

(2) If G carries an invariant metric $\cdot$, then

$$\eta^G(\xi) = \eta + \tfrac{1}{2}(\theta^L + \theta^R) \cdot \xi$$

defines a closed equivariant 3-form $\eta^G \in \Omega_G^3(M)$. Conditions (i),(ii) in the definition of a q-Hamiltonian G-space may be combined into a single condition $\mathrm{d}_G\omega = -\Phi^*\eta^G$.

THEOREM 3.17 (Kirwan [**48**]). *Let (M, ω, Φ) be a Hamiltonian G-space, with 0 a regular value of the moment map Φ. Then the pull-back map*

$$H_G(M) \to H_G(\Phi^{-1}(0)) \cong H(M/\!\!/G)$$

is a surjective ring homomorphism.

Thus, all cohomology classes on the symplectic quotient are obtained from equivariant cohomology classes on the unreduced space. For instance, the class $[\omega_G]$ descends to the class of the symplectic form on $M/\!\!/G$. This result is the starting point for the calculation of intersection pairings on $M/\!\!/G$ using localization on M, see e.g. [**43**], [**68**].

For a q-Hamiltonian G-space, the map $H_G(M) \to H_G(\Phi^{-1}(e)) = H(M/\!\!/G)$ need not be surjective, in general. There are in fact examples where $H_G^2(M) = 0$, so that the class of the symplectic form on $M/\!\!/G$ need not lie in the image of this map. It turns out that the correct version of the surjectivity theorem involves the topology of the group G. We assume that G is compact and simply connected. As is well-known, the inclusion of bi-invariant differential forms $(\wedge\mathfrak{g}^*)^G \cong \Omega(G)^{G\times G} \hookrightarrow \Omega(G)$ induces an isomorphism in cohomology. Since the de Rham differential restricts to zero on bi-invariant differential forms, it follows that

$$H(G) = (\wedge\mathfrak{g}^*)^G.$$

On the other hand, it is known that the invariants $(\wedge\mathfrak{g}^*)^G$ are an exterior algebra over a graded subspace $P^\bullet \subset (\wedge^\bullet\mathfrak{g}^*)^G$ of *primitive elements*.

$$(\wedge\mathfrak{g}^*)^G = \wedge P.$$

Here $\dim P = l$ equals the rank of G, and all homogeneous elements in P are of odd degree. Let $\eta_1, \ldots, \eta_l \in \Omega^{2d_i - 1}(G)$ be a homogeneous basis of P, where η_1 is the Cartan 3-form. For instance, if $G = \mathrm{SU}(n+1)$, the generators of $(\wedge \mathfrak{g}^*)^G$ are of degree $3, 5, 7, \ldots, 2n+1$. It turns out that each of the η_i admits an extension $\eta_i^G \in \Omega_G^{2d_i-1}(G)$ to an equivariantly closed form. These may be constructed using an equivariant version of the Bott-Shulman complex [**41**] (see also [**55**]). In particular, $\eta_1^G = \eta^G$.

Suppose now that (M, ω, Φ) is a q-Hamiltonian G-space. Define a new complex,

$$\widetilde{\Omega}_G(M) = \Omega_G(M)[u_1, \ldots, u_l],$$

where $[u_1, \ldots, u_l]$ denotes the graded ring of polynomials in given variables u_i of degree $2d_i - 2$, and with the differential

$$\widetilde{\mathrm{d}}_G = \mathrm{d}_G + \sum_{i=1}^{l} \Phi^* \eta_i^G \tfrac{\partial}{\partial u_i}.$$

(Here $\Phi^*\eta_i^G$ acts by exterior multiplication, raising the degree by $2d_i - 1$, while the differentiation $\frac{\partial}{\partial u_i}$ lowers the degree by $2d_i - 2$. We hence see that $\widetilde{\mathrm{d}}_G$ raises the degree by 1, as required.) The cohomology of this complex is denoted $\widetilde{H}_G^\bullet(M)$. Let

$$\widetilde{\Omega}_G^\bullet(M) \to \widetilde{\Omega}_G^\bullet(\Phi^{-1}(e)) \to \Omega_G^\bullet(\Phi^{-1}(e)) \tag{3.2}$$

be the cochain map, given by pull-back followed by the augmentation map for $[u_1, \ldots, u_l]$ (setting these variables equal to zero). For instance, the element

$$\omega + u_1$$

is a cocycle (since $\mathrm{d}_G\omega = -\Phi^*\eta^G$), and its image under the map (3.2) is simply the pull-back of ω to $\Phi^{-1}(e)$ (a closed, basic form).

Theorem 3.18 (Kirwan surjectivity for q-Hamiltonian G-spaces). *Suppose (M, ω, Φ) is a q-Hamiltonian G-space, where G is compact and simply connected, and suppose e is a regular value of Φ. Then the map*

$$\widetilde{H}_G^\bullet(M) \to H_G^\bullet(\Phi^{-1}(e)) = H^\bullet(M/\!/G)$$

is a surjective ring homomorphism.

The surjectivity result was originally proved by Bott, Tolman and Weitsman [**16**] in terms of Hamiltonian loop group actions. In unpublished work with A. Alekseev, we obtained the reformulation above, using a 'small model' for the equivariant cohomology of the loop group space. As an application, one obtains generators for the cohomology rings of moduli spaces, see e.g. [**55**].

4. Quantization of Hamiltonian G-spaces

Our aim in these lectures is to explain the quantization of q-Hamiltonian G-spaces. In this Section, we set the stage by reviewing aspects of the quantization of ordinary Hamiltonian G-spaces. The term 'quantization' will be used in a wide sense. Ideally, the quantization of a symplectic manifold should be Hilbert space, and a Hamiltonian G-action (thought of as classical symmetries) should be quantized to define a representation of G by unitary operators on the Hilbert space (thought of as quantum symmetries). The method of geometric quantization produces such G-representations, but requires further data and additional assumptions.

Rather than dealing with concrete Hilbert spaces, we will be content with *isomorphism classes* of G-representations. That is, we will take the quantization of a Hamiltonian G-space to be a certain element of the representation ring of G.

4.1. Background in representation theory. In this Section, we take G to be compact and connected. For any G-representation $\pi\colon G \to \mathrm{Aut}(V)$, let $\chi_V \in C^\infty(G)$ be its character, $\chi_V(g) = \mathrm{tr}(\pi(g))$. Characters have the properties

$$\chi_{V\oplus W} = \chi_V + \chi_W, \quad \chi_{V\otimes W} = \chi_V\chi_W, \quad \chi_{V^*} = \chi_V^*,$$

hence they form a subring $R(G) \subset C^\infty(G)$ of the ring of complex-valued functions, invariant under the involution $*$. As an additive group, $R(G)$ is the $\mathbb{Z}$-module spanned by the characters of irreducible representations, also called the *irreducible characters*.

Fix a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$, and let $P \subset \mathfrak{t}^*$ be the *(real) weight lattice*. Thus $\mu \in \mathfrak{t}^*$ lies in P if and only if the Lie algebra homomorphism

$$\mathfrak{t} \to \mathfrak{u}(1), \ \xi \mapsto 2\pi\sqrt{-1}\langle\mu, \xi\rangle$$

exponentiates to a group homomorphism $e_\mu\colon T \to \mathrm{U}(1)$. For any G-representation $\pi\colon G \to \mathrm{Aut}(V)$, we define the *weight spaces* $V_\mu = \{v \in V|\ \forall t \in T\colon \pi(t)v = e_\mu(t)v\}, \ \ \mu \in P$, and the set of weights

$$P(V) = \{\mu \in P|\ V_\mu \neq 0\}.$$

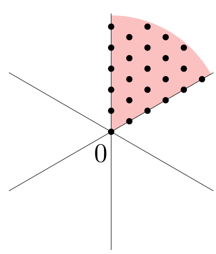
Let $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ be a choice of fundamental Weyl chamber. It is known that if V is irreducible, then there is a unique weight $\mu \in P(V)$ such that $\mu + \epsilon$ has maximal length, for any $\epsilon \in \mathrm{int}(\mathfrak{t}^*_+)$. This element $\mu \in P(V) \cap \mathfrak{t}^*_+$ is called the *highest weight* of V. By *H. Weyl's theorem*, this sets up a 1-1 correspondence between the set of irreducible representations and the set

$$P_+ = P \cap \mathfrak{t}^*_+$$

of *dominant weights* of G. Thus, as a $\mathbb{Z}$-module we have

$$R(G) = \mathbb{Z}[P_+],$$

with basis the irreducible characters χ_μ indexed by dominant weights $\mu \in P_+$. In the figure below, the shaded area is the fundamental Weyl chamber for the group SU(3), while the dominant weights are indicated as dots.



4.2. Quantization of Hamiltonian G-spaces. Suppose now that (M, ω, Φ) is a Hamiltonian G-space, with moment map $\Phi\colon M \to \mathfrak{g}^*$.

DEFINITION 4.1. A pre-quantum line bundle $L \to M$ is a G-equivariant Hermitian line bundle with connection ∇, such that

(1) $\mathrm{curv}(\nabla) = \omega$,

(2) The $\mathfrak{g}$-action on L is given by Kostant's formula

$$\xi_L = \mathrm{Lift}_\nabla(\xi_M) + \langle \Phi, \xi \rangle \partial_\theta$$

where $\partial_\theta \in \mathfrak{X}(L)$ generates the S^1-action on L.

Remarks 4.2. (1) The existence of a pre-quantum line bundle is equivalent to the integrality of the 2-form ω.

(2) If G is simply connected, the existence of the pre-quantum lift of the G-action from M to L is automatic. Indeed, the formula for ξ_L defines a Lie algebra action of $\mathfrak{g}$ on L by infinitesimal Hermitian automorphisms, and this Lie algebra action integrates to a Lie group action.

(3) If a G-equivariant pre-quantum line bundle exists, then the choice of L is unique up to a *flat* G-equivariant line bundle.

Given an equivariant pre-quantization, we obtain an element $\mathcal{Q}(M)$ of the representation ring, as follows. Let $J\colon TM \to TM$ be a G-invariant compatible almost complex structure, i.e. $g(v,w) = \omega(Jv,w)$ is a Riemannian metric. (In other words, every tangent space admits an isomorphism $T_mM \to \mathbb{C}^n = \mathbb{R}^{2n}$ taking ω_m to the standard symplectic structure $\sum_{i=1}^n e_{2i-1} \wedge e_{2i}$ and J_m to the standard complex structure $e_{2i-1} \mapsto e_{2i},\ \ e_{2i} \mapsto -e_{2i-1}$.) The space of G-invariant compatible almost complex structures is well-known to be contractible; hence the particular choice of J is unimportant for what follows. Let $TM^{\mathbb{C}} = T^{1,0}M \oplus T^{0,1}M$ be the decomposition into $\pm i$ eigenbundles of J. Then $\wedge T^{0,1}M$ is a spinor module over the Clifford bundle $\mathbb{C}\mathrm{l}(TM)$, where the Clifford action of $T^{0,1}M$ is by exterior multiplication and that of $T^{1,0}M$ is by contraction. (See Section 7 below.) Tensoring with L one obtains a new spinor module,

$$\mathsf{S} = \wedge T^{0,1}M \otimes L$$

Let $\partial\!\!\!/\colon \Gamma(\mathsf{S}) \to \Gamma(\mathsf{S})$ be the associated Dirac operator, given by the covariant derivative $\nabla\colon \Gamma(\mathsf{S}) \to \Gamma(T^*M \otimes \mathsf{S})$ followed by the Clifford action of $T^*M \cong TM \subset \mathbb{C}\mathrm{l}(TM)$ on S. Then $\partial\!\!\!/$ is a G-equivariant elliptic operator, and hence it has a G-equivariant index. Let $\mathsf{S}^+,\ \mathsf{S}^-$ be the even, odd part of the spinor bundle.

Definition 4.3. The *quantization* $\mathcal{Q}(M) \in R(G)$ of the pre-quantized Hamiltonian G-space (M,ω,Φ) is the G-index

$$\mathcal{Q}(M) = \mathrm{index}_G(\partial\!\!\!/) = \chi_{\ker(\partial\!\!\!/|_{\mathsf{S}^+})} - \chi_{\ker(\partial\!\!\!/|_{\mathsf{S}^-})}.$$

For any given L, the construction of $\partial\!\!\!/$ involves a few choices such as the choice of J and of connections; however, the stability property of indices guarantees that $\mathcal{Q}(M)$ is independent of those choice. (In fact, it turns out that for G connected, even the choice of L does not affect $\mathcal{Q}(M)$. This is immediate from the equivariant index formula of Berline and Vergne [**14**], cf. [**54**].) The basic properties of the quantization are as follows:

(1) $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
(2) $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
(3) $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
(4) *Borel-Weil-Bott* (weak version): $G.\mu,\ \mu \in \mathfrak{t}^*_+$ is pre-quantized if and only if $\mu \in P_+$. In this case,

$$\mathcal{Q}(G.\mu) = \chi_\mu.$$

Property (d) is a weak version of the Borel-Weil-Bott theorem: the strong version uses *Kähler quantization*, and realizes the irreducible representation corresponding to μ as a space of holomorphic sections of the pre-quantum line bundle.

Let $R(G) \to \mathbb{Z},\ \chi \mapsto \chi^G$ be the group homomorphism defined on basis elements by $\chi_\mu^G = \delta_{\mu,0}$. That is, χ^G is the coefficient of the basis element χ_0 in χ. The map $\chi \mapsto \chi^G$ may be regarded as the 'quantum counterpart' to symplectic reduction (taking the coefficient of $\mu = 0$ corresponds to setting the moment map value equal to 0). The following fact was conjectured by Guillemin-Sternberg in the 1980s. (In [**33**], Guillemin and Sternberg gave a full proof of a similar statement for Kähler quantization.)

Theorem 4.4 (Quantization commutes with reduction).

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

Remark 4.5. The right hand side of this result requires some explanation. If 0 is a regular value of the moment map, and G acts freely on the zero level set $\Phi^{-1}(0)$, then $M/\!\!/G$ is a symplectic manifold with pre-quantum line bundle $L/\!\!/G = L|_{\Phi^{-1}(0)}/G$. In this case, the right hand side is defined as the (non-equivariant) index of the corresponding Spin_c-Dirac operator. If the action on the zero level set is only locally free, then $L/\!\!/G$ becomes an orbifold line bundle over the orbifold $M/\!\!/G$, and the index has to be interpreted accordingly (using Kawasaki's index theorem for orbifolds). In the most general case, one can define the right hand side by a partial desingularization of $M/\!\!/G$, reducing to the orbifold case. In this generality, the result was proved in [**58**].

Example 4.6. Let $N_{\mu_1\mu_2\mu_3}$ for $\mu_1, \mu_2, \mu_3 \in P_+$ be the tensor coefficients, defined by

$$\chi_{\mu_1}\chi_{\mu_2} = \sum_{\mu_3 \in P_+} N_{\mu_1\mu_2\mu_3}\chi_{\mu_3}^*.$$

Equivalently, $N_{\mu_1\mu_2\mu_3} = (\chi_{\mu_1}\chi_{\mu_2}\chi_{\mu_3})^G$. Let $\mathcal{O}_i$ be the coadjoint orbits of $\mu_i \in P_+$. Then

$$N_{\mu_1\mu_2\mu_3} = \mathcal{Q}(\mathcal{O}_1 \times \mathcal{O}_2 \times \mathcal{O}_3/\!\!/G),$$

realizing the tensor coefficients as an index.

Given a pre-quantized Hamiltonian G-space (M, ω, Φ), let $N(\mu) \in \mathbb{Z}$ be the multiplicity of χ_μ in the quantization $\mathcal{Q}(M)$,

$$\mathcal{Q}(M) = \sum_{\mu \in P_+} N(\mu)\chi_\mu.$$

Thus $N(0) = \mathcal{Q}(M)^G$. For any $\mu \in \mathfrak{t}_+^*$, let $M/\!\!/_\mu G = \Phi^{-1}(\mu)/G_\mu$ be the symplectic quotient at level $\mu \in \mathfrak{g}^*$. The *shifting trick* expresses $M/\!\!/_\mu G$ as a reduction at 0:

$$M/\!\!/_\mu G = (M \times (G.\mu)^*)/\!\!/G;$$

here $(G.\mu)^*$ denotes the coadjoint orbit $G.\mu$ with minus the standard symplectic structure and minus the inclusion as a moment map. Suppose $\mu \in P_+ \subset \mathfrak{g}^*$. Since

$$\mathcal{Q}((G.\mu)^*) = \mathcal{Q}(G.\mu)^* = \chi_\mu^*,$$

Theorem 4.4 shows that the multiplicity of 0 in $\mathcal{Q}(M \times (G.\mu)^*)$ equals the multiplicity $N(\mu)$ of μ in $\mathcal{Q}(M)$. Thus

$$N(\mu) = \mathcal{Q}(M/\!\!/_\mu G).$$

4.3. Localization. In most cases, the 'quantization commutes with reduction' theorem is not very practical for the calculation of weight multiplicities in $\mathcal{Q}(M)$. Instead, the result is often used in the opposite direction: One obtains the indices of symplectic quotients $\mathcal{Q}(M/\!\!/_\mu G)$ from the knowledge of $\mathcal{Q}(M)$. The main technique for the computation of $\mathcal{Q}(M)$ is *localization*.

The Atiyah-Segal-Singer equivariant index theorem for elliptic operators, specialized to the case of Spin_c-Dirac operators, gives the formula

$$\mathcal{Q}(M)(g) = \sum_{F\subset M^g} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}}{D_{\mathbb{R}}(\nu_F, g)}$$

where the sum is over fixed point manifolds $F \subset M^g$ for the action of g. Here $\mathcal{L}$ is the 'Spin_c-line bundle' $\mathcal{L} = L^2 \otimes K^{-1}$, with K the canonical bundle, and ν_F is the normal bundle to F. The terms $\widehat{A}(F), \mathrm{Ch}(\mathcal{L}|_F, g)^{1/2}$, and $D_{\mathbb{R}}(\nu_F, g)$ are certain characteristic classes of $TF,\ \mathcal{L}|_F,\ \nu_F$. (For details, see e.g. [**57**, Section 5.3].)

REMARK 4.7. The fixed point formula can also be written in 'Riemann-Roch form',

$$\mathcal{Q}(M)(g) = \sum_{F\subset M^g} \int_F \frac{\mathrm{Td}(F)\,\mathrm{Ch}(L|_F, g)}{D_{\mathbb{C}}(\nu_F, g)},$$

which is often easier to use for computations. However, the 'Spin_c-form' will be more convenient for our discussion.

REMARK 4.8. If one tries develop a similar quantization procedure for q-Hamiltonian G-spaces (M, ω, Φ), one is faced with several obstacles. First, the 2-form ω need not be closed, hence it cannot be the curvature form of a line bundle. Secondly, since ω can be degenerate, there is no obvious notion of 'compatible complex structure'. (In fact, there are examples of conjugacy classes $\mathcal{C}$ of compact, simply connected Lie groups not admitting any Spin_c-structure.) Hence, there is no suitable Dirac operator in sight. In the following sections we will explain how to get around these problems.

5. The level k fusion ring

From the correspondence with Hamiltonian loop group spaces, we expect that the result of the quantization procedure of q-Hamiltonian spaces should be an element not of the representation ring but of the *fusion ring* of G, at suitable level.

For the remainder of these lecture notes, we will assume that G *is compact, simply connected and* simple. We fix a maximal torus T and a fundamental Weyl chamber $\mathfrak{t}^*_+$. Recall that $P_+ = P \cap \mathfrak{t}^*_+$ are the dominant weights. Let $\theta \in P_+$ be the highest root, i.e. the highest weight of the adjoint representation of G on $\mathfrak{g}^{\mathbb{C}}$. The fundamental alcove has the following description

$$\mathfrak{A} = \{\xi \in \mathfrak{t}_+ |\ \langle\theta, \xi\rangle \leq 1\}.$$

We denote by $\rho \in P_+$ the unique shortest weight in $P_+ \cap \mathrm{int}(\mathfrak{t}^*_+)$; equivalently 2ρ is the sum of the positive roots of G. The *basic inner product* $\cdot$ on $\mathfrak{g}$ is the unique invariant inner product such that $\theta\cdot\theta = 2$ (using the identification $\mathfrak{g} \cong \mathfrak{g}^*$ given by the inner product). We will use the basic inner product to identify $\mathfrak{g}$ and $\mathfrak{g}^*$. The *dual Coxeter number* of G is the positive integer defined by

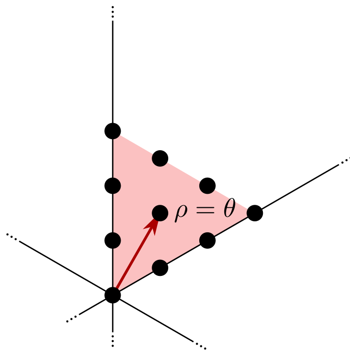
$$\mathsf{h}^\vee = 1 + \theta\cdot\rho.$$

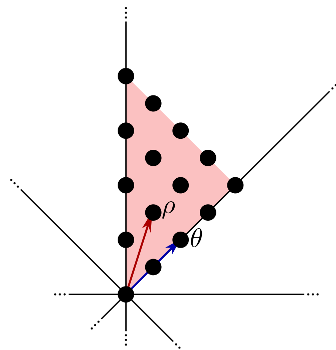
For $G = \mathrm{SU}(N)$ one has $\mathsf{h}^\vee = N$.

DEFINITION 5.1. Let $k \in \{1, 2, \ldots\}$. The *level k weights* of G are the elements of

$$P_k = P \cap k\mathfrak{A}.$$

The following pictures show the set of level k weights, as well as the weights ρ, θ, in two examples. The shaded area is $k\mathfrak{A}$.





For $\lambda \in P_k$ define the special element

$$t_\lambda = \exp(\tfrac{\lambda+\rho}{k+\mathsf{h}^\vee}) \in T.$$

DEFINITION 5.2. The *level k fusion ring* (Verlinde algebra) is the quotient

$$R_k(G) = R(G)/I_k(G)$$

by the *level k fusion ideal*, $I_k(G) = \{\chi \in R(G) |\ \chi(t_\lambda) = 0\ \ \forall\ \lambda \in P_k\}$.

The fusion ring $R_k(G)$ plays an important role in conformal field theory (see e.g. [**28**]). It is also known as the *level k Verlinde algebra*, after the physicist Erik Verlinde [**69**].

REMARK 5.3. $R_k(G)$ is also the fusion ring of level k projective representations of the loop group LG. However, we will not need this interpretation here.

Some basic properties of the level k fusion ring are as follows:

(1) the unit and involution of $R(G)$ descend to a unit and involution of $R_k(G)$,
(2) $R_k(G)$ has *finite* $\mathbb{Z}$-basis the images τ_μ of $\chi_\mu, \mu \in P_k$. Thus

$$R_k(G) = \mathbb{Z}[P_k].$$

(3) $R_k(G)$ has a trace,

$$R_k(G) \to \mathbb{Z},\ \ \tau \mapsto \tau^G$$

where $\tau_\mu^G = \delta_{\mu,0}$.
(4) The integers

$$N^{(k)}_{\mu_1\mu_2\mu_3} = (\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G, \quad \mu_i \in P_k$$

are called the *level k fusion coefficients*. They encode the multiplication in $R_k(G)$:

$$\tau_{\mu_1}\tau_{\mu_2} = \sum_{\mu_3 \in P_k} N^{(k)}_{\mu_1\mu_2\mu_3}\tau^*_{\mu_3}.$$

If $\mu_1, \mu_2, \mu_3 \in P_+$ are fixed, the fusion coefficients become independent of k for sufficiently large k, and coincide with the tensor coefficients:

$$N^{(k)}_{\mu_1\mu_2\mu_3} = N_{\mu_1\mu_2\mu_3}, \qquad k >> 0.$$

EXAMPLE 5.4. For $G = \mathrm{SU}(2)$, it is not difficult to determine the level k fusion ring 'by hand'. Identify $\mathfrak{t} \cong \mathbb{R}$ in such a way that $P_+ = \{0, 1, \ldots\}$. Here $m \geq 0$ is realized as the dominant weight for the m-th symmetric power of the defining representation, $S^m\mathbb{C}^2$. We have $\rho = 1$, $\theta = 2$, and the alcove is the interval $[0, 1] \subset \mathbb{R}$. Hence $P_k = \{0, 1, \ldots, k\}$. The product in $R(\mathrm{SU}(2))$ is given by the well-known formula

$$\chi_l\chi_m = \chi_{l+m} + \chi_{l+m-2} + \ldots + \chi_{|l-m|}.$$

Equivalently, the tensor coefficients are given by

$$N_{m_1m_2m_3} = 1$$

if $m_1 + m_2 + m_3$ is even with $m_i \leq \frac{1}{2}(m_1 + m_2 + m_3)$ for $i = 1, 2, 3$, and are zero in all other cases. One finds that the level k fusion ideal is $I_k(\mathrm{SU}(2)) = \langle\chi_{k+1}\rangle$, and the quotient map $R(G) \to R_k(G)$ is 'signed reflection' across indices $k + 1,\ 2k + 3,\ 3k + 5, \ldots$.

To illustrate, if $k = 5$ we find $\tau_3\tau_4 = \tau_3 + \tau_1$ since

$$\chi_3\chi_4 = \chi_7 + \chi_5 + \chi_3 + \chi_1,$$

and because $\chi_7 \mapsto -\tau_5,\ \chi_5 \mapsto \tau_5$ under the quotient map. For $m_1, m_2, m_3 \in \{0, 1, \ldots, k\}$, the SU(2) fusion coefficients at level k are given by

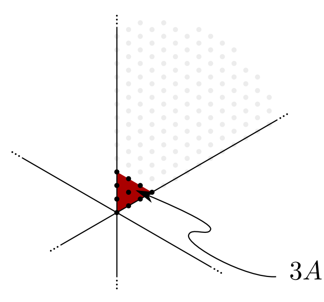
$$N^{(k)}_{m_1m_2m_3} = 1$$

provided $m_1 + m_2 + m_3$ is even with

$$m_i \leq \tfrac{1}{2}(m_1 + m_2 + m_3) \leq k$$

for $i = 1, 2, 3$, and are zero in all other cases.

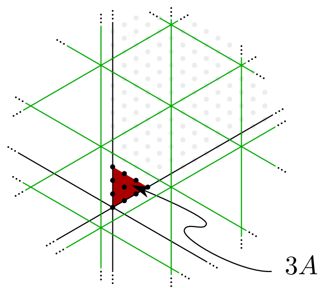
For a general compact simple Lie group G, the quotient map $R(G) \to R_k(G)$ is a 'signed reflection' for a shifted Stiefel diagram. We illustrate the quotient map for $G = \mathrm{SU}(3)$ and level $k = 3$. Consider the set P_k of level k weights



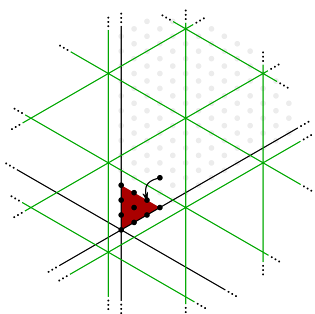
One can show that the weights P_k, shifted by ρ, are exactly the weights in the interior the shifted alcove at level $k + \mathsf{h}^\vee$:

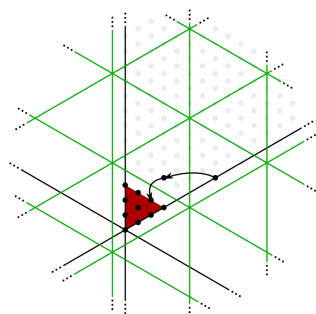
$$P_k + \rho = P \cap (k + \mathsf{h}^\vee)\,\mathrm{int}(\mathfrak{A}).$$

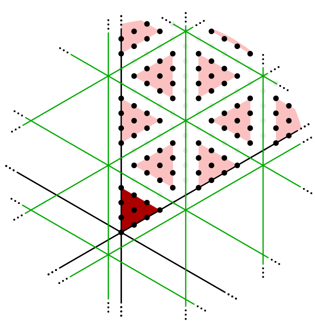
The affine reflections across the faces of the shifted alcove $(k + \mathsf{h}^\vee)\mathfrak{A} - \rho$ alcove generate the *ρ-shifted level $k + \mathsf{h}^\vee$ Stiefel diagram*, shown in the following picture.



The shifted affine Weyl group is the group of transformations of $\mathfrak{t}$, generated by reflections across these affine hyperplanes:







The last picture shows the weights that can be reflected into P_k. If $\mu \in P_+$ lies on the walls of the shifted Stiefel diagram, then χ_μ lies in the kernel of the quotient map $R_k(G)$. Otherwise, the quotient map takes χ_μ to $\pm\tau_\nu$, where $\nu \in P_k$ is the unique level k weight related to μ by a sequence of affine reflections, and where the sign (plus or minus) is given by the parity (even or odd) of the required number of reflections.

REMARK 5.5. It was shown by Gepner [**31**] and Bouwknegt-Ridout [**18**] that for $G = \mathrm{SU}(N)$, the level k fusion ideal has the description

$$I_k(G) = \langle \chi_{(k+1)\varpi_1}, \ldots, \chi_{(k+N-1)\varpi_1} \rangle,$$

where ϖ_1 (the first fundamental weight) is the highest weight of the defining representation of $\mathrm{SU}(N)$ on $\mathbb{C}^N$. There is a similar presentation of the fusion ideal for the symplectic group $\mathrm{Sp}(r)$. Explicit presentations of the fusion rings for the other simple groups, with small numbers of generators, were obtained by C. Douglas in [**25**].

By definition of the ideal $I_k(G)$, the evaluation maps

$$R(G) \to \mathbb{C}, \ \chi \mapsto \chi(t_\lambda)$$

for $\lambda \in P_k$ vanishes on $I_k(G)$, hence they descend to the fusion ring:

$$R_k(G) \to \mathbb{C}, \quad \tau \mapsto \tau(t_\lambda).$$

After complexification, we obtain a new additive basis $\tilde{\tau}_\mu, \ \mu \in P_k$ of $R_k(G) \otimes \mathbb{C}$, characterized by the property

$$\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}.$$

In the new basis, the product is diagonalized: $\tilde{\tau}_\mu \tilde{\tau}_\nu = \delta_{\mu,\nu} \tilde{\tau}_\nu$. The two bases are related by the *S-matrix*

$$S \in \mathrm{End}(\mathbb{C}[P_k]).$$

The S-matrix is the unique *unitary* matrix with properties

$$S_{\mu,\nu} = S_{\nu,\mu}, \quad S_{0,\nu} > 0$$

for $\mu, \nu \in P_k$, and such that

$$\tau_\mu = \sum_{\nu \in P_k} S_{0,\nu}^{-1} S_{\mu,\nu}^* \tilde{\tau}_\nu;$$

In terms of the S-matrix, the fusion coefficients take on the form,

$$N^{(k)}_{\mu_1\mu_2\mu_3} = \sum_{\nu \in P_k} \frac{S_{\mu_1,\nu} S_{\mu_2,\nu} S_{\mu_3,\nu}}{S_{0,\nu}}.$$

6. Pre-quantization of q-Hamiltonian spaces

While the 2-form ω for a q-Hamiltonian G-space (M, ω, Φ) is not closed, in general, the pair $(\omega, -\eta)$ defines a *relative cocycle* for the map Φ. To explain in more detail, we recall the cone construction from homological algebra.

6.1. Relative cohomology.

DEFINITION 6.1. Let $F^\bullet \colon S^\bullet \to R^\bullet$ be a cochain map between cochain complexes. The *algebraic mapping cone* is the cochain complex

$$\mathrm{cone}^k(F) = R^{k-1} \oplus S^k, \quad \mathrm{d}(x, y) = (f(y) - \mathrm{d}x, \mathrm{d}y).$$

Its cohomology is denoted $H^\bullet(F)$, and is called the relative cohomology of the cochain map $F^\bullet$.

The short exact sequence of cochain complexes $0 \to R^{k-1} \to \mathrm{cone}^k(F) \to S^k \to 0$ gives rise to a long exact sequence of cohomology groups,

$$\cdots \to H^{k-1}(R) \to H^k(F) \to H^k(S) \to H^k(R) \to \cdots .$$

The connecting homomorphism $H^\bullet(S) \to H^\bullet(R)$ is just the map induced by F.

Given a smooth map $\Phi\colon M \to N$ between manifolds, we define $\Omega^\bullet(\Phi) = \mathrm{cone}^\bullet(\Phi^*)$ to be the algebraic mapping cone for the pull-back of differential forms, $\Phi^*\colon \Omega^\bullet(N) \to \Omega^\bullet(M)$. Its cohomology $H^\bullet(\Phi) := H^\bullet(\Phi^*)$ is called the relative de Rham cohomology of the map Φ. The usual isomorphism with the singular cohomology with real coefficients carries over the the relative setting, and there is a *coefficient homomorphism*

$$H^\bullet(\Phi, \mathbb{Z}) \to H^\bullet(\Phi) = H^\bullet(\Phi, \mathbb{R}).$$

6.2. Definition of pre-quantization. For a q-Hamiltonian G-space, we have $\mathrm{d}\omega = -\Phi^*\eta$ and $\mathrm{d}\eta = 0$. Hence

$$(\omega, -\eta) \in \Omega^3(\Phi)$$

is a cocycle. (In fact, working with equivariant forms the pair $(\omega, -\eta_G)$ is an equivariant relative cocycle in $\Omega^3_G(\Phi)$, using the algebraic mapping cone for the Cartan complexes.) Suppose G simple, simply connected, $\cdot$ the basic inner product.

Definition 6.2. [**52, 57**] Let (M, ω, Φ) be a q-Hamiltonian G-space, $\Phi\colon M \to G$. A *level k pre-quantization* of (M, ω, Φ) is an integral lift of

$$k[(\omega, -\eta)] \in H^3(\Phi, \mathbb{R}).$$

There is an equivariant version of this condition, but for simply connected compact groups G the equivariance is automatic. Indeed, in this case the natural map $H^\bullet_G(X, \mathbb{Z}) \to H^\bullet(X, \mathbb{Z})$ for a G-space X is an isomorphism in degrees ≤ 2, while for any G-map Φ the map $H^\bullet_G(\Phi, \mathbb{Z}) \to H^\bullet(\Phi, \mathbb{Z})$ is an isomorphism in degrees ≤ 3. Cf. Krepski [**52**, Section 3].

Remark 6.3. The geometric interpretation of the pre-quantization condition involves 'gerbes'. Loosely speaking, the pre-quantization of the condition $\mathrm{d}(k\omega) = -k\Phi^*\eta$ is given by a gerbe over G, with 3-curvature form $k\eta$, together with a trivialization of the pull-back of this gerbe to M, with $k\omega$ the curvature form of the trivialization. See Shahbazi [**65**] for further details.

6.3. Basic properties, examples. One has the following criterion for the integrality of the relative form $k(\omega, -\eta) \in \Omega^3(\Phi)$. For any manifold M, let $C_\bullet(M)$ be the chain complex of *smooth* singular chains on M (i.e. $C_k(M)$ consists of $\mathbb{Z}$-linear combinations of smooth maps $\Delta^k \to M$, where Δ^k is the k-simplex). Recall that a closed differential form $\alpha \in \Omega^k(M)$ is integral (i.e. its class $[\alpha] \in H^k(M, \mathbb{R})$ lies in the image of $H^k(M, \mathbb{Z})$) if and only if $\int_\Sigma \alpha \in \mathbb{Z}$ for all k-cycles $\Sigma \in Z_k(M)$. This criterion extends to the relative case, so that we have:

Proposition 6.4. *A q-Hamiltonian G-space (M, ω, Φ) is pre-quantizable at level k if and only if for all $\Sigma \in Z_2(M)$, and any $X \in C_3(G)$ with $\Phi(\Sigma) = \partial X$,*

$$k\Big(\int_\Sigma \omega + \int_X \eta\Big) \in \mathbb{Z}.$$

Note that for given Σ, it suffices to check for *any* X, due to the integrality of η. In particular, the criterion is satisfied if the second homology group $H_2(M,\mathbb{Z})$ is zero. Indeed, in this case we can take $X = \Phi(Y)$ with $Y \in C_3(M)$, $\partial Y = \Sigma$, and the criterion holds true by Stokes' theorem.

EXAMPLE 6.5. The double $\mathbf{D}(G) = G\times G$, $\Phi(a,b) = aba^{-1}b^{-1}$ is pre-quantizable for all $k \in \mathbb{N}$, since $H_2(D(G),\mathbb{Z}) = 0$.

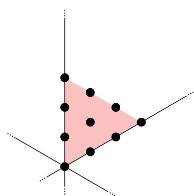
EXAMPLE 6.6. The q-Hamiltonian $\mathrm{SU}(n)$-space $M = S^{2n}$ and the q-Hamiltonian $\mathrm{Sp}(n)$-space $M = \mathbb{H}\mathrm{P}(n)$ are pre-quantized at all levels $k \in \mathbb{N}$, since $H_2(M,\mathbb{Z}) = 0$ in these examples.

Recall that conjugacy classes $\mathcal{C} \subset G$ are parametrized by points in the alcove, where $\xi \in \mathfrak{A}$ corresponds to the conjugacy class $\mathcal{C} = G.\exp\xi$. We have:

EXAMPLE 6.7. The level k pre-quantized conjugacy classes $\mathcal{C} \subset G$ are those indexed by

$$\xi \in \tfrac{1}{k}P_k \subset A.$$

The following picture shows the pre-quantized conjugacy classes for SU(3) at level $k = 3$.



$$\begin{aligned} G &= \mathrm{SU}(3) \\ k &= 3 \end{aligned}$$

In all these examples, the torsion subgroup of $H^2(M,\mathbb{Z})$ is trivial, hence the pre-quantization is unique.

Pre-quantizations are well-behaved with respects to products: If M_1, M_2 are level k-pre-quantized q-Hamiltonian G-spaces, then their fusion product $M_1 \times M_2$ inherits a level k pre-quantization. In particular, the q-Hamiltonian G-space $D(G)^h \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ has a level k pre-quantization, provided the conjugacy classes $\mathcal{C}_j$ have level k pre-quantizations.

Furthermore, if M is a level k pre-quantized q-Hamiltonian G-space, then the symplectic quotient $M/\!\!/G$ inherits a pre-quantization at level k, i.e. for the k-th multiple of the symplectic form. (If the symplectic quotient is singular, this statement should be interpreted as in [**58**].)

7. Twisted Spin_c-structures on q-Hamiltonian spaces

Besides the notion of pre-quantization, a key ingredient in the quantization of Hamiltonian G-spaces is the existence of a canonical Spin_c-structure (defined by a compatible almost complex structure). For q-Hamiltonian G-spaces, there need not be a Spin_c-structure in general, but it turns out that there is a canonical *twisted* Spin_c-structure.

7.1. Spin_c**-structures.** We will use the following viewpoint toward Spin_c-structures. Given a Euclidean vector space V, let $\mathbb{C}\mathrm{l}(V)$ denote its complex Clifford algebra. Thus $\mathbb{C}\mathrm{l}(V)$ is the complex unital algebra with generators $v \in V$ and relations $v_1v_2 + v_2v_1 = 2\langle v_1, v_2\rangle$. Using a basis $e_1,\ldots,e_n \in V$ to identify $V \cong \mathbb{R}^n$, the Clifford algebra has basis the products $e_I = e_{i_1}\ldots e_{i_k}$ for

$I = \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}$ with $i_1 < \cdots < i_k$, with the convention $e_\emptyset = 1$. Thus, $\mathbb{C}\mathrm{l}(V) = \wedge(V) \otimes \mathbb{C}$ as a vector space. The Clifford algebra carries a $\mathbb{Z}_2$-grading, where the even (resp. odd) part is spanned by products of an even (resp. odd) number of elements of V.

Definition 7.1. Suppose $\dim V$ is even. A *spinor module* over $\mathbb{C}\mathrm{l}(V)$ is a $\mathbb{Z}_2$-graded Hermitian vector space S, together with an isomorphism $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathsf{S})$ preserving $\mathbb{Z}_2$-gradings and involutions $*$.

A concrete spinor module is obtained by the choice of an orthogonal complex structure $J \in \mathrm{End}(V)$: Let $V^{\mathbb{C}} = V^{1,0} \oplus V^{0,1}$ be the decomposition into $\pm i$ eigenspaces of J, the space $\wedge V^{0,1}$ is a spinor module, with Clifford action of generators $v = v^{1,0} + v^{0,1}$ given by $\iota(v^{1,0}) + \epsilon(v^{0,1})$ (here ι denotes contraction, ϵ is exterior multiplication). One has the following fact:

Proposition 7.2. *For any two spinor modules S, S' over $\mathbb{C}\mathrm{l}(V)$, the space*

$$\mathrm{Hom}_{\mathbb{C}\mathrm{l}(V)}(\mathsf{S}, \mathsf{S}')$$

of linear maps $\mathsf{S} \to \mathsf{S}'$ intertwining the Clifford actions is 1-dimensional.

These definitions generalize to Euclidean vector bundles $V \to M$ in an obvious way. In particular, we define a spinor module over $\mathbb{C}\mathrm{l}(V)$ to be a $\mathbb{Z}_2$-graded Hermitian vector bundle $\mathsf{S} \to M$, together with an even isomorphism of $*$-algebra bundles $\mathbb{C}\mathrm{l}(V) \to \mathrm{End}(\mathsf{S})$. Given an orthogonal complex structure J on V, the bundle $\wedge V^{0,1} \to M$ is such a spinor module.

Definition 7.3. A Spin_c-structure on an even rank Euclidean vector bundle $V \to M$ is a spinor module S over $\mathbb{C}\mathrm{l}(V)$. A Spin_c-structure on an even-dimensional Riemannian manifold is a Spin_c-structure on TM.

Remarks 7.4.
(1) A Spin_c-structure on a Euclidean vector bundle V of rank n can also be defined as an orientation on V together with a lift of the structure group from $\mathrm{SO}(n)$ to the group $\mathrm{Spin}_c(n)$. The two definitions are equivalent [**60**].
(2) For Euclidean vector bundle of odd rank, one can define a Spin_c-structure on V to be a Spin_c-structure on $V \oplus \mathbb{R}$.
(3) There are two topological obstructions to the existence of a Spin_c-structure on V. The first obstruction $w^1(V) \in H^1(M, \mathbb{Z}_2)$ is simply the obstruction to orientability of V. The second obstruction $W^3(V) \in H^3(M, \mathbb{Z})$ is the *third integral Stiefel-Whitney class*, given as the image of $w^2(V) \to H^2(M, \mathbb{Z}_2)$ under the Bockstein homomorphism. Note that $W^3(V)$ is 2-torsion, which is consistent with the fact that $V \oplus V = V \otimes \mathbb{C}$ carries a Spin_c-structure.
(4) If S is spinor module, then so is the graded $\mathsf{S} \otimes L$ for any $\mathbb{Z}_2$-graded Hermitian line bundle L. (A $\mathbb{Z}_2$-grading on a complex line bundle $L \to M$ is just the assignment of an even or odd parity over each component of M.) Proposition 7.2 generalizes to the fact that any two Spin_c-structures S, S' on V differ by a $\mathbb{Z}_2$-graded Hermitian line bundle:

$$\mathsf{S}' = \mathsf{S} \otimes L; \quad L = \mathrm{Hom}_{\mathbb{C}\mathrm{l}(V)}(\mathsf{S}, \mathsf{S}').$$

7.2. Dixmier-Douady theory. Given a separable Hilbert space H, we denote by $\mathbb{K}(\mathsf{H})$ the $*$-algebra of *compact operators* on H, i.e. the norm closure of the algebra of operators of finite rank. One may think of $\mathbb{K}(\mathsf{H})$ as an appropriate notion of infinite matrices. The action of the unitary group $\mathrm{U}(\mathsf{H})$ by conjugation on $\mathbb{K}(\mathsf{H})$ descends to the projective unitary group $\mathrm{PU}(\mathsf{H}) = \mathrm{U}(\mathcal{H})/\mathrm{U}(1)$, and in fact it is known that

$$\mathrm{Aut}(\mathbb{K}(\mathsf{H})) = \mathrm{PU}(\mathsf{H}),$$

where $\mathrm{PU}(\mathsf{H})$ carries the strong topology.[2]

DEFINITION 7.5. A ($\mathbb{Z}_2$-graded) *Dixmier-Douady bundle* over M is a ($\mathbb{Z}_2$-graded) bundle of $*$-algebras $\mathcal{A} \to M$, with typical fiber $\mathbb{K}(\mathsf{H})$ for some ($\mathbb{Z}_2$-graded) Hilbert space H. A *Morita trivialization* of the Dixmier-Douady bundle $\mathcal{A}$ is a bundle of ($\mathbb{Z}_2$-graded) Hilbert spaces $\mathcal{E} \to M$ with an even isomorphism of $*$-algebra bundles $\mathcal{A} \to \mathbb{K}(\mathcal{E})$.

EXAMPLE 7.6. For an even rank Euclidean vector bundle $V \to M$, the Clifford bundle $\mathbb{C}\mathrm{l}(V)$ is a $\mathbb{Z}_2$-graded Dixmier-Douady bundle. A $\mathbb{Z}_2$-graded Morita trivialization of $\mathbb{C}\mathrm{l}(V)$ is the same thing as a spinor module S over $\mathbb{C}\mathrm{l}(V)$, i.e. it is a Spin_c-structure on V.

Generalizing this example, one finds that for any two Morita trivializations $\mathcal{E}, \mathcal{E}'$ of a $\mathbb{Z}_2$-graded DD-bundle $\mathcal{A} \to M$, the bundle $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}')$ is a $\mathbb{Z}_2$-graded Hermitian line bundle, and conversely any two Morita trivializations differ by such a line bundle

$$\mathcal{E}' = \mathcal{E} \otimes L; \qquad L = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E}').$$

Given a DD-bundle $\mathcal{A} \to M$, there is an obstruction $\mathrm{DD}(\mathcal{A}) \in H^3(M, \mathbb{Z})$ to the existence of a Morita trivialization $\mathcal{E}$, called the *Dixmier-Douady class*. In the $\mathbb{Z}_2$-graded setting, there is an additional obstruction in $H^1(M, \mathbb{Z}_2)$ to introducing a compatible $\mathbb{Z}_2$-grading on $\mathcal{E}$.

REMARK 7.7. One viewpoint towards the DD-class is as follows. Consider the principal $\mathrm{PU}(\mathsf{H})$-bundle $P \to M$ associated to $\mathcal{A}$. Choose a trivializing open cover U_α of M, so that P is described by transition functions $\chi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{PU}(\mathsf{H})$. Over triple overlaps, $\chi_{\alpha\beta}\chi_{\beta\gamma}\chi_{\alpha\gamma} = 1$. Lift to $\mathrm{U}(\mathsf{H})$-valued functions $\widetilde{\chi}_{\alpha\beta}$. Then $\psi_{\alpha\beta\gamma} = \widetilde{\chi}_{\beta\gamma}\widetilde{\chi}_{\alpha\gamma}^{-1}\widetilde{\chi}_{\alpha\beta}$ is a $\mathrm{U}(1)$-valued function on triple overlaps. On quadruple overlaps one has, by definition of ψ,

$$\psi_{\beta\gamma\delta}\psi_{\alpha\gamma\delta}^{-1}\psi_{\alpha\beta\delta}\psi_{\alpha\beta\gamma}^{-1} = 1,$$

which means that ψ is a Čech cocycle, defining a class in $H^2(M, \underline{\mathrm{U}(1)}) = H^3(M, \mathbb{Z})$. For a detailed discussion of Dixmier-Douady theory, see [**62**].

More generally, if $\mathsf{H}_1, \mathsf{H}_2$ are two Hilbert spaces, we have the Banach space $\mathbb{K}(\mathsf{H}_1, \mathsf{H}_2)$ of compact operators from H_1 to H_2, again defined as the norm closure of finite rank operators. It is a bimodule:

$$\mathbb{K}(\mathsf{H}_2) \circlearrowright \mathbb{K}(\mathsf{H}_1, \mathsf{H}_2) \circlearrowleft \mathbb{K}(\mathsf{H}_1).$$

If H_i carry $\mathbb{Z}_2$-gradings, then this bimodule structure is compatible with $\mathbb{Z}_2$-gradings.

[2] In the following discussion, some subtleties are being ignored. See [**57**], and references given there, for a more careful treatment.

Definition 7.8. Suppose $\mathcal{A}_i \to M_i$, $i = 1, 2$ are two ($\mathbb{Z}_2$-graded) Dixmier-Douady bundles, with typical fiber $\mathbb{K}(\mathsf{H}_i)$. A ($\mathbb{Z}_2$-graded) *Morita morphism*

$$(\Phi, \mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$$

is a map $\Phi\colon M_1 \to M_2$ together with a ($\mathbb{Z}_2$-graded) Banach bundle $\mathcal{E} \to M_1$ of bimodules

$$\Phi^*\mathcal{A}_2 \circlearrowright \mathcal{E} \circlearrowleft \mathcal{A}_1$$

locally modeled on $\mathbb{K}(\mathsf{H}_2) \circlearrowright \mathbb{K}(\mathsf{H}_1, \mathsf{H}_2) \circlearrowleft \mathbb{K}(\mathsf{H}_1)$.

The composition of two Morita morphisms $(\Phi, \mathcal{E})\colon \mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ and $(\Phi', \mathcal{E}')\colon \mathcal{A}_2 \dashrightarrow \mathcal{A}_3$ has underlying map the composition $\Phi' \circ \Phi$, and bimodule a completion of the tensor product $\Phi^*\mathcal{E}' \otimes_{\Phi^*\mathcal{A}_2} \mathcal{E}$. A Morita morphism is invertible if Φ is; the inverse is defined using an 'opposite' bimodule.

A Morita trivialization of $\mathcal{A} \to M$ is equivalent to a Morita morphism $(p, \mathcal{E})\colon \mathcal{A} \dashrightarrow \mathbb{C}$, where $\mathbb{C} \to \mathrm{pt}$ is the trivial DD bundle. Again, Morita morphisms may be twisted by line bundles, and any two Morita morphisms $\mathcal{A}_1 \dashrightarrow \mathcal{A}_2$ differ by a Hermitian line bundle:

$$L = \mathrm{Hom}_{\Phi^*\mathcal{A}_2 - \mathcal{A}_1}(\mathcal{E}, \mathcal{E}') \quad \longleftrightarrow \quad \mathcal{E}' = \mathcal{E} \otimes L.$$

The Dixmier-Douady theorem states that DD-bundles $\mathcal{A} \to M$ are classified, up to Morita isomorphisms inducing the identity map on the base, by $H^3(M, \mathbb{Z})$. The result extends to G-equivariant DD-bundles, see [**12**].

7.3. The Dixmier-Douady bundle $\mathcal{A}_G^{\mathrm{Spin}}$. It is known that

$$H^2(\mathrm{SO}(n), \mathbb{Z}) = 0,\ H^3(\mathrm{SO}(n), \mathbb{Z}) = \mathbb{Z},\ H^1(\mathrm{SO}(n), \mathbb{Z}_2) = \mathbb{Z}_2$$

for $n = 3$ and all $n \geq 5$. If V is a Euclidean vector space of dimension $\dim V \geq 5$, we denote by $\mathcal{A}_{\mathrm{SO}(V)} \to \mathrm{SO}(V)$ the $\mathrm{SO}(V)$-equivariant $\mathbb{Z}_2$-graded DD-bundle whose characteristic classes in $H^3(\mathrm{SO}(V), \mathbb{Z})$ and $H^1(\mathrm{SO}(V), \mathbb{Z}_2)$ represent the generators. Since $H^2(\mathrm{SO}(V), \mathbb{Z}) = 0$, the particular choice of this DD-bundle does not matter. If $V \subset V'$ is a subspace of a larger Euclidean vector space, then $\mathcal{A}_{\mathrm{SO}(V)}$ is canonically Morita isomorphic to the pull-back of $\mathcal{A}_{\mathrm{SO}(V')}$ under the inclusion $\mathrm{SO}(V) \hookrightarrow \mathrm{SO}(V')$. Consequently, we may extend the definition to $\dim V < 5$ by taking $\mathcal{A}_{\mathrm{SO}(V)}$ to be the pull-back of $\mathcal{A}_{\mathrm{SO}(V')}$, where $\dim V' \geq 5$. (E.g., take $V' = V \oplus \mathbb{R}^5$). An explicit construction of this bundle may be found in Atiyah-Segal [**12**], see also [**4**] for a discussion of their result.

Given a compact, connected Lie group G, with an invariant inner product on $\mathfrak{g}$, we let

$$\mathcal{A}_G^{\mathrm{Spin}} \to G$$

be the pull-back of $\mathcal{A}^{\mathrm{SO}(\mathfrak{g})}$ under the adjoint representation $G \to \mathrm{SO}(\mathfrak{g})$. (The notation is motivated by a relationship with the spin representation of the loop group.) A nice property of $\mathcal{A}_G^{\mathrm{Spin}}$ is that it is *multiplicative*: there is a Morita morphism $\mathcal{A}_G^{\mathrm{Spin}} \times \mathcal{A}_G^{\mathrm{Spin}} \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}$ covering group multiplication on G, and with an associativity property.

Remark 7.9. If G is compact, simple and simply connected, so that $H^3_G(G, \mathbb{Z}) = H^3(G, \mathbb{Z}) = \mathbb{Z}$, it is known that $\mathrm{DD}(\mathcal{A}_G^{\mathrm{Spin}})$ represents the $\mathsf{h}^\vee$-th multiple of the generator of $H^3(G, \mathbb{Z})$, where $\mathsf{h}^\vee$ is the dual Coxeter number.

7.4. Twisted Spin_c-structure.

THEOREM 7.10 (Alekseev-M [**4**]). *Let G be a compact Lie group, with a positive definite invariant metric $\cdot$ on its Lie algebra. For any q-Hamiltonian G-space (M, ω, Φ), there is a distinguished G-equivariant $\mathbb{Z}_2$-graded Morita morphism*

$$(\Phi, \mathcal{E}^{\mathrm{Spin}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}_G^{\mathrm{Spin}}.$$

Keeping in mind that a Morita trivialization $\mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathbb{C}$ is a Spin_c-structure, we think of this morphism as a *twisted* Spin_c*-structure* (following the terminology from [**70**]).

REMARK 7.11. The restriction of $\mathcal{A}_G^{\mathrm{Spin}}$ to the group unit e is G-equivariantly Morita trivial, and the Morita trivialization is essentially unique (since there are no non-trivial G-equivariant line bundles over pt.) By composing the twisted Spin_c-structure with this Morita trivialization, it follows that the restriction $TM|_{\Phi^{-1}(e)}$ inherits an *ordinary* Spin_c-structure. It turns out [**4**] that this is equal to the Spin_c-structure defined by the non-degenerate 2-form given by the restriction of ω to $TM|_{\Phi^{-1}(e)}$, hence it induces the correct Spin_c-structure on $M/\!\!/G$.

In the Hamiltonian setting, the next step is to twist the Spin_c-structure coming from the almost complex structure by the pre-quantum line bundle L. Similarly, for q-Hamiltonian spaces we can twist by the pre-quantization. To simplify the discussion, we will return to the assumption that G is simple and simply connected. Let $\mathcal{A}_G^{(k)} \to G$ be any G-DD bundle over G whose Dixmier-Douady class is k times the generator of $H_G^3(G, \mathbb{Z}) = \mathbb{Z}$. For example, $\mathcal{A}_G^{\mathrm{Spin}}$ may be used as $\mathcal{A}_G^{(\mathsf{h}^\vee)}$. A level k pre-quantization of (M, ω, Φ) determines a Morita morphism,

$$(\Phi, \mathcal{E}^{\mathrm{Preq}})\colon M \times \mathbb{C} \dashrightarrow \mathcal{A}_G^{(k)}.$$

(Classes in $H^3(\Phi, \mathbb{Z})$ may be realized in terms of DD bundles over the target, together with Morita trivializations of the pull-back under Φ.) Tensoring the two Morita morphisms, we obtain a G-equivariant Morita morphism,

$$(\Phi, \mathcal{E}^{\mathrm{Spin}} \otimes \mathcal{E}^{\mathrm{Preq}})\colon \mathbb{C}\mathrm{l}(TM) \dashrightarrow \mathcal{A}_G^{(k+\mathsf{h}^\vee)}. \tag{7.1}$$

In the following section we will use this Morita morphism to obtain a push-forward in twisted K-homology.

8. Quantization of q-Hamiltonian G-spaces

8.1. Twisted K-homology. Recall that a C^*-algebra is a Banach algebra with a conjugate linear involution $*$, isomorphic to a norm closed subalgebra of the algebra $\mathbb{B}(\mathsf{H})$ of bounded operators on a Hilbert space, with $*$ induced by the adjoint. For instance, $\mathbb{K}(\mathsf{H})$ is a C^*-algebra. If $\mathcal{A} \to X$ is a G-equivariant Dixmier-Douady bundle, the space

$$\mathsf{A} = \Gamma_0(X, \mathcal{A})$$

of sections vanishing at infinity (i.e. the closure of the space of sections of compact support) is a G-equivariant C^*-algebra.

DEFINITION 8.1 (Donovan-Karoubi [**24**], Rosenberg [**63**]). The *twisted G-equivariant K-homology* of X with coefficients in $\mathcal{A}$ is defined as

$$K_\bullet^G(X, \mathcal{A}) := K_G^\bullet(\Gamma_0(X, \mathcal{A})),$$

the equivariant K-homology of the G-C^*-algebra $\Gamma_0(X, \mathcal{A})$.

REMARK 8.2. Here we are working with Kasparov's definition of the K-homology of G-C^*-algebras [**46, 47**]. Let us very briefly sketch Kasparov's approach; an excellent reference for this material is the book [**36**] by Higson and Roe. Let A be a $\mathbb{Z}_2$-graded C^* algebra. A *Fredholm module over* A is a $\mathbb{Z}_2$-graded Hilbert space H with a $*$-representation $\pi\colon \mathsf{A} \to \mathbb{B}(H)$, together with an odd element $F \in \mathbb{B}(H)$, s.t. $\forall a \in \mathsf{A}$

(1) $[\pi(a), F] \in \mathbb{K}(H)$,
(2) $(F^2 + I)\pi(a) \in \mathbb{K}(H)$.

Kasparov defines the K-homology group $K^0(\mathsf{A})$ as the set of all Fredholm modules over A, modulo a suitable notion of 'homotopy'. (For $\mathsf{A} = C(X)$ the continuous functions on a compact Hausdorff space, a definition along similar lines had been proposed by Atiyah [**8**].) One puts $K^1(\mathsf{A}) = K^0(\mathsf{A} \otimes \mathbb{C}\mathrm{l}(\mathbb{R}))$. It is a contravariant functor in C^*-algebras, hence $K_\bullet(X) = K^\bullet(C(X))$ is a covariant functor in spaces X. The definition has a straightforward extension to G-C^*-algebras, defining groups $K_G^\bullet(\mathsf{A})$.

The twisted K-homology groups are functorial with respect to Morita morphisms of Dixmier-Douady-bundles.

EXAMPLE 8.3. There is a canonical ring isomorphism $K_0^G(\mathrm{pt}) = R(G)$, where the ring structure on the left hand side is given by push-forward under the map $\mathrm{pt} \times \mathrm{pt} \to \mathrm{pt}$.

EXAMPLE 8.4. Suppose D is an equivariant skew-adjoint odd elliptic differential operator acting on a $\mathbb{Z}_2$-graded Hermitian vector bundle $V = V^+ \oplus V^- \to M$ over a compact manifold M. It has an equivariant index $\mathrm{index}_G(D) := \chi_{\ker D|_{V^+}} - \chi_{\ker D|_{V^-}}$. The pair

$$H = \Gamma_{L^2}(M, V),\ F = \tfrac{D}{\sqrt{1+D^*D}}$$

with the natural action of $C(M)$ defines a K-homology class $[D] \in K_0^G(M)$. The index is a push-forward under the map $p\colon M \to \mathrm{pt}$ to a point:

$$p_*[D] = \mathrm{index}_G(D).$$

EXAMPLE 8.5. Let M be a compact Riemannian G-manifold of even dimension. Then there is a *fundamental class*

$$[M] \in K_0^G(M, \mathbb{C}\mathrm{l}(TM)),$$

represented by the de Rham Dirac operator on $\Gamma(M, \wedge T^*M)$. A Spin_c-structure on M defines a Morita trivialization of $\mathbb{C}\mathrm{l}(TM)$ and a Spin_c-Dirac operator $\not\partial_M$. The class $[\not\partial_M]$ is the image of $[M]$ under the resulting isomorphism $K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(M)$. Thus $\mathbb{C}\mathrm{l}(TM)$ plays the role of an 'orientation bundle' in K-theory. Compare with singular homology: Any compact manifold, regardless of orientation, has a fundamental class in the homology group $H_{\dim M}(M, o_M)$ with coefficients in the orientation bundle $o_M = \det(TM)$. An orientation on M trivializes the bundle o_M, and identifies the fundamental class as an element of $H_{\dim M}(M)$. Recall also that there is an isomorphism $H_{\dim M}(M, o_M) \cong H^0(M, \mathbb{Z})$, taking $[M]$ to 1. Similarly, for an even-dimensional Riemannian manifold there is an isomorphism

$$K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \cong K_G^0(M) \tag{8.1}$$

with equivariant K-theory, taking $[M]$ to the element $1 \in K_G^0(M)$.

EXAMPLE 8.6. Let G be compact, simply connected, and simple. Denote by $\mathcal{A}_G^{(l)} \to G$ a G-Dixmier-Douady bundle at level $l \in \mathbb{Z} \cong H^3(G, \mathbb{Z})$. $K_0^G(G, \mathcal{A}_G^{(l)})$ has a ring structure defined by $(\mathrm{Mult}_G)_*$. (Note that $\mathrm{Mult}_G^* \mathcal{A}^{(l)}$ is Morita isomorphic to $\mathrm{pr}_1^* \mathcal{A}_G^{(l)} \otimes \mathrm{pr}_2^* \mathcal{A}_G^{(l)}$ since the two bundles have the same Dixmier-Douady class; the specific choice of Morita isomorphism is unimportant since $H_G^2(G \times G, \mathbb{Z}) = 0$.) The theorem of *Freed-Hopkins-Teleman* [**29**] shows that for all non-negative integers $k \geq 0$, there is a canonical isomorphism of rings

$$K_0^G(G, \mathcal{A}_G^{(k+\mathsf{h}^\vee)}) \cong R_k(G) \tag{8.2}$$

where $R_k(G)$ is the level k fusion ring (Verlinde ring).

8.2. Quantization as a push-forward. Suppose G is a compact, simple, simply connected Lie group, and (M, ω, Φ) is a level k pre-quantized q-Hamiltonian G-space. The Morita morphism (7.1) defines a push-forward in twisted K-homology,

$$\Phi_* \colon K_0^G(M, \mathbb{C}\mathrm{l}(TM)) \to K_0^G(G, \mathcal{A}_G^{(k+\mathsf{h}^\vee)}).$$

Using the isomorphism (8.1) and the Freed-Hopkins-Teleman result (8.2), we have constructed an $R(G)$-module homomorphism

$$\Phi_* \colon K_0^G(M) \to R_k(G).$$

DEFINITION 8.7. [**57**] The *quantization* of a level k pre-quantized q-Hamiltonian G-space (M, ω, Φ) is the element

$$\mathcal{Q}(M) = \Phi_*(1) \in R_k(G).$$

As shown in [**57**], the quantization of q-Hamiltonian spaces has properties parallel to those for Hamiltonian spaces:

(1) $\mathcal{Q}(M_1 \cup M_2) = \mathcal{Q}(M_1) + \mathcal{Q}(M_2)$,
(2) $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$,
(3) $\mathcal{Q}(M^*) = \mathcal{Q}(M)^*$,
(4) Let $\mathcal{C}$ be the conjugacy class of $\exp(\frac{1}{k}\mu)$, $\mu \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}) = \tau_\mu.$$

Recall the trace $R_k(G) \to \mathbb{Z},\ \tau \mapsto \tau^G$ where $\tau_\mu^G = \delta_{\mu,0}$.

THEOREM 8.8 (Quantization commutes with reduction). *Let (M, ω, Φ) be a level k prequantized q-Hamiltonian G-space. Then*

$$\mathcal{Q}(M)^G = \mathcal{Q}(M/\!\!/G).$$

This result is a combination of the $[Q, R] = 0$ theorem for Hamiltonian loop group actions, due to Alekseev-M-Woodward [**6**], with the localization theorem for q-Hamiltonian spaces, Theorem 8.15 below. In [**6**] the quantization of a Hamiltonian loop group space is essentially *defined* in terms of fixed point data, and the $[Q, R] = 0$ theorem was proved in those terms. On the other hand, Theorem 8.15 identifies the fixed point formula with the more satisfactory definition of $\mathcal{Q}(M)$ as a K-homology push-forward.

Similar to Example 4.6 we have:

EXAMPLE 8.9. Let $\mathcal{C}_i$ be the conjugacy classes of $\exp(\frac{1}{k}\mu_i),\ \mu_i \in P_k$. Then

$$\mathcal{Q}(\mathcal{C}_1 \times \mathcal{C}_2 \times \mathcal{C}_3/\!\!/G) = (\tau_{\mu_1}\tau_{\mu_2}\tau_{\mu_3})^G = N_{\mu_1\mu_2\mu_3}^{(k)}.$$

EXAMPLE 8.10. The double $\mathbf{D}(G) = G \times G$, $\Phi(a,b) = aba^{-1}b^{-1}$ has level k quantization

$$\mathcal{Q}(D(G)) = \sum_{\mu \in P_k} \tau_\mu \tau_\mu^*.$$

REMARK 8.11. The Hamiltonian analogue of the double is the *non-compact* Hamiltonian G-space T^*G, with the cotangent lift of the conjugation action. Any reasonable quantization scheme for non-compact spaces gives

$$\mathcal{Q}(T^*G) = \sum_{\mu \in P_+} \chi_\mu \chi_\mu^*,$$

the character for conjugation action on $L^2(G)$, defined as an element of a completion of $R(G)$.

Since $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$, we also get the quantization of iterated fusions of copies of $D(G)$ and of level k prequantized conjugacy classes $\mathcal{C}_j$. To work out the product, it is convenient to re-write these results in terms of the basis $\tilde{\tau}_\mu$ of $R_k(G) \otimes \mathbb{C}$, where $\tilde{\tau}_\mu(t_\lambda) = \delta_{\lambda,\mu}$:

$$\mathcal{Q}\big(G.\exp(\tfrac{1}{k}\mu)\big) = \tau_\mu = \sum_{\nu \in P_k} \frac{S^*_{\mu,\nu}}{S_{0,\nu}} \tilde{\tau}_\nu.$$

$$\mathcal{Q}(D(G)) = \sum_{\nu \in P_k} \frac{1}{S^2_{0,\nu}} \tilde{\tau}_\nu$$

Using $\mathcal{Q}(M_1 \times M_2) = \mathcal{Q}(M_1)\mathcal{Q}(M_2)$ this gives

PROPOSITION 8.12. *Let $\mu_1, \ldots, \mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. Then the level k quantization of $D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r$ is given by the formula,*

$$\mathcal{Q}\Big(D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r\Big) = \sum_{\nu \in P_k} \frac{S^*_{\mu_1,\nu} \cdots S^*_{\mu_r,\nu}}{S^{2g+r}_{0,\nu}} \tilde{\tau}_\nu$$

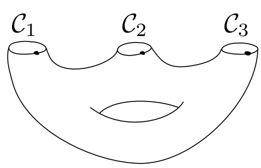
Hence, using the q-Hamiltonian 'quantization commutes with reduction' theorem, we obtain,

THEOREM 8.13 (Symplectic Verlinde formulas). *Let $\mu_1, \ldots, \mu_r \in P_k$, and $\mathcal{C}_j = G.\exp(\frac{1}{k}\mu_j)$. The level k quantization of the moduli space*

$$\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r) = (D(G)^g \times \mathcal{C}_1 \times \cdots \times \mathcal{C}_r)/\!\!/G$$

is given by the formula

$$\mathcal{Q}\Big(\mathcal{M}(\Sigma_g^r, \mathcal{C}_1, \ldots, \mathcal{C}_r)\Big) = \sum_{\nu \in P_k} S_{\mu_1,\nu} \cdots S_{\mu_r,\nu}\ S_{0,\nu}^{-(2g+r-2)}$$



REMARK 8.14. The choice of a complex structure on Σ, compatible with the orientation, defines a Kähler structure on the moduli space, and its pre-quantization is given by a holomorphic line bundle (using an appropriate interpretation in case the moduli space is singular). The more common setting for the Verlinde formulas

is as the dimension of the space of holomorphic sections of the pre-quantum line bundle (i.e. the Kähler quantization). Provided the higher cohomology groups vanish, this dimension equals the index computed above.

8.3. Localization. As in the case of Hamiltonian spaces, the main technique for actually calculating the quantization of q-Hamiltonian spaces is by localization. Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. Since the pull-back of the Cartan 3-form $\eta \in \Omega^3(G)$ to the maximal torus $T \subset G$ vanishes, the map in cohomology $H^3(G, \mathbb{R}) \to H^3(T, \mathbb{R})$ is the zero map. Due to the absence of torsion, this is also true with integer coefficients, proving that $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$ is Morita trivial. In fact, by considering the pull-back of the generator of $H^3_G(G, \mathbb{Z}) = \mathbb{Z}$ to a class in $H^3_T(T, \mathbb{Z})$ (see [**57**, Section 5.1]), one finds that it is $T_{k+\mathsf{h}^\vee}$-equivariantly Morita trivial, where $T_{k+\mathsf{h}^\vee} \subset T$ is the finite subgroup generated by the elements $t_\lambda, \lambda \in P_k$. (Note that while the conjugation action of $T_{k+\mathsf{h}^\vee}$ on T is trivial, there is still a non-trivial action on $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_T$.) Let us choose any such Morita trivialization, with the additional property that the resulting Morita trivialization of $\mathcal{A}^{(k+\mathsf{h}^\vee)}|_e$ is G-equivariant. Even with this additional normalization the choice is not quite canonical: One may still twist by a line bunde over T with a trivial T-action.

Suppose now that $t \in T$ is a regular element (i.e. $G_t = T$), and let (M, ω, Φ) be a q-Hamiltonian G-space. If $F \subset M^t$ is a component of the fixed point set, then $\Phi(F) \subseteq T$, by T-equivariance of the moment map. By composition, we obtain a $T_{k+\mathsf{h}^\vee}$-equivariant Morita morphism

$$\mathbb{C}\mathrm{l}(TM|_F) \dashrightarrow \mathcal{A}^{(k+\mathsf{h}^\vee)}|_T \dashrightarrow \mathbb{C},$$

or equivalently a $T_{k+\mathsf{h}^\vee}$-equivariant Spin_c-structure on $TM|_F$. Thus, even though M itself does not carry a global Spin_c-structure, one does have Spin_c-structures along the fixed point manifolds. Consequently, the fixed point contributions from the equivariant index theorem for Spin_c-Dirac operators are well-defined, even though there is no globally defined operator.

We specialize to the case $t = t_\lambda,\ \lambda \in P_k$. Recall again (Section 5) that the evaluation of elements $\tau \in R_k(G)$ at the points t_λ is well-defined, and τ can be recovered from the values $\tau(t_\lambda)$.

THEOREM 8.15. *Let (M, ω, Φ) be a level k pre-quantized q-Hamiltonian G-space. For $\lambda \in P_k$,*

$$\mathcal{Q}(M)(t_\lambda) = \sum_{F \subset M^{t_\lambda}} \int_F \frac{\widehat{A}(F)\,\mathrm{Ch}(\mathcal{L}_F, t_\lambda)^{1/2}}{D_{\mathbb{R}}(\nu_F, t_\lambda)}$$

where $\mathcal{L}_F$ is the Spin_c*-line bundle for $TM|_F$.*

REMARK 8.16. As mentioned before, in [**6**] quantization of a Hamiltonian loop group space was *defined* in terms of fixed point data, and this definition was used to establish its main properties. However, it was unclear in [**6**] what the 'equivariant object' might be of which the right hand side of this formula are the localization contributions. The definition of $\mathcal{Q}(M)$ as a push-forward in twisted equivariant K-homology provides an answer to this question, avoiding the use of infinite-dimensional manifolds and loop groups.

As a typical application of Theorem 8.15, consider the case $M = D(G)^h = G^{2h}$. Since the G-action on M is just conjugation, and t_λ is regular, the fixed point set is simple $F = T^{2h} \subset G^{2h}$, with $\Phi(T^{2h}) = \{e\}$ and with a trivial normal bundle $\nu_F = (\mathfrak{g}/\mathfrak{t})^{2h}$. Since the geometry is so simple, the evaluation of the fixed points contributions poses no problems. See [**6**] or [**57**] for the calculation.

REMARK 8.17. Since G is compact and simply connected, its center $Z(G)$ is finite. Let $Z \subset Z(G)$ be a central subgroup, and $G' = G/Z$. The 2-form and moment map for the fused double $\mathbf{D}(G) = G \times G$ (cf. Section 3.1.2) are $Z \times Z$-invariant, hence they descend to give the structure of a q-Hamiltonian G-space on

$$\mathbf{D}(G') = G' \times G', \quad \Phi \colon \mathbf{D}(G') \to G.$$

Put differently, the q-Hamiltonian G'-space $\mathbf{D}(G')$ may be regarded as a q-Hamiltonian G-space, using the canonical lifting of the moment map. The symplectic quotients $\mathbf{D}(G') \times \cdots \times \mathbf{D}(G') /\!\!/ G$ may be interpreted as moduli spaces of flat G'-bundles over closed surfaces. More generally, one can consider surfaces with boundary, and moduli spaces of flat G'-bundles with prescribed holonomies around boundary circles. The quantizations of thus spaces are expected to be given by Verlinde-type formulas, conjectured by Fuchs-Schweigert [**30**]. However, pre-quantizibility conditions, and the evaluation of the fixed point contributions, are much more involved than in the case of simply connected groups. For $G' = \mathrm{SO}(3)$ the results are worked out in full generality in Krepski-M [**53**]; the higher rank case will be considered in forthcoming work.

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Quantization and automorphic forms

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Abstract. This short survey is based on my talk given at the Conference on Mathematical Aspects of Quantization at the Center for Mathematics at Notre Dame in June 2011.

1. Introduction

The relationship between automorphic forms and quantization is a recurrent topic in quantization literature. See, for example, monographs [**Hu**], [**P**], [**U1**], [**U2**].

The main purpose of this paper is to give a brief explanation (which, hopefully, is easy to understand) of how exactly automorphic forms appear in Berezin-Toeplitz quantization. We also mention some results that provide explicit constructions of automorphic forms via Poincaré series. At the end of the paper there are some remarks on deformation quantization.

This survey emphasizes the complex geometry point of view rather than representation-theoretic or arithmetic or analytic aspects of the relationship between automorphic forms and quantization.

Section 2 contains relevant definitions and facts from the theory of automorphic forms. The discussion that involves quantization is in Section 3.

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2. Automorphic forms

2.1. One complex variable. The upper half plane $H = \{z \in \mathbb{C} \mid Im(z) > 0\}$ is biholomorphic to the unit disc $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. The group $SL(2,\mathbb{R})$ acts on H by fractional-linear transformations: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$, $z \in H$

$$\gamma : z \mapsto \gamma z = \frac{az+b}{cz+d}$$

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(we denote the automorphism $H \to H$ defined by a matrix γ by the same letter, γ, to simplify notations). Note that the complex Jacobian of γ

$$J(\gamma, z) = \frac{d(\gamma z)}{dz} = \frac{1}{(cz+d)^2}.$$

The group $SL(2,\mathbb{R})$ is isomorphic to the group $SU(1,1) = \{\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C}, |a|^2 - |b|^2 = 1\}$ which acts on $\mathcal{D}$ by

$$z \mapsto \gamma z = \frac{az+b}{\bar{b}z+\bar{a}}$$

$z \in \mathcal{D}$, $\gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in SU(1,1)$. Note that the complex Jacobian is $1/(\bar{b}z+\bar{a})^2$.

Let Γ be a discrete subgroup of $SL(2,\mathbb{R})$ such that the quotient $X = \Gamma\backslash H$ is smooth and compact. Thus X is a compact Riemann surface of genus $g > 1$.

Denote by K_H the holomorphic cotangent bundle on H, by $K_{\mathcal{D}}$ the holomorphic cotangent bundle on $\mathcal{D}$, and by K_X the holomorphic cotangent bundle on X.

Let k be a non-negative integer.

DEFINITION 2.1. A function $f : H \to \mathbb{C}$ is called a **(holomorphic) Γ-automorphic form of weight** k if f is holomorphic and satisfies

$$f(\gamma z) = f(z)(cz+d)^{2k} \ \forall \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, z \in H. \tag{2.1}$$

Condition (2.1) is often called *the automorphy law* and can be restated as

$$f(\gamma z)J(\gamma, z)^k = f(z) \ \forall \gamma \in \Gamma, z \in H.$$

The assumption that f is holomorphic may be modified (e.g. it can be replaced by the requirement that f is an eigenfunction of the hyperbolic Laplacian). The assumption that X is smooth and compact can be removed (in fact, this has to be done in order to consider the case when Γ is $SL(2,\mathbb{Z})$ or a congruence subgroup), but then another condition on f is usually added. Also, instead of $(cz+d)^{2k}$ the automorphy factor can be taken to be $(cz+d)^k\chi(\gamma)$, where $\chi : \Gamma \to \mathbb{C}$ is a normalized character on Γ. See, for example, [**B**] and [**Bu**] for details.

If $\Gamma = SL(2,\mathbb{Z})$ then the term "automorphic form" is usually replaced by "modular form". Also, if $k = 0$ then the terms "automorphic function" and "modular function" are used.

Finally, let's observe that a holomorphic function f satisfies (2.1) if and only if the holomorphic k-differential $f(z)dz^k$ is Γ-invariant. A holomorphic Γ-invariant section $f(z)dz^k$ of $K_H^{\otimes k}$ can also be viewed as a holomorphic section of $K_X^{\otimes k}$, i.e. an element of the complex vector space $H^0(X, K_X^{\otimes k})$ (which has dimension g if $k = 1$ and dimension $k(2g-2)-(g-1)$ for $k \geq 2$).

Here is a quick example of a function that satisfies an automorphy law.

EXAMPLE 2.2. Let Γ be the cyclic subgroup generated by the matrix $\gamma_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ that corresponds to the transformation $H \to H$ defined by $z \mapsto z+1$. The quotient $X = \Gamma\backslash H$ is an infinite cylinder. The automorphy law (2.1) becomes the condition $f(z+1) = f(z) \ \forall z \in H$. Note that k does not play any role. The function $f(z) = e^{2\pi i z}$ is a holomorphic function on H that satisfies the automorphy law, and $e^{2\pi i z} dz^k$ is a holomorphic Γ-invariant section of $K_H^{\otimes k}$. $\square$

In this example the quotient $\Gamma \backslash H$ is of infinite volume. Explicit examples of Γ that provide compact quotients can be found in [**I**], [**K**], see also discussion and references in [**VGS**].

The definition of an automorphic form is similar for the disk:

DEFINITION 2.3. A function $f : \mathcal{D} \to \mathbb{C}$ is called a **(holomorphic) Γ-automorphic form of weight** k if f is holomorphic and satisfies

$$f(\gamma z) = f(z)(\bar{b}z + \bar{a})^{2k} \ \forall \gamma = \begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix} \in \Gamma, z \in \mathcal{D}. \tag{2.2}$$

As before, a holomorphic function f satisfies (2.2) if and only if the holomorphic k-differential $f(z)dz^k$ is Γ-invariant.

2.2. Several complex variables. For $n \geq 2$ there are several kinds of automorphic forms of n complex variables. In this note we shall consider only $\mathbb{C}$-valued holomorphic automorphic forms on irreducible bounded symmetric domains.

Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$ (n is a positive integer). Thus $D = G/K$, where G is a real semisimple Lie group and K is a maximal compact subgroup of G. There is a classification of bounded symmetric domains (four series corresponding to the classical matrix Lie groups plus two exceptional domains). For $n = 1$, D is isomorphic to the unit disc $SU(1,1)/U(1)$ (i.e., informally speaking, there is only one bounded symmetric domain for $n = 1$). The action of G on D is induced by the action of G on itself by left multiplication. Let's denote an element of G and the corresponding automorphism $D \to D$ by the same letter. For $\gamma \in G$ $J(\gamma, z)$ will denote the determinant of the $n \times n$ Jacobi matrix $(\frac{\partial(\gamma z)_j}{\partial z_k})$.

EXAMPLE 2.4. Let $G = SU(n,1) = \{A \in SL(n+1, \mathbb{C}) \mid A^t \sigma \bar{A} = \sigma\}$, $K = S(U(n) \times U(1))$, where $\sigma = \begin{pmatrix} 1_{n \times n} & 0 \\ 0 & -1 \end{pmatrix}$. Then D is the unit ball in $\mathbb{C}^n$. It is also called the n-dimensional complex hyperbolic space. G acts on D by fractional-linear transformations: for $\gamma = (a_{jk})$

$$\gamma : D \to D$$

$$z = (z_1, ..., z_n) \mapsto (\frac{a_{11}z_1 + ... + a_{1n}z_n + a_{1,n+1}}{a_{n+1,1}z_1 + ... + a_{n+1,n}z_n + a_{n+1,n+1}}, ...,$$

$$(\frac{a_{n1}z_1 + ... + a_{nn}z_n + a_{n,n+1}}{a_{n+1,1}z_1 + ... + a_{n+1,n}z_n + a_{n+1,n+1}}).$$

and $J(\gamma, z) = 1/(a_{n+1,1}z_1 + ... + a_{n+1,n}z_n + a_{n+1,n+1})^{n+1}$. $\square$

Let Γ be a discrete subgroup of G such that the quotient $X = \Gamma \backslash D = \Gamma \backslash G/K$ is smooth and compact. There are explicit examples of such manifolds X for any n. D has a G-invariant Kähler structure, so X is naturally a compact Kähler manifold. Let K_D denote the canonical bundle of D, i.e. $K_D = \bigwedge^n T^{*'} D$, where $T^{*'} D$ is the holomorphic cotangent bundle. Let K_X denote the canonical bundle on X. Denote by ω the Kähler form on X, normalized so that $c_1(K_X) = [\omega]$.

As before, let k be a non-negative integer.

DEFINITION 2.5. ([**Ba**], [**PS**]) A function $f : D \to \mathbb{C}$ is called a **(holomorphic) Γ-automorphic form of weight** k if f is holomorphic and satisfies

$$f(\gamma z)J(\gamma, z)^k = f(z) \ \forall \gamma \in \Gamma, z \in D. \tag{2.3}$$

As in the case $n = 1$, the assumptions of f being holomorphic and X being smooth and compact can be relaxed. Also, f is a holomorphic Γ-automorphic form of weight k on D if and only if the corresponding holomorphic section $f(z)(dz_1 \wedge \ldots \wedge dz_n)^{\otimes k}$ of $K_D^{\otimes k}$ is Γ-invariant. Thus, there is a natural isomorphism between $H^0(X, K_X^{\otimes k})$ and the space $S_k(\Gamma)$ of holomorphic Γ-automorphic forms of weight k on D.

2.3. Poincaré series. A well-known way to construct automorphic forms is via Poincaré series. For a function $F : D \to \mathbb{C}$ the associated Poincaré series is *formally* defined as

$$\Theta_F^{(k)}(z) = \sum_{\gamma \in \Gamma} F(\gamma z) J(\gamma, z)^k. \tag{2.4}$$

If F is holomorphic and integrable (i.e. F is in the weighted Bergman space $A^{(k)}(D)$), and k is sufficiently large (e.g. for $n = 1$, $k \geq 2$), then the series converges absolutely and uniformly on compact sets, $\Theta_F^{(k)}$ is in $S_k(\Gamma)$, and the Poincaré series map $A^{(k)}(D) \to S_k(\Gamma)$ is surjective.

REMARK 2.6. The essense of the idea utilized in (2.4) can be seen in the following simple observation. Suppose N is a finite group that acts on a set S. How can one construct an N-invariant (complex-valued) function on S ? Take any function $F : S \to \mathbb{C}$ and define a new function as follows: $\theta_F(x) = \sum_{g \in N} F(gx)$, $x \in S$. It's easy to see that θ_F is N-invariant.

Similarly one can begin with an integrable holomorphic section s of $K_D^{\otimes k}$ and construct a Γ-invariant holomorphic section of $K_D^{\otimes k}$ as $\sum_{\gamma \in \Gamma} s(\gamma z)$ (assuming that k is sufficiently large, to ensure convergence). Representing sections of $K_D^{\otimes k}$ by functions on D, one obtains the Poincaré series (2.4). $\square$

REMARK 2.7. There is a modification of this construction, called *relative Poincaré series*, with the summation taken over $\Gamma_0 \backslash \Gamma$, where Γ_0 is a subgroup of Γ. $\square$

The following theorem, due to Petersson, tells us how to construct explicitly all automorphic forms for $n = 1$, $k \geq 2$.

THEOREM 2.8. ([**Pe**], [**H**]) *Let $\Gamma \subset SU(1,1)$ be such that $X = \Gamma \backslash \mathcal{D}$ is smooth and compact and let $k \geq 2$. Let g be the genus of X. Then any holomorphic Γ-automorphic form of weight k on $\mathcal{D}$ is $\sum_{\gamma \in \Gamma} p(\gamma z) J(\gamma, z)^k$, where p is a polynomial in z of degree $\leq k(2g - 2)$.*

The following generalization of this statement was proven in [**F2**]:

THEOREM 2.9. *Let D, Γ be as above and k be sufficiently large. Let $f \in S_k(\Gamma)$. Then $f(z) = \sum_{\gamma \in \Gamma} p(\gamma z) J(\gamma, z)^k$ where $p = p(z)$ is a polynomial in $z_1, .., z_n$ of degree not higher than k^n $vol_{\omega^n}(X)$.*

Note that for $n = 1$ the upper bound for the degree of the polynomial in Theorem 2.9 is $k(2g - 2)$, i.e. we recover Theorem 2.8.

3. Quantization

3.1. Automorphic forms as quantum-mechanical states. Recall that the setting of Berezin-Toeplitz quantization (which is also called Kähler quantization) is as follows. Let X be a compact Kähler manifold. Assume that the Kähler form ω is integral. Let $L \to X$ be a holomorphic Hermitian line bundle such that the curvature of the Hermitian connection is equal to $-2\pi i\omega$ (we observe: it follows that $c_1(L) = [\omega]$ and also that L is ample). X, with the symplectic form on it, is viewed as the classical phase space. The complex vector space $H^0(X, L^{\otimes k})$ is interpreted as the space of quantum-mechanical wave functions. The positive integer k is formally regarded as $1/\hbar$, where $\hbar$ is the Planck constant, and the limit $k \to \infty$ is called the semi-classical limit. "Quantization" means a way to associate a linear operator T_f to a function $f \in C^\infty(X)$ so that a version of Dirac's quantization conditions holds. The quantum observables $T_f^{(k)}$ are Berezin-Toeplitz operators acting on $H^0(X, L^{\otimes k})$. See [**Bo**], [**S2**] for details.

Note that Berezin-Toeplitz quantization is, in a sense, a version of geometric quantization. In geometric quantization the manifold is symplectic but not necessarily Kähler. See [**T**], [**BMS**], [**S2**] for the relation between operators of geometric quantization and Berezin-Toeplitz operators.

In the context of Section 2 the compact Kähler manifold X is $\Gamma\backslash D$, $L = K_X$ and the space of quantum-mechanical wave functions is $S_k(\Gamma)$.

Interpretation of wave functions as automorphic forms leads to a more explicit understanding of the space of wave functions (comparing to other situations of Kähler quantization), and, perhaps, gives a better control over analytic properties of the wave functions.

3.2. More on Poincaré series. Relative Poincaré series in the context of Berezin-Toeplitz quantization appeared in [**BPU**], where certain sequences of sections $u_k \in H^0(X, L^{\otimes k})$ were studied. Here $L \to X$ is a quantizing line bundle on an integral compact Kähler manifold X (not necessarily a quotient of a bounded symmetric domain). These sections have meaning of "delta-functions" in appropriate sense and are associated to Bohr-Sommerfeld Lagrangian submanifolds of X, equipped with a half-form. The semiclassical asymptotics obtained in this paper provided valuable information about relative Poincaré series for $n = 1$. Later these asymptotics were used in [**F1**], [**F3**], to study relative Poincaré series on compact smooth quotients of the unit ball $SU(n, 1)/S(U(n) \times U(1))$.

It was mentioned earlier that the Poincaré series map is a useful tool for constructing automorphic forms. There are a number of results that give conditions under which the Poincaré series of a given function is not identically zero (non-vanishing problem) or, alternatively, results that provide information about the kernel of the Poincaré series operator. For $n = 1$ see, in particular, [**K1**], [**K2**], [**L**], [**Ma**], [**Me**].

In a recent paper [**AF**] we worked in the following setting.

Let Σ be a hyperbolic Riemann surface, with a covering map $\pi : \mathcal{D} \to \Sigma$. Let Λ be a closed subset of Σ such that $\pi^{-1}(\Sigma - \Lambda)$ is connected. Denote $V = \Sigma - \Lambda$ and $U = \mathcal{D} - \pi^{-1}(\Lambda)$. Let $k \geq 2$ be an integer.

In this paper we study spaces of integrable, square-integrable, and bounded holomorphic k-differentials on V.

For example, we observe that if Λ is finite, then the space of integrable holomorphic sections of $K_V^{\otimes k}$ is isomorphic to the space of integrable meromorphic sections of $K_\Sigma^{\otimes k}$, with at most simple poles, all in Λ.

One of the main results of this paper is a description of the kernel of the Poincaré series operator $\Theta : A^{(1)}(U) \to A^{(1)}(V)$, where $A^{(1)}(U)$ (resp. $A^{(1)}(V)$) denotes the space of integrable holomorphic sections of $K_U^{\otimes k}$ (resp. of $K_V^{\otimes k}$).

The interpretation in terms of quantization is as before: V is the classical phase space, $A^{(1)}(V)$ is the space of wave functions (or we can say that $\mathbb{P}(A^{(1)}(V))$ is the space of quantum-mechanical states), k is $1/\hbar$. Note that if Λ is non-empty then V is non-compact. Hence, this paper provides information about the space of quantum states for a class of non-compact phase spaces. From the point of view of physics the case when the classical phase space is non-compact is most interesting.

3.3. Automorphic forms and deformation quantization. Berezin-Toeplitz quantization is closely related to deformation quantization. Recall that the goal of deformation quantization is to construct a star product on an algebra of functions on X. There is a star product on $C^\infty(X)$ defined via symbols of Berezin-Toeplitz operators [**S1**].

Another situation when automorphic forms appear in the context of deformation quantization arises in the discussion of Rankin-Cohen brackets [**Z**], [**CMZ**].

Let $n = 1$ and $\Gamma = SL(2,\mathbb{Z})$. Denote by M_k the space of weight k modular forms. The j-th Rankin-Cohen bracket of two modular forms $f \in M_r$ and $g \in M_s$ is

$$[f,g]_j = \sum_{l=0}^{j} (-1)^l \binom{2r+j-1}{j-l} \binom{2s+j-1}{l} \frac{d^l f}{dz^l} \frac{d^{j-l} g}{dz^{j-l}} \in M_{r+s+j}.$$

Rankin-Cohen brackets define a star product on the algebra of modular forms $\oplus_{k=0}^\infty M_k$. It was shown in [**BTY**] that this star product is equivalent to the Moyal product. In [**P1**] it is explained how the Rankin-Cohen brackets arise in the framework of quantization of coadjoint orbits. One possible higher-dimensional generalization is presented in [**DP**]. See also surveys [**P2**] and [**P3**].

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Noncommutative Poisson structures, derived representation schemes and Calabi-Yau algebras

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Abstract. In this paper, we introduce and study the notion of a derived Poisson structure on an associative algebra A. This structure is characterized by the property of being the weakest structure on A that induces natural (graded commutative) Poisson structures on the derived moduli spaces of finite-dimensional representations of A. A derived Poisson structure is represented by a graded (super) Lie algebra bracket on the cyclic homology $\mathrm{HC}_\bullet(A)$ and can be viewed as a higher homological extension of the H_0-Poisson structure introduced by W. Crawley-Boevey (2011). In the second part of the paper, we construct a large class of examples of derived Poisson structures arising from finite-dimensional n-cyclic coalgebras. These examples include linear duals of finite-dimensional n-cyclic algebras which are n-Calabi-Yau categories in the sense of Kontsevich and Soibelman (2009).

1. Introduction

Recall that a Poisson structure on a commutative algebra A is a Lie bracket $\{-,-\} : A \times A \to A$ satisfying the Leibniz rule $\{a, bc\} = b\{a, c\} + \{a, b\}c$ for all $a, b, c \in A$. For noncommutative algebras, this definition is known to be too restrictive: if A is a noncommutative domain (more generally, a prime ring), any Poisson bracket on A is a multiple of the commutator $[a, b] = ab - ba$ (see [FL], Theorem 1.2). Motivated by recent work on noncommutative geometry (see [**Ko, G, BL, CBEG, VdB**]), Crawley-Boevey [**CB**] proposed a different notion of the Poisson structure on an associative algebra A that agrees with the above definition for commutative algebras and has surprisingly nice categorical properties. His idea was to find the weakest structure on A that induces natural Poisson structures on the moduli spaces of finite-dimensional semisimple representations of A. It turns out that such a weak Poisson structure is given by a Lie bracket on the 0-th cyclic homology $\mathrm{HC}_0(A) = A/[A, A]$ satisfying some extra conditions; it is thus called an H_0-*Poisson structure* in [**CB**]. The very terminology of [**CB**] suggests that there might exist a 'higher' homological extension of this construction. The aim of the present paper is to show that this is indeed the case: our main construction yields

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a graded (super) Lie algebra structure on the full cyclic homology of A:

$$\{-,-\}:\ \mathrm{HC}_\bullet(A)\times\mathrm{HC}_\bullet(A)\to\mathrm{HC}_\bullet(A)$$

that satisfies certain properties and restricts to Crawley-Boevey's H_0-Poisson structure on $\mathrm{HC}_0(A)$. We call such structures the *derived Poisson structures* on A.

To explain our results in more detail we first recall the main theorem of [**CB**]. Let A be an associative unital algebra and let V be a finite-dimensional vector space, both defined over a field k of characteristic zero. The classical representation scheme $\mathrm{Rep}_V(A)$ parametrizing the k-linear representations of A in V can be defined as a functor on the category of commutative algebras:

$$\mathrm{Rep}_V(A):\ \mathtt{Comm\,Alg}_k\to\mathtt{Sets}\ ,\quad B\mapsto\mathrm{Hom}_{\mathtt{Alg}_k}(A,\,B\otimes_k\mathrm{End}\,V)\ . \tag{1}$$

It is well known that (1) is representable, and we denote the corresponding commutative algebra by $k[\mathrm{Rep}_V(A)]$. The group $\mathrm{GL}(V)$ acts naturally on $\mathrm{Rep}_V(A)$, with orbits corresponding to the isomorphism classes of representations. The closed orbits correspond to the classes of semisimple representations and are parametrized by the affine quotient scheme $\mathrm{Rep}_V(A)/\!/\mathrm{GL}(V)=\mathrm{Spec}\,k[\mathrm{Rep}_V(A)]^{\mathrm{GL}(V)}$ (see, for example, [**K**]). Now, there is a natural trace map

$$\mathrm{Tr}_V:\ \mathrm{HC}_0(A)\to k[\mathrm{Rep}_V(A)]^{\mathrm{GL}(V)} \tag{2}$$

defined by taking characters of representations. In terms of (2), we may state the main result of [**CB**] as

THEOREM 1 ([**CB**], Theorem 1.6). *Given an H_0-Poisson structure on A, for each V, there exists a unique Poisson structure on $\mathrm{Rep}_V(A)/\!/\mathrm{GL}(V)$ satisfying*

$$\{\mathrm{Tr}_V(a),\,\mathrm{Tr}_V(b)\}=\mathrm{Tr}_V(\{a,\,b\})\ ,\quad\forall\ a,b\in\mathrm{HC}_0(A)\ .$$

Our generalization of Theorem 1 is based on results of the recent paper [**BKR**], where the trace map (2) is extended to higher cyclic homology. We briefly review these results referring the reader to [**BKR**] (and Section 2 below) for details. Varying A (while keeping V fixed) one can regard the representation functor (1) as a functor on the category $\mathtt{Alg}_k$ of algebras. This functor can then extended to the category $\mathtt{DGA}_k$ of differential graded (DG) algebras, and the scheme $\mathrm{Rep}_V(A)$ can be derived by replacing A by its cofibrant resolution in $\mathtt{DGA}_k$. The fact that the result is independent of the choice of resolution was first proved in [**CK**]. In [**BKR**], we gave a more conceptual proof, using Quillen's theory of model categories [**Q1**], and found a simple algebraic construction for the total derived functor of Rep_V. When applied to A, this derived functor is represented (in the homotopy category of DG algebras) by a commutative DG algebra $\mathrm{DRep}_V(A)$. The homology of $\mathrm{DRep}_V(A)$ depends only on A and V, with $\mathrm{H}_0[\mathrm{DRep}_V(A)]$ being isomorphic to $k[\mathrm{Rep}_V(A)]$. Following [**BKR**], we will refer to $\mathrm{H}_\bullet[\mathrm{DRep}_V(A)]$ as the *representation homology* of A and denote it by $\mathrm{H}_\bullet(A,V)$. The action of $\mathrm{GL}(V)$ on $\mathrm{Rep}_V(A)$ extends naturally to $\mathrm{DRep}_V(A)$, and we have an isomorphism of graded algebras $\mathrm{H}_\bullet[\mathrm{DRep}_V(A)^{\mathrm{GL}(V)}]\cong\mathrm{H}_\bullet(A,V)^{\mathrm{GL}(V)}$. Now, one of the key results of [**BKR**] is the construction of canonical trace maps

$$(\mathrm{Tr}_V)_n:\ \mathrm{HC}_n(A)\to\mathrm{H}_n(A,V)^{\mathrm{GL}(V)}\ ,\quad\forall\,n\geq 0\ , \tag{3}$$

extending (2) to the higher cyclic homology[1]. In terms of (3), the main result of the present paper may be stated as a direct generalization of Theorem 1.

[1] We will review this construction in Section 2 below.

THEOREM 2. *Given a derived Poisson structure on A, for each V, there is a unique graded Poisson bracket on the graded commutative algebra* $\mathrm{H}_\bullet(A,V)^{\mathrm{GL}(V)}$ *such that*

$$\{(\mathrm{Tr}_V)_\bullet(\alpha),\,(\mathrm{Tr}_V)_\bullet(\beta)\} = (\mathrm{Tr}_V)_\bullet(\{\alpha,\,\beta\})\ ,\quad \forall\ \alpha,\beta\in\mathrm{HC}_\bullet(A)\ .$$

In fact, we will prove a more refined result (Theorem 10), of which Theorem 2 is an easy consequence. Our key observation is that, when extended properly to the category of DG algebras, the weak Poisson structures behave well with respect to homotopy (in the sense that the homotopy equivalent Poisson structures on A induce, via the derived representation functor, homotopy equivalent DG Poisson algebra structures on $\mathrm{DRep}_V(A)$). Working in the homotopy-theoretic framework allows us to give a precise meaning to the claim that the derived Poisson structures are indeed the *weakest* structures on A inducing the usual (graded) Poisson structures under the representation functor (see Remark 3.3).

The paper is organized as follows. In Section 2, we review basic definitions and results of [**BKR**] and [**BR**] needed for the present paper. In Section 3, we extend Crawley-Boevey's definition of a NC Poisson structure to the category of DG algebras and introduce a relevant notion of homotopy for such structures. We also prove our first main result (Theorem 10) in this section. In Section 4, we then propose the definition of a noncommutative P_∞-algebra extending the results of Section 3 to strong homotopy algebras. We show that a noncommutative P_∞-algebra structure on A induces a P_∞-structure on $\mathrm{DRep}_V(A)$ and that the homotopy equivalent noncommutative P_∞-algebra structures induce homotopy equivalent P_∞-structures on $\mathrm{DRep}_V(A)$. This result is part of Theorem 12, which is the second main result of this paper. The proof of Theorem 12 is parallel to the proof of Theorem 10, however the calculations are technically more complicated. Finally, Section 5 provides an interesting class of examples of derived Poisson structures. These examples arise from n-cyclic coalgebras (through Van den Bergh's double bracket construction) and include, in particular, linear duals of finite-dimensional n-cyclic algebras. The main result of Section 5 – Theorem 15 – shows that there is a natural double Poisson algebra structure on the cobar construction of any cyclic coassociative DG coalgebra. The finite-dimensional n-cyclic algebras are known to be a special case of n-Calabi-Yau categories in the sense of [**KS, Cos**]; our results imply that these algebras carry noncommutative $(2-n)$-Poisson structures. We conclude with a few remarks on string topology clarifying the relation of the present paper to the recent work of two of the current authors (X. Ch. and F. E.) with W. L. Gan (see [**CEG**]).

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Notation and Conventions

Throughout this paper, k denotes a base field of characteristic zero. Unless stated otherwise, all differential graded (DG) objects (complexes, algebras, modules, etc.) are equipped with differentials of degree -1. The Koszul sign rule is always assumed when we operate with such objects: whenever two DG maps (or operations) of degrees p and q are permuted, the sign is multiplied by $(-1)^{pq}$.

2. The Derived Representation Functor and Higher Trace Maps

In this section, we review basic definitions and results of [**BKR**] and [**BR**]. Our aim is to give a short and readable survey which goes slightly beyond the preliminaries for the present paper. This survey evolved from notes of the talk given by the first author at the Notre Dame conference on quantization.

2.1. Representation functors. Let $\mathtt{DGA}_k$ be the category of associative unital DG k-algebras, and let $\mathtt{CDGA}_k$ be its full subcategory consisting of commutative DG algebras. The inclusion functor $\mathtt{CDGA}_k \hookrightarrow \mathtt{DGA}_k$ has a natural left adjoint which assigns to a DG algebra A its maximal commutative quotient; we denote it by

$$(4)\qquad (-)_{\natural\natural}\,:\ \mathtt{DGA}_k \to \mathtt{CDGA}_k\ ,\quad A \mapsto A/\langle [A,A] \rangle\ .$$

Now, given a finite-dimensional k-vector space V, we introduce the following functor

$$(5)\qquad \sqrt[V]{-}\ :\ \mathtt{DGA}_k \to \mathtt{DGA}_k\ ,\quad A \mapsto (A *_k \operatorname{End} V)^{\operatorname{End} V}\ .$$

Here $A *_k \operatorname{End} V$ denotes the free product of A with the endomorphism algebra of V and $(-)^{\operatorname{End} V}$ stands for the centralizer of $\operatorname{End} V$ as the subalgebra in that free product. Combining (4) and (5), we define

$$(6)\qquad (-)_V\,:\ \mathtt{DGA}_k \to \mathtt{CDGA}_k\ ,\quad A \mapsto A_V := (\sqrt[V]{A})_{\natural\natural}\ .$$

Theorem 3 ([**BKR**], Theorem 2.2). *For any $A \in \mathtt{DGA}_k$, the DG algebra A_V represents the functor*

$$\operatorname{Rep}_V(A)\,:\ \mathtt{CDGA}_k \to \mathtt{Sets}\ ,\quad B \mapsto \operatorname{Hom}_{\mathtt{DGA}_k}(A,\, B \otimes_k \operatorname{End} V)\ .$$

Theorem 3 implies that there is a bijection

$$(7)\qquad \operatorname{Hom}_{\mathtt{CDGA}_k}(A_V,\, B) = \operatorname{Hom}_{\mathtt{DGA}_k}(A,\, B \otimes_k \operatorname{End} V)\ ,$$

functorial in $A \in \mathtt{DGA}_k$ and $B \in \mathtt{CDGA}_k$. Informally, it suggests that $A_V = k[\operatorname{Rep}_V(A)]$ should be thought of as a DG algebra of functions on the affine DG scheme parametrizing the representations of A in V. Letting $B = A_V$ in (7), we get a canonical DG algebra homomorphism

$$(8)\qquad \pi_V\ :\ A \to A_V \otimes \operatorname{End} V\ ,$$

which is the universal representation of A. Furthermore, for $g \in \mathrm{GL}_k(V)$, we have a unique automorphism of A_V corresponding under the adjunction (7) to the composite map

$$A \xrightarrow{\pi_V} A_V \otimes \operatorname{End} V \xrightarrow{\mathrm{Id}\otimes \mathrm{Ad}(g)} A_V \otimes \operatorname{End} V\ .$$

This defines an action of $\mathrm{GL}_k(V)$ on A_V by DG algebra automorphisms, that is functorial in A. Thus, we can introduce the $\mathrm{GL}_k(V)$-invariant subfunctor of (6):

$$(9)\qquad (-)_V^{\mathrm{GL}}\,:\ \mathtt{DGA}_k \to \mathtt{CDGA}_k\ ,\quad A \mapsto A_V^{\mathrm{GL}_k(V)}\ .$$

Next, we recall that the categories $\mathtt{DGA}_k$ and $\mathtt{CDGA}_k$ carry natural model structures[2] in the sense of Quillen [**Q1**]. The weak equivalences in these model categories are the quasi-isomorphisms and the fibrations are the degreewise surjective maps. The cofibrations are characterized in abstract terms: as the morphisms satisfying the left lifting property with respect to the acyclic fibrations (see, e.g., [**H**]). Every DG algebra $A \in \mathtt{DGA}_k$ has a cofibrant resolution which is given by a surjective quasi-isomorphism $QA \stackrel{\sim}{\twoheadrightarrow} A$, with QA being a cofibrant object in $\mathtt{DGA}_k$. In particular, if A is concentrated in non-negative degrees (for example, an ordinary algebra $A \in \mathtt{Alg}_k$), any almost free resolution $R \stackrel{\sim}{\twoheadrightarrow} A$ is cofibrant in $\mathtt{DGA}_k$. Replacing DG algebras by their cofibrant resolutions one defines the homotopy category $\mathtt{Ho}(\mathtt{DGA}_k)$, in which the morphisms are given by the homotopy classes of morphisms between cofibrant objects in $\mathtt{DGA}_k$. The category $\mathtt{Ho}(\mathtt{DGA}_k)$ is equivalent to the (abstract) localization of the category $\mathtt{DGA}_k$ at the class of weak equivalences. The corresponding localization functor $\mathtt{DGA}_k \to \mathtt{Ho}(\mathtt{DGA}_k)$ acts as the identity on objects while mapping each morphism $f : A \to B$ in $\mathtt{DGA}_k$ to the homotopy class of its cofibrant lifting $Qf : QA \to QB$.

We can now state one of the main results of [**BKR**] which combines (part of) Theorem 2.2 and Theorem 2.6 of *loc. cit.*

THEOREM 4 ([**BKR**]). (*a*) *The functor* (6) *has a total left derived functor*

$$\boldsymbol{L}(\,\text{–}\,)_V :\ \mathtt{Ho}(\mathtt{DGA}_k) \to \mathtt{Ho}(\mathtt{CDGA}_k)\ , \quad A \mapsto (QA)_V\ ,\ f \mapsto (Qf)_V\ ,$$

which is adjoint to the derived functor $\mathrm{End}\, V \otimes \text{–} : \mathtt{Ho}(\mathtt{CDGA}_k) \to \mathtt{Ho}(\mathtt{DGA}_k)$.

(*b*) *The functor* (9) *has a total left derived functor*

$$\boldsymbol{L}(\,\text{–}\,)_V^{\mathrm{GL}} :\ \mathtt{Ho}(\mathtt{DGA}_k) \to \mathtt{Ho}(\mathtt{CDGA}_k)\ , \quad A \mapsto (QA)_V^{\mathrm{GL}}\ ,\ f \mapsto (Qf)_V^{\mathrm{GL}}\ .$$

Here QA is any cofibrant replacement of A and Qf is the corresponding cofibrant lifting of f.

The point of Theorem 4 is that the DG algebras $(QA)_V$ and $(QA)_V^{\mathrm{GL}}$ depend only on A and V, provided we view them as objects in the homotopy category $\mathtt{Ho}(\mathtt{CDGA}_k)$. In particular, for $A \in \mathtt{Alg}_k$, we set $\mathrm{DRep}_V(A) := \boldsymbol{L}(A)_V$ and define[3] $\mathrm{H}_\bullet(A, V) := \mathrm{H}_\bullet[\mathrm{DRep}_V(A)]$. This last object is a graded commutative algebra which we call the *representation homology* of A. Using the standard adjunction (7), it is not difficult to show that $\mathrm{H}_0[\mathrm{DRep}_V(A)] \cong k[\mathrm{Rep}_V(A)]$ whenever A is an ordinary algebra (see [**BKR**], 2.3.4). In addition, we have the following property which shows that homology commutes with taking invariants.

PROPOSITION 5 ([**BKR**], Theorem 2.6). *For any $A \in \mathtt{DGA}_k$, there is a natural isomorphism of graded commutative algebras*

$$\mathrm{H}_\bullet[\boldsymbol{L}(A)_V^{\mathrm{GL}}] \cong \mathrm{H}_\bullet(A, V)^{\mathrm{GL}_k(V)}\ .$$

[2]For basic definitions of the theory of model categories and results needed for the present paper we refer the reader to [**BKR**], Appendix **A**. A leisurely introduction to this theory can be found in [**DS**].

[3]Sometimes, we will abuse this notation letting $\mathrm{H}_\bullet(A, V)$ denote $\mathrm{H}_\bullet[\boldsymbol{L}(A)_V]$ for any DG algebra $A \in \mathtt{DGA}_k$.

2.2. Higher traces. We now construct the trace maps (3) relating cyclic homology to representation homology. Given an associative DG algebra R with the identity element $1_R \in R$, we write

$$R_\natural := R/[R,R] \quad , \quad \mathcal{C}(R) := R/(k \cdot 1_R + [R,R]) \ .$$

Both $R_\natural$ and $\mathcal{C}(R)$ are complexes of vector spaces with differentials induced from R. If $A \in \mathtt{Alg}_k$ is an ordinary algebra, we let $\mathrm{HC}_\bullet(A)$ and $\overline{\mathrm{HC}}_\bullet(A)$ denote its cyclic and reduced cyclic homology, respectively. The precise relation between the two is explained in [**L**], Sect. 2.2.13; here, we only recall a canonical map

$$\mathrm{HC}_\bullet(A) \to \overline{\mathrm{HC}}_\bullet(A) \tag{10}$$

which is induced by the projection of complexes $\mathrm{CC}_\bullet(A) \twoheadrightarrow \mathrm{CC}_\bullet(A)/\mathrm{CC}_\bullet(k)$, where $\mathrm{CC}_\bullet(A)$ is the Connes cyclic complex computing $\mathrm{HC}_\bullet(A)$.

The starting point for our construction is the following well-known result due to Feigin and Tsygan.

THEOREM 6 ([**FT**], Theorem 1). *For any $A \in \mathtt{Alg}_k$, there is an isomorphism of graded vector spaces*

$$\overline{\mathrm{HC}}_\bullet(A) \cong \mathrm{H}_\bullet[\mathcal{C}(R)] \ ,$$

where $R = QA$ is a(ny) cofibrant resolution of A in $\mathtt{DGA}_k$.

For a simple conceptual proof of this theorem, we refer to [**BKR**], Section 3.

Now, for any $R \in \mathtt{DGA}_k$, consider the composite map

$$R \xrightarrow{\pi_V} R_V \otimes \mathrm{End}\, V \xrightarrow{\mathrm{Id}\otimes\mathrm{Tr}} R_V$$

where π_V is the universal representation of R in V (see (8)), and $\mathrm{Tr} : \mathrm{End}\, V \to k$ is the usual matrix trace. It is clear that this map factors through $R_\natural$ and its image lies in R_V^{GL}. Hence, we get a map of complexes

$$\mathrm{Tr}_V(R)_\bullet : \ R_\natural \to R_V^{\mathrm{GL}} \ , \tag{11}$$

which extends by multiplicativity to the map of graded commutative algebras

$$\mathrm{Tr}_V(R)_\bullet : \ \mathbf{\Lambda}(R_\natural) \to R_V^{\mathrm{GL}} \ , \tag{12}$$

where $\mathbf{\Lambda}$ denotes the graded symmetric algebra over k. We will need the following result which is a generalization of a well-known theorem of Procesi [**P**] to the case of DG algebras.

THEOREM 7 ([**BR**], Theorem 3.1). *For any $R \in \mathtt{DGA}_k$, the algebra map* (12) *is degreewise surjective.*

Now, let $A \in \mathtt{Alg}_k$ be an ordinary algebra, and let $R = QA$ be a cofibrant replacement of A in $\mathtt{DGA}_k$. The trace map (11) descends to a map of complexes

$$\mathrm{Tr}_V(R)_\bullet : \ \mathcal{C}(R) \to \bar{R}_V^{\mathrm{GL}} \ , \tag{13}$$

where $\bar{R}_V^{\mathrm{GL}}$ is the deunitalization of R_V^{GL}. The long homology sequence arising from the short exact sequence

$$0 \to k \to R_V^{\mathrm{GL}} \to \bar{R}_V^{\mathrm{GL}} \to 0$$

shows that $\mathrm{H}_n(\bar{R}_V^{\mathrm{GL}}) \cong \mathrm{H}_n(R_V^{\mathrm{GL}})$ for all $n \geq 1$. Then, with the identifications of Proposition 5 and Theorem 6 and in combination with (10), the map (13) induces

$$\mathrm{Tr}_V(A)_n : \ \mathrm{HC}_n(A) \to \mathrm{H}_n(A, V)^{\mathrm{GL}_k(V)} \ , \quad n \geq 1 \ , \tag{14}$$

which are the higher trace maps (3) discussed in the Introduction. We remark that these trace maps may be nontrivial even when A is a commutative algebra (see, e. g., Section 5.4 below).

2.3. Stabilization. We now explain how to 'stabilize' the family of maps (14) passing to the infinite-dimensional limit $\dim_k V \to \infty$. We will work with unital DG algebras A which are *augmented* over k. We recall that the category of such DG algebras is naturally equivalent to the category of non-unital DG algebras, with A corresponding to its augmentation ideal $\bar{A}$. We identify these two categories and denote them by $\mathtt{DGA}_{k/k}$. Further, to simplify the notation we take $V = k^d$ and identify $\mathrm{End}\, V = M_d(k)$, $\mathrm{GL}(V) = \mathrm{GL}_k(d)$; in addition, for $V = k^d$, we will write A_V as A_d. Bordering a matrix in $M_d(k)$ by 0's on the right and on the bottom gives an embedding $M_d(k) \hookrightarrow M_{d+1}(k)$ of non-unital algebras. As a result, for each $B \in \mathtt{CDGA}_k$, we get a map of sets

$$\mathrm{Hom}_{\mathtt{DGA}_{k/k}}(\bar{A},\, M_d(B)) \to \mathrm{Hom}_{\mathtt{DGA}_{k/k}}(\bar{A},\, M_{d+1}(B)) \tag{15}$$

defining a natural transformation of functors from $\mathtt{CDGA}_k$ to $\mathtt{Sets}$. Since B's are unital and A is augmented, the restriction maps

$$\mathrm{Hom}_{\mathtt{DGA}_k}(A,\, M_d(B)) \xrightarrow{\sim} \mathrm{Hom}_{\mathtt{DGA}_{k/k}}(\bar{A},\, M_d(B)) \ , \quad \varphi \mapsto \varphi|_{\bar{A}} \tag{16}$$

are isomorphisms for all $d \in \mathbb{N}$. Combining (15) and (16), we thus have natural transformations

$$\mathrm{Hom}_{\mathtt{DGA}_k}(A,\, M_d(-)) \to \mathrm{Hom}_{\mathtt{DGA}_k}(A,\, M_{d+1}(-)) \ . \tag{17}$$

By standard adjunction (7), (17) yield an inverse system of morphisms $\{\mu_{d+1,d} : A_{d+1} \to A_d\}$ in $\mathtt{CDGA}_k$. Taking the limit of this system, we define

$$A_\infty := \varprojlim_{d \in \mathbb{N}} A_d \ .$$

Next, we recall that the group $\mathrm{GL}(d)$ acts naturally on A_d, and it is easy to check that $\mu_{d+1,d} : A_{d+1} \to A_d$ maps the subalgebra A_{d+1}^{GL} of GL-invariants in A_{d+1} to the subalgebra A_d^{GL} of GL-invariants in A_d. Defining $\mathrm{GL}(\infty) := \varinjlim \mathrm{GL}(d)$ through the standard inclusions $\mathrm{GL}(d) \hookrightarrow \mathrm{GL}(d+1)$, we extend the actions of $\mathrm{GL}(d)$ on A_d to an action of $\mathrm{GL}(\infty)$ on A_∞ and let $A_\infty^{\mathrm{GL}(\infty)}$ denote the corresponding invariant subalgebra. Then one can prove (see [**T-TT**])

$$A_\infty^{\mathrm{GL}(\infty)} \cong \varprojlim_{d \in \mathbb{N}} A_d^{\mathrm{GL}(d)} \ . \tag{18}$$

This isomorphism allows us to equip $A_\infty^{\mathrm{GL}(\infty)}$ with a natural topology: namely, we put first the discrete topology on each $A_d^{\mathrm{GL}(d)}$ and equip $\prod_{d\in\mathbb{N}} A_d^{\mathrm{GL}(d)}$ with the product topology; then, identifying $A_\infty^{\mathrm{GL}(\infty)}$ with a subspace in $\prod_{d\in\mathbb{N}} A_d^{\mathrm{GL}(d)}$ via (18), we put on $A_\infty^{\mathrm{GL}(\infty)}$ the induced topology. The corresponding topological DG algebra will be denoted A_∞^{GL}.

Now, for each $d \in \mathbb{N}$, we have the commutative diagram

$$\begin{tikzcd} & \mathcal{C}(A) \arrow[dl, "\mathrm{Tr}_{d+1}(A)_\bullet"'] \arrow[dr, "\mathrm{Tr}_d(A)_\bullet"] & \\ A_{d+1}^{\mathrm{GL}} \arrow[rr, "\mu_{d+1,d}"] & & A_d^{\mathrm{GL}} \end{tikzcd}$$

where $\mathcal{C}(A)$ is the cyclic functor restricted to $\mathtt{DGA}_{k/k}$ (*cf.* Section 2.2). Hence, by the universal property of inverse limits, there is a morphism of complexes $\mathrm{Tr}_\infty(A)_\bullet$: $\mathcal{C}(A) \to A_\infty^{\mathrm{GL}}$ that factors $\mathrm{Tr}_d(A)_\bullet$ for each $d \in \mathbb{N}$. We extend this morphism to a homomorphism of commutative DG algebras:

(19) $$\mathrm{Tr}_\infty(A)_\bullet :\ \mathbf{\Lambda}[\mathcal{C}(A)] \to A_\infty^{\mathrm{GL}} \ .$$

The following lemma is one of the key technical results of [**BR**]; it should be compared to Theorem 7 in the finite-dimensional case ($d = \dim_k V$).

LEMMA 1 ([**BR**], Lemma 3.1). *The map* (19) *is* topologically *surjective: i.e., its image is dense in* A_∞^{GL}.

Letting A_∞^{Tr} denote the image of (19), we define the functor

(20) $$(\,-\,)_\infty^{\mathrm{Tr}} \ :\ \mathtt{DGA}_{k/k} \to \mathtt{CDGA}_k \ , \quad A \mapsto A_\infty^{\mathrm{Tr}} \ .$$

The algebra maps (19) then give a morphism of functors

(21) $$\mathrm{Tr}_\infty(\,-\,)_\bullet :\ \mathbf{\Lambda}[\mathcal{C}(\,-\,)] \to (\,-\,)_\infty^{\mathrm{Tr}} \ .$$

Now, to state the main result of [**BR**] we recall that the category of augmented DG algebras $\mathtt{DGA}_{k/k}$ has a natural model structure induced from $\mathtt{DGA}_k$. We also recall the derived Feigin-Tsygan functor $\boldsymbol{L}\mathcal{C}(\,-\,) : \mathtt{Ho}(\mathtt{DGA}_{k/k}) \to \mathtt{Ho}(\mathtt{CDGA}_k)$ inducing the isomorphism of Theorem 6.

THEOREM 8 ([**BR**], Theorem 4.2). (*a*) *The functor* (20) *has a total left derived functor* $\boldsymbol{L}(\,-\,)_\infty^{\mathrm{Tr}} \ :\ \mathtt{Ho}(\mathtt{DGA}_{k/k}) \to \mathtt{Ho}(\mathtt{CDGA}_k)$.

(*b*) *The morphism* (21) *induces an isomorphism of functors*

$$\mathrm{Tr}_\infty(\,-\,)_\bullet :\ \mathbf{\Lambda}[\boldsymbol{L}\mathcal{C}(\,-\,)] \xrightarrow{\sim} \boldsymbol{L}(\,-\,)_\infty^{\mathrm{Tr}} \ .$$

By definition, $\boldsymbol{L}(-)_\infty^{\mathrm{Tr}}$ is given by $\boldsymbol{L}(A)_\infty^{\mathrm{Tr}} = (QA)_\infty^{\mathrm{Tr}}$, where QA is a cofibrant resolution of A in $\mathtt{DGA}_{k/k}$. For an ordinary augmented k-algebra $A \in \mathtt{Alg}_{k/k}$, we set

$$\mathrm{DRep}_\infty(A)^{\mathrm{Tr}} := (QA)_\infty^{\mathrm{Tr}} \ .$$

By part (a) of Theorem 8, $\mathrm{DRep}_\infty(A)^{\mathrm{Tr}}$ is well defined. On the other hand, part (b) implies

COROLLARY 1. *For any* $A \in \mathtt{Alg}_{k/k}$, $\mathrm{Tr}_\infty(A)_\bullet$ *induces an isomorphism of graded commutative algebras*

(22) $$\mathbf{\Lambda}[\overline{\mathrm{HC}}(A)] \cong \mathrm{H}_\bullet[\mathrm{DRep}_\infty(A)^{\mathrm{Tr}}] \ .$$

In fact, one can show that $\mathrm{H}_\bullet[\mathrm{DRep}_\infty(A)^{\mathrm{Tr}}]$ has a natural structure of a graded Hopf algebra, and the isomorphism of Corollary 1 is actually an isomorphism of Hopf algebras. This isomorphism is analogous to the famous Loday-Quillen-Tsygan isomorphism computing the stable homology of matrix Lie algebras $\mathfrak{gl}_n(A)$ in terms of cyclic homology (see [**LQ, T**]). Heuristically, it implies that the cyclic homology of an augmented algebra is determined by its representation homology.

3. NC Poisson structures and DG representation schemes

In this section, we propose a definition of a NC Poisson structure on an associative DG algebra $A \in \mathtt{DGA}_k$. Our definition generalizes the notion of a noncommutative Poisson structure in the sense of [**CB**]. We show that our noncommutative DG Poisson structures induce (via the natural trace maps) DG Poisson algebra structures on $A_V^{\mathrm{GL}(V)}$ for all V. Subsequently, in the next section, we will introduce an NC P_∞-structure, which is a strong homotopy version of the notion of a NC Poisson structure. We note that our definition of and results relating to NC Poisson algebras and NC P_∞-algebras may be mimicked to give definitions of, and corresponding results for, NC n-Poisson algebras and NV $n-P_\infty$ algebras for every n. At the level of homology, our construction gives a higher extension of Crawley-Boevey's notion of an H_0-Poisson structure on an algebra A. Indeed, suppose that A has a cofibrant resolution $R \in \mathtt{DGA}_k$ which is equipped with a NC n-Poisson structure. Then, this last structure on R induces a graded Lie algebra structure $\{-,-\}_\natural$ on the (shifted) cyclic homology $\mathrm{HC}_\bullet(A)[n]$, and we will refer to $\{-,-\}_\natural$ as a *derived n-Poisson structure* on A.

The following result is a direct generalization of Theorem 2 stated in the Introduction.

THEOREM 9. *Let A be an algebra equipped with a derived n-Poisson structure. Then, there exists a unique graded n-Poisson algebra structure $\{-,-\}$ on $\mathrm{H}_\bullet(A,V)^{\mathrm{GL}}$ such that*

$$(\mathrm{Tr}_V)_\bullet(\{\alpha,\beta\}_\natural) = \{(\mathrm{Tr}_V)_\bullet(\alpha),(\mathrm{Tr}_V)_\bullet(\beta)\}$$

for all $\alpha,\beta \in \mathrm{HC}_\bullet(A)$.

Theorem 9 is a consequence of the more fundamental Theorem 10 that we will prove in this section. In fact, we show (see Theorem 10 (i)) that a NC n-Poisson structure on a DGA R induces DG n-Poisson structures on R_V^{GL} (via natural trace maps) in a functorial manner. In Theorem 9, the graded n-Poisson structure on $\mathrm{H}_\bullet(A,V)^{\mathrm{GL}}$ is precisely the one induced on homology by the DG n-Poisson structure on R_V^{GL} coming from Theorem 10(i).

Further, we give a reasonable definition of the "homotopy category" of NC Poisson algebras and prove a stronger statement (Theorem 10 (ii)) at the level of homotopy categories.

3.1. NC Poisson algebras. Fix $A \in \mathtt{DGA}_k$, and let $\underline{\mathrm{Der}}(A)^\natural$ denote the subcomplex of the DG Lie algebra $\underline{\mathrm{Der}}(A)$ comprising those derivations whose image is contained in $[A,A]$. It is easy to see that $\underline{\mathrm{Der}}(A)^\natural$ is a DG Lie ideal of $\underline{\mathrm{Der}}(A)$, so that $\underline{\mathrm{Der}}(A)_\natural := \underline{\mathrm{Der}}(A)/\underline{\mathrm{Der}}(A)^\natural$ is a DG Lie algebra.

Now, let V be a representation of $\underline{\mathrm{Der}}(A)_\natural$, i. e. a DG Lie algebra homomorphism $\varrho : \underline{\mathrm{Der}}(A)_\natural \to \underline{\mathrm{End}}_k V$.

3.1.1. *Definitions.* By *Poisson structure* on V we will mean a DG Lie algebra structure on V whose adjoint representation $\mathrm{ad} : V \to \underline{\mathrm{End}}_k V$ factors through ϱ: i. e., there is a morphism of DG Lie algebras $i : V \to \underline{\mathrm{Der}}(A)_\natural$ such that $\mathrm{ad} = \varrho \circ i$.

For any DG algebra A, the natural action of $\underline{\mathrm{Der}}(A)$ on A induces a Lie algebra action of $\underline{\mathrm{Der}}(A)_\natural$ on $A_\natural$. A *NC Poisson structure* on A is then, by definition, a Poisson structure on the representation $A_\natural$. It is easy to see that if A is a commutative DG algebra, a NC Poisson structure on A is exactly the same thing as a Poisson bracket on A.

Let A and B be NC Poisson DG algebras, i.e. objects in $\mathtt{DGA}_k$ equipped with NC Poisson structures. A *morphism* $f : A \to B$ of NC Poisson DG algebras is then a morphism $f : A \to B$ in $\mathtt{DGA}_k$ such that $f_\natural : A_\natural \to B_\natural$ is a morphism of DG Lie algebras. We can therefore define the category $\mathtt{NCPoiss}_k$.

Further, note that if B is a NC Poisson DG algebra and Ω is the de Rham algebra of the affine line (*cf.* [**BKR**], Section B.4), then $B \otimes \Omega$ can be given the structure of a NC Poisson DG algebra via extension of scalars. Indeed, since $[B \otimes \Omega, B \otimes \Omega] = [B, B] \otimes \Omega$, we have $(B \otimes \Omega)_\natural \cong B_\natural \otimes \Omega$. The DG Lie structure on $B_\natural \otimes \Omega$ is simply the one obtained by extending the corresponding structure on $B_\natural$. The structure map $i : B_\natural \otimes \Omega \to \underline{\mathrm{Der}}(B \otimes \Omega)_\natural$ is simply the composite map

$$B_\natural \otimes \Omega \xrightarrow{i \otimes \Omega} \underline{\mathrm{Der}}(B \otimes \Omega)_{\Omega,\natural} \to \underline{\mathrm{Der}}(B \otimes \Omega)_\natural \ ,$$

where $\underline{\mathrm{Der}}(B \otimes \Omega)_{\Omega,\natural} := \underline{\mathrm{Der}}_\Omega(B \otimes \Omega)/\underline{\mathrm{Der}}_\Omega(B \otimes \Omega)^\natural$ and $\underline{\mathrm{Der}}_\Omega(B \otimes \Omega)$ denotes the DG Lie algebra of Ω-linear derivations from $B \otimes \Omega$ into itself, with $\underline{\mathrm{Der}}_\Omega(B \otimes \Omega)^\natural$ being the Lie ideal of derivations whose image is contained in $B_\natural \otimes \Omega$.

We can now introduce the notion of P-homotopy along the lines of [**BKR**], Proposition B.2 and Remark B.4.3. To be precise, we call two morphisms $f, g : A \to B$ in $\mathtt{NCPoiss}$ *P-homotopic* if there is a morphism $h : A \to B \otimes \Omega$ such that $h(0) = f$ and $h(1) = g$ (see Proposition B.2 of [**BKR**] for the definitions of $h(0)$, $h(1)$). It is easy to check that P-homotopy is an equivalence relation on $\mathrm{Hom}_{\mathtt{NCPoiss}}(A, B)$ for any A and B in $\mathtt{NCPoiss}_k$. Thus, we can define $\mathtt{Ho}^*(\mathtt{NCPoiss})$ to be the category whose objects are the cofibrant (in $\mathtt{DGA}_k$) DG algebras equipped with NC Poisson structures, with $\mathrm{Hom}_{\mathtt{Ho}^*(\mathtt{NCPoiss})}(A, B)$ being the space of P-homotopy classes of morphisms in $\mathrm{Hom}_{\mathtt{NCPoiss}}(A, B)$.

Notation. In what follows, for a DG algebra A with a NC Poisson structure, the symbol $[-, -]$ shall be used to denote the corresponding Lie bracket on $A_\natural$. The symbol $\{-, -\}_\natural$ shall be used to denote the induced Lie bracket on $\mathrm{H}_\bullet(A_\natural)$.

3.2. The main theorem. The following theorem is the first main result of this paper.

THEOREM 10. (*a*) *The functor* $(\,-\,)_V^{\mathrm{GL}} : \mathtt{DGA}_k \to \mathtt{CDGA}_k$ *enriches to give the following commutative diagram*

$$\begin{array}{ccc} \mathtt{NCPoiss}_k & \xrightarrow{(\,-\,)_V^{\mathrm{GL}}} & \mathtt{Poiss}_k \\ \downarrow & & \downarrow \\ \mathtt{DGA}_k & \xrightarrow{(\,-\,)_V^{\mathrm{GL}}} & \mathtt{CDGA}_k \end{array} \tag{23}$$

where the vertical arrows are the forgetful functors.

(*b*) *The functor* $(\,-\,)_V^{\mathrm{GL}} : \mathtt{NCPoiss}_k \to \mathtt{Poiss}_k$ *descends to a functor* $\boldsymbol{L}^*(\,-\,)_V^{\mathrm{GL}} : \mathtt{Ho}^*(\mathtt{NCPoiss}_k) \to \mathtt{Ho}(\mathtt{Poiss}_k)$. *Further,* $\boldsymbol{L}(\,-\,)_V^{\mathrm{GL}} : \mathtt{Ho}(\mathtt{DGA}_k) \to \mathtt{Ho}(\mathtt{CDGA}_k)$ *enriches to give a commutative diagram*

$$\begin{array}{ccc} \mathtt{Ho}^*(\mathtt{NCPoiss}_k) & \xrightarrow{\boldsymbol{L}^*(\,-\,)_V^{\mathrm{GL}}} & \mathtt{Ho}(\mathtt{Poiss}_k) \\ \downarrow & & \downarrow \\ \mathtt{Ho}(\mathtt{DGA}_k) & \xrightarrow{\boldsymbol{L}(\,-\,)_V^{\mathrm{GL}}} & \mathtt{Ho}(\mathtt{CDGA}_k) \end{array} \tag{24}$$

The next section, Section 3.2.1 shall have certain preliminaries we require for the proof of Theorem 10. Section 3.2.2 shall contain the proof of Theorem 10.

3.2.1. *From DG Lie to DG Poisson algebras.* The following proposition is a (minor) generalization of Theorem 3.3 of [**S**].

PROPOSITION 11. *(i) One has a functor*

$$\texttt{Lie} \to \texttt{Poiss}, \quad V \mapsto \mathbf{\Lambda}(V)$$

from the category of DG Lie algebras to the category of DG Poisson algebras.
(ii) Further, for any DG Poisson algebra A and a morphism $f : V \to A$ of DG Lie algebras, one has a unique morphism $\tilde{f} : \mathbf{\Lambda}(V) \to A$ of DG Poisson algebras such that $f = \tilde{f} \circ \iota$ where $\iota : V \to \mathbf{\Lambda}(V)$ is the obvious inclusion.

PROOF. Clearly, $\mathbf{\Lambda}(V)$ is a commutative DG algebra. We extend the Lie bracket on V to a Poisson bracket on $\mathbf{\Lambda}(V)$ via the rules[4]

$$[u, v.w] \;=\; [u,v].w + (-1)^{|u||v|} v.[u,w], \quad [u,v] = (-1)^{|u||v|}[v,u]\,.$$

That this indeed gives a well defined DG Poisson structure on $\mathbf{\Lambda}(V)$ is a special case of Proposition 13 from Section 4 (which in turn is a part of Theorem 3.15 of [**S**]). Given a morphism $f : V \to W$ of DG Lie algebras, one gets the morphism $\mathbf{\Lambda}(f) : \mathbf{\Lambda}(V) \to \mathbf{\Lambda}(W)$ in $\texttt{CDGA}_k$. We verify that $F := \mathbf{\Lambda}(f)$ is a morphism of DG Poisson algebras as follows.

First, suppose that $F([u,v]) = [F(u),F(v)]$ and $F([u,w]) = [F(u),F(w)]$. Then,

$$\begin{aligned} F([u,v.w]) \;&=\; F([u,v].w) + (-1)^{|u||v|} F(v.[u,w]) \\ &=\; F([u,v]).F(w) + (-1)^{|F(u)||F(v)|} F(v).F([u,w]) \\ &=\; [F(u),F(v)].F(w) + (-1)^{|F(u)||F(v)|} F(v).[F(u),F(w)] \\ &=\; [F(u), F(v).F(w)] \;=\; [F(u), F(vw)]\,. \end{aligned}$$

Hence, by induction, it suffices to verify that $F([x,y]) = [F(x),F(y)]$ on V. Since $F|_V = f$, the latter is indeed true. This proves (i).

Note that there exists a unique extension $\tilde{f}$ of f to a morphism $\mathbf{\Lambda}(V) \to A$ in $\texttt{CDGA}_k$. By the above computation, $\tilde{f}$ is a homomorphism DG Poisson algebras. This proves (ii). □

Two morphisms $f, g : V \to W$ of DG Lie algebras are *L-homotopic* if there exists a morphism $h : V \to W \otimes \Omega$ of DG Lie algebras[5] such that $h(0) = f$, $h(1) = g$.

LEMMA 2. *If f is L-homotopic to g, $\mathbf{\Lambda}(f)$ is P-homotopic to $\mathbf{\Lambda}(g)$.*

PROOF. Indeed, the natural inclusion $W \otimes \Omega \hookrightarrow \mathbf{\Lambda}(W) \otimes \Omega$ is a morphism of DG Lie algebras. Hence, its composition with h is a morphism of DG Lie algebras. By Proposition 11 (ii), there exists a unique morphism $\tilde{h} : \mathbf{\Lambda}(V) \to \mathbf{\Lambda}(W) \otimes \Omega$ of DG Poisson algebras extending h. Since $\tilde{h}(0)|_V = \mathbf{\Lambda}(f)|_V$, $\tilde{h}(0) = \mathbf{\Lambda}(f)$ by Proposition 11 (ii). Similarly, $\tilde{h}(1) = \mathbf{\Lambda}(g)$. This proves the desired proposition. □

[4] $|u|$ denotes the degree of a homogenous element u in $\mathbf{\Lambda}(V)$.

[5] The DG Lie structure on $W \otimes \Omega$ is obtained from that on W by extension of scalars.

3.2.2. *Proof of Theorem* 10. It follows from Proposition 11 and Lemma 2 that if $f, g : A \to B$ are P-homotopic morphisms of NC Poisson algebras, then, $\mathbf{\Lambda}(f_\natural), \mathbf{\Lambda}(g_\natural) : \mathbf{\Lambda}(A_\natural) \to \mathbf{\Lambda}(B_\natural)$ are P-homotopic morphisms of DG Poisson algebras. The following proposition shows that the DG Poisson structure of $\mathbf{\Lambda}(A_\natural)$ induces one on A_V^{GL} via $\mathrm{Tr}_V(A)_\bullet$.

LEMMA 3. *If* $(\mathrm{Tr}_V)_\bullet(\beta) = 0$, *then for any* $\alpha \in \mathbf{\Lambda}(A_\natural)$, $(\mathrm{Tr}_V)_\bullet([\alpha, \beta]) = 0$.

PROOF. Since

$$[\alpha_1\alpha_2, \beta] = \pm\alpha_1.[\alpha_2, \beta] \pm [\alpha_1, \beta].\alpha_2\,,$$

and since $(\mathrm{Tr}_V)_\bullet$ is a morphism in $\mathtt{CDGA}_k$, it suffices to prove the desired proposition for $\alpha \in A_\natural$. In this case, viewing α as an element of $A_\natural[1]$, consider the element $i(\alpha) \in \mathtt{Der}(A)_\natural$. Choose any $\partial_\alpha \in \mathtt{Der}(A)$ whose image in $\mathtt{Der}(A)_\natural$ is $i(\alpha)$. By Lemma 5 of Section 4 (which may be read independently of the rest of that section), there is a (graded) derivation ψ_α of A_V^{GL} such that

$$(\mathrm{Tr}_V)_\bullet(\partial_\alpha(\beta)) = \psi_\alpha((\mathrm{Tr}_V)_\bullet(\beta))$$

for all $\beta \in A_\natural$. Hence,

$$(\mathrm{Tr}_V)_\bullet([\alpha, \beta]) = \psi_\alpha((\mathrm{Tr}_V)_\bullet(\beta))$$

for all $\beta \in A_\natural$. Since $(\mathrm{Tr}_V)_\bullet([\alpha, -])$ as well as $\psi_\alpha((\mathrm{Tr}_V)_\bullet(-))$ are derivations with respect to $(\mathrm{Tr}_V)_\bullet$, it follows that

$$(\mathrm{Tr}_V)_\bullet([\alpha, \beta]) = \psi_\alpha((\mathrm{Tr}_V)_\bullet(\beta))$$

for all $\beta \in \mathbf{\Lambda}(A_\natural)$. The right hand side of the above equation indeed vanishes when $(\mathrm{Tr}_V)_\bullet(\beta) = 0$. $\square$

By Lemma 3, the antisymmetric pairing on A_V^{GL} given by

$$\{f, g\} := (\mathrm{Tr}_V)_\bullet([(\mathrm{Tr}_V)_\bullet^{-1}(f), (\mathrm{Tr}_V)_\bullet^{-1}(g)])$$

is well defined. That $\{-, -\}$ equips A_V^{GL} with the structure of a DG Poisson algebra follows from Proposition 11 (i) and Theorem 7.

Further, the following argument shows that if $f : A \to B$ is a morphism of NC Posson algebras, then $f_V^{\mathrm{GL}} : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}}$ is a morphism of DG Poisson algebras. Indeed, since $\mathbf{\Lambda}(f_\natural)$ is a morphism of DG Poisson algebras, $(\mathrm{Tr}_V)_\bullet \circ \mathbf{\Lambda}(f_\natural) = f_V^{\mathrm{GL}} \circ (\mathrm{Tr}_V)_\bullet$ and

$$\{(\mathrm{Tr}_V)_\bullet(\alpha), (\mathrm{Tr}_V)_\bullet(\beta)\} = (\mathrm{Tr}_V)_\bullet([\alpha, \beta])$$

for all $\alpha, \beta \in \mathbf{\Lambda}(A_\natural)$ (and similarly for B),

$$\begin{aligned}\{f_V^{\mathrm{GL}}((\mathrm{Tr}_V)_\bullet(\alpha)), f_V^{\mathrm{GL}}((\mathrm{Tr}_V)_\bullet(\beta))\} &= \{(\mathrm{Tr}_V)_\bullet(\mathbf{\Lambda}(f_\natural)(\alpha)), (\mathrm{Tr}_V)_\bullet(\mathbf{\Lambda}(f_\natural)(\beta))\} \\ = (\mathrm{Tr}_V)_\bullet([\mathbf{\Lambda}(f_\natural)(\alpha), \mathbf{\Lambda}(f_\natural)(\beta)]) &= (\mathrm{Tr}_V)_\bullet \circ \mathbf{\Lambda}(f_\natural)([\alpha, \beta]) \\ &= f_V^{\mathrm{GL}}(\{(\mathrm{Tr}_V)_\bullet(\alpha), (\mathrm{Tr}_V)_\bullet(\beta)\}).\end{aligned}$$

Theorem 7 then completes the verification that $f_V^{\mathrm{GL}} : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}}$ is a morphism of DG Poisson algebras. This completes the proof of Theorem 10 (i). Note that the same argument also shows that if $f, g : A \to B$ are P-homotopic morphisms of NC Poisson algebras, then $f_V^{\mathrm{GL}}, g_V^{\mathrm{GL}}$ are P-homotopic morphisms of DG Poisson algebras: indeed, if $h : A \to B \otimes \Omega$ is a P-homotopy between f and g, then $h_V^{\mathrm{GL}} : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}} \otimes \Omega$ is a morphism of DG Poisson algebras by (a trivial modification of) the same argument as above.

Therefore, to complete the proof of Theorem 10 (ii), we only need to verify two assertions:

(a) If $f, g : A \to B$ are P-homotopic morphisms in $\mathtt{NCPoiss}$, then $\gamma(f) = \gamma(g)$ in $\mathtt{Ho}(\mathtt{DGA}_k)$.
(b) If $f, g : C_1 \to C_2$ are P-homotopic morphisms in $\mathtt{Poiss}$, then $\gamma(f) = \gamma(g)$ in $\mathtt{Ho}(\mathtt{Poiss})$.
We check (a): let $p_A : QA \overset{\sim}{\to} A$ be a cofibrant resolution of A. Then, $\gamma(f) = \gamma(fp_A)$ and $\gamma(g) = \gamma(gp_A)$ in $\mathtt{Ho}(\mathtt{DGA}_k)$. Since $f, g : A \to B$ are polynomially M-homotopic to each other (see [**BKR**] Remark B.4.3), so are fp_A and gp_A. By [**BKR**], Proposition B.2, $\gamma(fp_A) = \gamma(gp_A)$ in $\mathtt{Ho}(\mathtt{DGA}_k)$. Thus, $\gamma(f) = \gamma(g)$ in $\mathtt{Ho}(\mathtt{DGA}_k)$. (b) is checked similarly, using the natural analogue of [**BKR**], Proposition B.2 for $\mathtt{Poiss}$ (see [**BKR**], the end of Remark B.4.3).

3.3. Remark. As mentioned in [**BKR**], Section 5.6, we expect that the category $\mathtt{NCPoiss}_k$ has a natural model structure compatible with the standard (projective) model structure on $\mathtt{DGA}_k$. The notation $\mathtt{Ho}^*$ is to remind the reader that, since $\mathtt{NCPoiss}_k$ is not yet proven to be a model category, $\mathtt{Ho}^*(\mathtt{NCPoiss}_k)$ is not yet confirmed to be an abstract homotopy category in Quillen's sense. One way to remedy this problem is to use the construction of a fibre product (homotopy pullback) of model categories due to Toën (see [**To**]). First, passing to infinite-dimensional limit $V \to V_\infty$ (see [**BR**]), we can stabilize the family of representation functors replacing $(-)_V^{\mathrm{GL}}$ in (23) by

$$(-)_\infty^{\mathrm{Tr}} : \mathtt{DGA}_k \to \mathtt{CDGA}_k \ . \tag{25}$$

By [**BR**], Theorem 4.2, (25) is a left Quillen functor having the total left derived functor $\boldsymbol{L}(-)_\infty^{\mathrm{Tr}}$. Then, by [**To**], the homotopy pullback of (25) along the forgetful functor $\mathtt{Poiss}_k \to \mathtt{CDGA}_k$ is a model category $\mathtt{DGA}_k \times^h_{\mathtt{CDGA}_k} \mathtt{Poiss}_k$, which, in view of (23), comes together with a functor

$$\mathtt{NCPoiss}_k \to \mathtt{DGA}_k \times^h_{\mathtt{CDGA}_k} \mathtt{Poiss}_k \ .$$

It is easy to show that this last functor is homotopy invariant, so it induces a functor

$$\mathtt{Ho}^*(\mathtt{NCPoiss}_k) \to \mathtt{Ho}(\mathtt{DGA}_k \times^h_{\mathtt{CDGA}_k} \mathtt{Poiss}_k) \ . \tag{26}$$

Our expectation is that (26) is an equivalence of categories. Since Toën's construction is known to give a correct notion of 'homotopy fibre product' (see [**Be**]), such an equivalence would mean that our *ad hoc* definition of NC Poisson structures is the correct one from homotopical point view. This would also give a precise meaning to the claim that the NC Poisson structures are the weakest structures on A inducing the usual Poisson structures under the representation functor (since the fibre product $\mathtt{DGA}_k \times^h_{\mathtt{CDGA}_k} \mathtt{Poiss}_k$ is exactly the category that has this property).

4. Noncommutative P_∞-algebras

The definition of an NC Poisson structure can be generalized to a definition of a NC P_∞-structure. In this subsection, we show that a NC P_∞-structure on A induces (in a functorial way) a P_∞-structure on A_V^{GL} for all finite dimensional V. Further, we show that homotopy equivalent NC P_∞-structures induce homotopy equivalent P_∞-structures on each A_V^{GL}. The main result in this subsection, i.e, Theorem 12, is therefore, a stronger version of Theorem 10. One can similarly define the notion of a NC n-P_∞ structure on a DGA. We remark here that Theorem 9 holds word for word with NC n-Poisson replaced by NC n-P_∞. The reader who is interested only in NC Poisson structures may skip this section and move on the next section.

4.1. Definitions. A P_∞*-structure* on a representation V of $\underline{\mathrm{Der}}(A)_\natural$ is a L_∞-algebra structure on V whose adjoint representation[6] factors through ϱ: i.e, there is a L_∞-morphism $i : V \to \underline{\mathrm{Der}}(A)_\natural$ (of L_∞-algebras) such that $\varrho \circ i = \mathrm{ad}$.

A *NC* P_∞*-structure* on an object A of $\mathtt{DGA}_k$ is, by definition, a NC P_∞- structure on $A_\natural$ (thought of as a representation of $\underline{\mathrm{Der}}(A)_\natural$ as in Section 3.1.1). Equivalently, a NC P_∞-algebra is a DG algebra A such that $A_\natural$ is equipped with a L_∞-structure $\{l_n : \wedge^n(A_\natural) \to (A_\natural)\}_{n\geq 1}$ such that l_n has degree $n-2$ and for all $a_1,..,a_{n-1} \in A$ homogenous, $l_n(\bar{a}_1,...,\bar{a}_{n-1},-) : A_\natural \to A_\natural$ is induced on $A_\natural$ by a derivation $\partial_{a_1,...,a_{n-1}} : A \to A$ [7] of degree $n-2+\sum_i |a_i|$.

A *morphism of NC* P_∞*-algebras* is a collection of maps $f_1, \bar{f}_2, ..$ such that $f_1 : A \to B$ is a morphism of DG algebras and $\{\bar{f}_n : \wedge^n(A_\natural) \to B_\natural\}_{n\geq 1}$ form an L_∞-morphism [8]. We further require that for all $a_1,..,a_{n-1} \in A$, $\bar{f}_n(\bar{a}_1,...,\bar{a}_{n-1},-) : A_\natural \to B_\natural$ is induced by a degree $\sum |a_i| + n - 1$ operator $\partial^f_{a_1,..,a_{n-1}} : A \to B$ such that the collection $\{\partial^f_{\mathbf{a}_S}\}_{S\subset\{1,..,n-1\}}$ is a f_1-polyderivation of multi-degree $(|a_1|+1,..,|a_{n-1}|+1)$. Here, for $S = \{i_1 < .. < i_p\}$, $\{\partial^f_{\mathbf{a}_S}\}_S := \partial^f_{a_{i_1},...,a_{i_p}}$. The reader is referred to [**BKR**], Section 5.5 for the definition of polyderivations and related material.

In addition, note that if B is a NC P_∞-algebra and Ω is the de-Rham algebra of the affine line, then $B \otimes \Omega$ naturally acquires the structure of a NC P_∞-algebra via extension of scalars.

This allows us to define the notion of homotopy along the lines of [**BKR**], Proposition B.2 and Remark B.4.3. A homotopy between morphisms $f, g : A \to B$ of P_∞-algebras is a morphism $h : A \to B \otimes \Omega$ of P_∞-algebras such that $h(0) = f$, $h(1) = g$.

We have thus, defined the category $\mathtt{NCP}_\infty$ of NC P_∞ algebras. The full subcategory of $\mathtt{NCP}_\infty$ algebras whose objects are commutative DGAs equipped with NC P_∞ structure will be called $\mathtt{P}_\infty$. We note that our definition of P_∞ algebra is more restrictive than the definition of P_∞ in the operadic sense (see [**CVdB**] for example). This definition, however, coincides with a definition of Poisson$_\infty$ algebras that has been studied earlier in the literature (see [**CF, S**] for example).

The category $\mathtt{Ho}^*(\mathtt{NCP}_\infty)$ is defined to be the category whose objects are objects of $\mathtt{NCP}_\infty$ that are cofibrant in $\mathtt{DGA}_k$ such that $\mathrm{Hom}_{\mathtt{Ho}^*(\mathtt{NCP}_\infty)}(A,B)$ is the space of homotopy equivalence classes in $\mathrm{Hom}_{\mathtt{NCP}_\infty}(A,B)$. The category $\mathtt{P}_\infty$ of P_∞ algebras and its "homotopy category" are analogously defined on the commutative side.

4.2. The main result. The following result is the main result of this subsection. It strengthens Theorem 10.

Theorem 12. *(i) The functor* $(-)^{\mathrm{GL}}_V : \mathtt{DGA}_k \to \mathtt{CDGA}_k$ *enriches to give the following commutative diagram of functors (vertical arrows being forgetful functors)*

$$\begin{tikzcd} \mathtt{NCGP}_\infty \arrow[r, "(-)^{\mathrm{GL}}_V"] \arrow[d] & \mathtt{P}_\infty \arrow[d] \\ \mathtt{DGA}_k \arrow[r, "(-)^{\mathrm{GL}}_V"] & \mathtt{CDGA}_k \end{tikzcd}$$

[6] Recall that is V is a L_∞-algebra, one has a (structure) L_∞-morphism $\mathrm{ad} : V \to \mathrm{End}_k(V)$.

[7] $\partial_{a_1,...,a_{n-1}}$ is any element of $\underline{\mathrm{Der}}(A)$ such that its image in $\underline{\mathrm{Der}}(A)_\natural$ coincides with $i(\bar{a}_1 \wedge ... \wedge \bar{a}_{n-1})$.

[8] $\bar{f}_1 : A_\natural \to B_\natural$ is the map induced by f_1.

(ii) Further, the functor $(-)^{\mathrm{GL}}_V : \mathtt{NCP}_\infty \to \mathtt{P}_\infty$ *descends to a functor*

$$\mathtt{Ho}^*(-)^{\mathrm{GL}}_V : \mathtt{Ho}^*(\mathtt{NCP}_\infty) \to \mathtt{Ho}^*(\mathtt{P}_\infty).$$

The proof of this theorem will be organized along the lines of the proof of Theorem 10.

4.3. From L_∞ to P_∞ algebras. The following proposition is (part of) Theorem 3.15 in [**S**].

PROPOSITION 13. *Let V be a L_∞-algebra. Then, $\mathbf{\Lambda}(V)$ (with the obvious multiplication) inherits the structure of a P_∞-algebra such that the L_∞-operations on $\mathbf{\Lambda}(V)$ extend those on V via the formula*

$$l_n(v_1, ., v_{n-1}, v.w) = l_n(v_1, .., v_{n-1}, v).w + (-1)^{(\sum |v_i|+n-2)|v|} v.l_n(v_1, ..., v_{n-1}, w).$$

Recall the definition of a polyderivation from [**BKR**], Section 5.5. Let $\{f_n\} : V \to G$ be a morphism of L_∞-algebras (where G is a P_∞-algebra). Equip $\mathbf{\Lambda}(V)$ with the P_∞-structures from Proposition 13. Each f_n uniquely extends to a map

$$f_n : \wedge^n(\mathbf{\Lambda}(V)) \to G$$

of degree $n-1$ such that for all $v_1, .., v_{n-1} \in \mathbf{\Lambda}(V)$, the maps $\{f_S(v_1, .., v_{n-1}, -) : \mathbf{\Lambda}(V) \to G\}_{S \subset \{1,...,n-1\}}$ constitute a polyderivation of multi-degree $(|v_1|+1, .., |v_{n-1}|+1)$ with respect to $f_1 : \mathbf{\Lambda}(V) \to G$ [9]. Here, for $S = \{i_1 < .. < i_k\} \subset \{1, ..., n-1\}$, $f_S(v_1, .., v_{n-1}, -)$ denotes the map $f_{k+1}(v_{i_1}, .., v_{i_k}, -) : \mathbf{\Lambda}(V) \to G$. The following key proposition is due to T. Schedler (in [**S1**]).

PROPOSITION 14. *The extended maps $\{f_n\}$ constitute the unique P_∞-morphism from $\mathbf{\Lambda}(V)$ to G extending $f : V \to G$.*

We remark that Propositions 13 and 14 together imply that $V \mapsto \mathbf{\Lambda}(V)$ is a functor from the category of L_∞-algebras to the category of P_∞-algebras. For a L_∞-morphism $f := \{f_n\} : V \to G$, the unique P_∞-morphism $\{f_n\}$ from $\mathbf{\Lambda}(V)$ to G extending $f : V \to G$ shall be denoted by $\mathbf{\Lambda}(f)$. Propositions 13 and 14 together imply that $\mathbf{\Lambda}((...))$ is a well defined functor from the category of L_∞-algebras to the category of P_∞-algebras. Recall that two L_∞-morphisms $f, g : V \to W$ are homotopic if there exists a L_∞-morphism $h : V \to W \otimes \Omega$ such that[10] $h(0) = f$ and $h(1) = g$. One has the following lemma.

LEMMA 4. *Suppose that $f, g : V \to W$ are homotopic L_∞-morphisms. Then, their extensions $\mathbf{\Lambda}(f), \mathbf{\Lambda}(g) : \mathbf{\Lambda}(V) \to \mathbf{\Lambda}(W)$ are homotopic P_∞-morphisms.*

PROOF. Let $h : V \to W \otimes \Omega$ be a L_∞-morphism with $h(0) = f$ and $h(1) = g$. The identification $\mathbf{\Lambda}(W) \otimes \Omega$ with $\mathbf{\Lambda}_\Omega(W \otimes \Omega)$ together with Proposition 14 shows that h extends to a P_∞-morphism $\mathbf{\Lambda}(h) : \mathbf{\Lambda}(V[) \to \mathbf{\Lambda}(W) \otimes \Omega$ such that $\mathbf{\Lambda}(h)(0) = \mathbf{\Lambda}(f) : \mathbf{\Lambda}(V) \to \mathbf{\Lambda}(W)$ and $\mathbf{\Lambda}(h)(1) = \mathbf{\Lambda}(g) : \mathbf{\Lambda}(V) \to \mathbf{\Lambda}(W)$. □

4.4. Proof of Theorem 12.

[9] $f_1 : \mathbf{\Lambda}(V) \to G$ is the obvious extension of the map of complexes $f_1 : V \to G$ to a morphism in $\mathtt{CDGA}_k$.

[10] The higher L_∞-structure maps of $W \otimes \Omega$ are extended from those of W by Ω-linearity.

4.4.1. *A technical lemma.* Our proof requires the following technical lemma.

LEMMA 5. *Let $\{\phi_S\}$ be a polyderivation of multi-degree $(d_1, .., d_k)$ with respect to $\phi : A \to B$. The induced polyderivation $\{(\phi_V)_S\}$ of multi-degree $(d_1, .., d_k)$ from* [**BKR**]*, Lemma 5.5 restricts to a polyderivation $\{(\phi_V)_S\}$ with respect to $\phi_V : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}}$.*

PROOF. We need to verify that $(\phi_V)_S$ maps A_V^{GL} to B_V^{GL}. For this, note that

$$(\phi_V)_S(a.b) = \sum_{T \sqcup T' = S} \pm (\phi_V)_T(a).(\phi_V)_{T'}(b).$$

The above equation implies that if $(\phi_V)_S(a)$ and $(\phi_V)_S(b)$ are in the subalgebra generated by the image of $(\mathrm{Tr}_V)_\bullet$ for all $S \subset \{1, .., k\}$, then $(\phi_V)_S(ab)$ is in the subalgebra generated by the image of $(\mathrm{Tr}_V)_\bullet$ for all $S \subset \{1, .., k\}$. This desired statement now follows from Theorem 7. □

4.4.2. *The main body of the proof.* It follows from Propositions 13, 14 and Lemma 4 that if $f, g : A \to B$ are homotopic morphisms of NC P_∞-algebras, $\mathbf{\Lambda}(f_\natural)$ and $\mathbf{\Lambda}(g_\natural)$ are homotopic morphisms of P_∞-algebras [11]. Let $l_n : \wedge^n(\mathbf{\Lambda}(A_\natural)) \to \mathbf{\Lambda}(A_\natural)$ denote the structure maps of $\mathbf{\Lambda}(A_\natural)$.

LEMMA 6. *For any $\beta \in \mathbf{\Lambda}(A_\natural)$ such that $(\mathrm{Tr}_V)_\bullet(\beta) = 0$,*

$$(\mathrm{Tr}_V)_\bullet(l_n(\alpha_1, ..., \alpha_{n-1}, \beta)) = 0$$

for any $\alpha_1, .., \alpha_{n-1} \in \mathbf{\Lambda}(A_\natural)$.

PROOF. Since

$$l_n(\alpha_1.\alpha_1', \alpha_2, .., \alpha_{n-1}, \beta) = \pm \alpha_1.l_n(\alpha_1', ..., \alpha_{n-1}, \beta) \pm \alpha_1'.l_n(\alpha_1, ..., \alpha_{n-1}, \beta),$$

since $(\mathrm{Tr}_V)_\bullet : \mathbf{\Lambda}(A_\natural) \to A_V^{\mathrm{GL}}$ is a ring homomorphism, and because the (anti)-symmetry of l_n, an inductive argument reduces the verification the required lemma to the case when $\alpha_1, ..., \alpha_{n-1} \in A_\natural$. Let $\partial_{\alpha_1, ..., \alpha_{n-1}}$ be any derivation of A whose image in $\mathtt{Der}(A)_\natural = i(\alpha_1 \wedge ... \wedge \alpha_{n-1})$. Then, by Lemma 5, there is a (graded) derivation $\psi_{\alpha_1, ..., \alpha_{n-1}}$ of A_V^{GL} such that

$$(\mathrm{Tr}_V)_\bullet(\partial_{\alpha_1, ..., \alpha_{n-1}}(\tilde{\beta}') \;=\; \psi_{\alpha_1, ..., \alpha_{n-1}}((\mathrm{Tr}_V)_\bullet(\beta'))$$

for any $\beta' \in A_\sharp$ and for any lift $\tilde{\beta}'$ of β to A. Hence,

$$(\mathrm{Tr}_V)_\bullet(l_n(\alpha_1, ..., \alpha_{n-1}, \beta')) \;=\; \psi_{\alpha_1, ..., \alpha_{n-1}}((\mathrm{Tr}_V)_\bullet(\beta'))$$

for any $\beta' \in A_\sharp$. It follows that

$$(\mathrm{Tr}_V)_\bullet(l_n(\alpha_1, ..., \alpha_{n-1}, \beta)) \;=\; \psi_{\alpha_1, ..., \alpha_{n-1}}((\mathrm{Tr}_V)_\bullet(\beta))$$

for any $\beta \in \mathbf{\Lambda}(A_\natural)$. The right hand side of the above equation is clearly 0 if $(\mathrm{Tr}_V)_\bullet(\beta) = 0$. □

[11] A short explanation is needed here: indeed, f consists of a morphism $f_1 : A \to B$ of DG-algebras and higher components $\bar{f}_2, ..., \bar{f}_n$ such that $\bar{f}_1, .., \bar{f}_n, ...$ are Taylor components of a L_∞-morphism from $A_\natural$ to $B_\natural$. The first component of $\mathbf{\Lambda}(f_\natural)$ is $\mathbf{\Lambda}((f_1)_\natural)$. The higher components are constructed as in the discussion before Proposition 14.

By Lemma 6 and Theorem 7, the operations $\bar{l}_n : \wedge^n(A_V^{\mathrm{GL}}) \to A_V^{\mathrm{GL}}$ defined by

$$\bar{l}_n(\alpha_1,..,\alpha_n) \,=\, (\mathrm{Tr}_V)_\bullet(l_n((\mathrm{Tr}_V)_\bullet^{-1}(\alpha_1),...,(\mathrm{Tr}_V)_\bullet^{-1}(\alpha_n)))$$

are well defined. Further, Theorem 7 and the fact that the l_n impose a P_∞ structure on $\mathbf{\Lambda}(A_\natural)$ together imply that the $\bar{l}_n$ impose a P_∞-structure on A_V^{GL}. Let $\{f_1, \bar{f}_2, ...,\} : A \to B$ be a homomorphism of NC P_∞-algebras.

Lemma 7. *As in Proposition* 14, *extend the* $\bar{f}_n$ *to maps* $f_n : \wedge^n(\mathbf{\Lambda}(A_\natural)) \to \mathbf{\Lambda}(B_\natural)$. *Then, for any* $\beta \in \mathbf{\Lambda}(A_\natural)$ *such that* $(\mathrm{Tr}_V)_\bullet(\beta) = 0$,

$$(\mathrm{Tr}_V)_\bullet(f_n(\alpha_1,...,\alpha_{n-1},\beta)) = 0$$

for all $\alpha_1,..,\alpha_{n-1} \in \mathbf{\Lambda}(A_\natural)$.

Proof. For $n = 1$, the required lemma is clear: indeed, $(\mathrm{Tr}_V)_\bullet$ is a natural transformation between the functors $A \mapsto \mathbf{\Lambda}(A_\natural)$ and $A \mapsto A_V^{\mathrm{GL}}$. Put $\alpha_n := \beta$. Note that $\mathbf{\Lambda}(A_\natural)$ has a polynomial grading where elements of $A_\natural$ may be viewed as the elements of polynomial degree 1. Put $J_n := \{2, 3, ..., n\}$. For $I := \{\alpha_{i_1} < ... < \alpha_{i_{|I|}}\} \subset J_n$ let $f_I(\alpha) := f_{|I|+1}(\alpha, \alpha_{i_1},..,\alpha_{i_{|I|}})$. Since

$$f_n(\alpha_1.\alpha_1', \alpha_2,..,\alpha_n) \,=\, \sum_{I\subset J_n} \pm f_I(\alpha).f_{J\setminus I}(\alpha')\,,$$

an induction on n as well as the polynomial degree of α_1, together with the (anti)-symmetry of f_n reduces the verification of the required lemma to the case when $\alpha_1,..,\alpha_{n-1} \in A_\natural$. In this case, let a_i be a lift of α_i for $1 \le i \le n-1$. Consider the polyderivation $\{\partial^f_{\mathbf{a}_S}\}_{S\subset\{1,..,n-1\}}$ from Section 4.1. By Lemma 5, there is a polyderivation $\{\psi^f_{\mathbf{a}_S} : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}}\}_{S\subset\{1,..,n-1\}}$ satisfying

$$(\mathrm{Tr}_V)_\bullet(\partial^f_{\mathbf{a}_S}(b)) \,=\, \psi^f_{\mathbf{a}_S}((\mathrm{Tr}_V)_\bullet(b))$$

for any $b \in A$, $S \subset \{1, ..., n-1\}$. Hence,

$$(\mathrm{Tr}_V)_\bullet(f_{|S|+1}(\alpha_{i_1},..,\alpha_{i_{|S|}},\beta')) \,=\, \psi^f_{\mathbf{a}_S}((\mathrm{Tr}_V)_\bullet(\beta'))$$

for all $\beta' \in A_\natural$ and $S := \{i_1 < ... < i_{|S|}\} \subset \{1, ..., n-1\}$. Since the operators $\{(\mathrm{Tr}_V)_\bullet(f_{|S|+1}(\alpha_{i_1},..,\alpha_{i_{|S|}},-))\}_S$ as well as $\{\psi^f_{\mathbf{a}_S}((\mathrm{Tr}_V)_\bullet(-))\}_S$ constitute polyderivations of the same multidegree with respect to $(\mathrm{Tr}_V)_\bullet \circ \mathbf{\Lambda}((f_1)_\natural)$, it follows that

$$(\mathrm{Tr}_V)_\bullet(f_{|S|+1}(\alpha_{i_1},..,\alpha_{i_{|S|}},\beta)) \,=\, \psi^f_{\mathbf{a}_S}((\mathrm{Tr}_V)_\bullet(\beta))$$

for all $\beta \in \mathbf{\Lambda}(A_\natural)$. Since the right hand side of the above equation vanishes when $(\mathrm{Tr}_V)_\bullet(\beta) = 0$, the required lemma follows. □

Lemma 7 and Theorem 7 imply that the maps

$$\wedge^n(A_V^{\mathrm{GL}}) \to B_V^{\mathrm{GL}}, \quad (\alpha_1,..,\alpha_n) \mapsto (\mathrm{Tr}_V)_\bullet(f_n((\mathrm{Tr}_V)_\bullet^{-1}(\alpha_1),...,(\mathrm{Tr}_V)_\bullet^{-1}(\alpha_n)))$$

are well defined. Theorem 7 together with Proposition 14 further implies that they constitute a P_∞-morphism f_V^{GL} from A_V^{GL} to B_V^{GL} (and that this construction is preserves compositions of morphisms). This proves Theorem 12 (i). A trivial modification of the same argument shows that if $h : A \to B \otimes \Omega$ is a P-homotopy between morphisms $f, g : A \to B \otimes \Omega$ of NC P_∞-algebras, then the above construction applied to h gives a morphism $h_V^{\mathrm{GL}} : A_V^{\mathrm{GL}} \to B_V^{\mathrm{GL}}$ that is a homotopy between the morphisms f_V^{GL} and g_V^{GL} of P_∞-algebras. This proves Theorem 12 (ii).

4.5. Remark. Another interesting question would be to relate $\mathtt{Ho}^*(\mathtt{NCP}_\infty)$ to $\mathtt{Ho}^*(\mathtt{NCPoiss})$. In this context, one may recall a theorem due to Munkholm ([**M**]), which states that $\mathtt{Ho}(\mathtt{DGA}_k)$ is equivalent to the full subcategory of $\mathtt{Ho}^*(A_\infty)$ whose objects are objects of $\mathtt{DGA}_k$. Here, $\mathtt{Ho}^*(A_\infty)$ is the category whose objects are A_∞-algebras and whose morphisms are homotopy classes of A_∞-morphisms. Given this, one may ask whether $\mathtt{Ho}^*(\mathtt{NCPoiss})$ is equivalent to the full subcategory of $\mathtt{Ho}^*(\mathtt{NCP}_\infty)$ whose objects are objects of $\mathtt{NCPoiss}$. If the answer to this question is negative, what is the precise relation between these two categories?

5. NC Poisson algebras from Calabi-Yau algebras

This section exhibits a large family of NC n-Poisson algebras: more generally, we show that if A is a finite dimensional graded n-cyclic algebra, the cobar construction applied to the linear dual $C := \mathrm{Hom}_k(A, k)$ of A is equipped with a $(2-n)$-Poisson double bracket. This Poisson double bracket induces a noncommutative $(2-n)$-Poisson structure. Of course, the cobar construction of C is cofibrant (in fact, free) as a DG algebra. In particular, a finite dimensional 2-cyclic algebra (for example, the cohomology of a compact smooth 2-manifold) gives rise to a cofibrant DG algebra with a noncommutative Poisson structure. We point out that a n-cyclic algebra is a special case of a n-Calabi-Yau A_∞-algebra in the sense of Kontsevich (see [**Cos**], Section 7.2).

In this section, we shall often use Sweedler's notation and write

$$\Delta(\alpha) \,=\, \alpha' \otimes \alpha''$$

for any element α in a coalgebra C with coproduct Δ.

5.1. Double Poisson (DG) algebras. Let A be an associative DG algebra over a field k. An A-bimodule M is a left $A \otimes A^{\mathrm{op}}$-module. On $A \otimes A$, there are two A-bimodule structures: one is the outer A-bimodule, namely

$$a \cdot (u \otimes v) \cdot b = au \otimes vb;$$

the other one is the inner A-bimodule, namely

$$a \cdot (u \otimes v) \cdot b = (-1)^{|a||u|+|a||b|+|b||v|} ub \otimes av\,.$$

Here, a, b, u, v are arbitrary homogenous elements of A.

Suppose that A is an associative (unital) DG algebra over a field k. A *double bracket* of degree n on A is a bilinear map

$$\{\!\{-,-\}\!\} : A \otimes A \to A \otimes A$$

which is a derivation of degree n (for the outer A-bimodule structure on $A \otimes A$) in its second argument and satisfies

$$\{\!\{a, b\}\!\} = -(-1)^{(|a|+n)(|b|+n)} \{\!\{b, a\}\!\}^\circ,$$

where $(u \otimes v)^\circ = (-1)^{|u||v|} v \otimes u$. For $a, b_1, ..., b_n$ homogeneous in A, let

$$\{\!\{a, b_1 \otimes \ldots \otimes b_n\}\!\}_L \,:=\, \{\!\{a, b_1\}\!\} \otimes b_2 \otimes \ldots \otimes b_n\,.$$

Further, for a permutation $s \in S_n$, let

$$\sigma_s(b_1 \otimes \ldots \otimes b_n) \,:=\, (-1)^t b_{s^{-1}(1)} \otimes \ldots \otimes b_{s^{-1}(n)}$$

where

$$t := \sum_{i<j;\, s^{-1}(i)>s^{-1}(j)} |a_{s^{-1}(i)}||a_{s^{-1}(j)}|\,.$$

Suppose that $\{\!\{-,-\}\!\}$ is a double bracket of degree n on A. If furthermore A satisfies the following *double Jacobi identity*

$$\big\{\!\!\big\{a, \{\!\{b,c\}\!\}\big\}\!\!\big\}_L + (-1)^{(|a|+n)(|b|+|c|)}\sigma_{(123)}\big\{\!\!\big\{b, \{\!\{c,a\}\!\}\big\}\!\!\big\}_L + (-1)^{(|c|+n)(|a|+|b|)}\sigma_{(132)}\big\{\!\!\big\{c, \{\!\{a,b\}\!\}\big\}\!\!\big\}_L = 0\,,$$

then A is called a *double n-Poisson algebra.*

Let $\mu : A \otimes A \to A$ denote the multiplication on A. Let $\{-,-\} := \mu \circ \{\!\{-,-\}\!\} : A \otimes A \to A$. The following is a direct generalization (to the DG setting) of Lemma 2.4.1 of [**VdB**].

LEMMA 8. *$\{-,-\}$ induces a noncommutative n-Poisson structure on A. In particular, when $n = 1$, $\{-,-\}$ induces a noncommutative Gerstenhaber structure on A.*

Notation. Using Sweedler's notation, we will often write

$$\{\!\{u,v\}\!\} \;=\; \{\!\{u,v\}\!\}' \otimes \{\!\{u,v\}\!\}''\,.$$

5.1.1. **Remark.** One thus has the category $n-\mathtt{DPoiss}$ of DG double n-Poisson algebras. We sketch how the analog of Theorem 10 holds for double n-Poisson algebras. The details in this subsubsection are left to the interested reader. Let A be a DG n-double Poisson algebra. Let Ω denote the de Rham algebra of the affine line, as in Section 3. Then, it is verified without difficulty that $A\otimes\Omega$ has a Ω-linear DG double n-Poisson structure. Indeed, after identifying $(A\otimes\Omega)\otimes_\Omega(A\otimes\Omega)$ with $A\otimes A\otimes\Omega$, the (Ω-linear) double bracket on $A\otimes\Omega$ is given by $\{\!\{-,-\}\!\}\otimes \mathrm{Id}_\Omega$.

One may therefore, define the "homotopy category" $\mathtt{Ho}^*(n-\mathtt{DPoiss})$ of $n-\mathtt{DPoiss}$: objects in $\mathtt{Ho}^*(n-\mathtt{DPoiss})$ are objects in $n-\mathtt{Poiss}$ that are cofibrant in $\mathtt{DGA}_k$. Morphisms in $\mathtt{Ho}^*(n-\mathtt{DPoiss})$ are homotopy classes of morphisms in $n-\mathtt{Poiss}$. Here, $f,g : A \to B$ in $n-\mathtt{Poiss}$ are *homotopic* if there exists $h : A \to B\otimes\Omega$ in $n-\mathtt{Poiss}$ such that $h(0) = f$ and $h(1) = g$. Here, when we say that $h : A \to B\otimes\Omega$ is in $n-\mathtt{Poiss}$, we mean that

$$\{\!\{-,-\}\!\}\circ(h\otimes_\Omega h) \;=\; (h\otimes_\Omega h)\circ\{\!\{-,-\}\!\}\,.$$

If A is a double n-Poisson algebra, a direct extension of the proofs of Propositions 7.5.1 and 7.5.2 of [**VdB**] shows that there exists a DG n-Poisson structure on A_V (which restricts to the one induced on A_V^{GL} by the corresponding noncommutative n-Poisson structure on A). One further shows without much difficulty that if A, B are double n-Poisson algebras and if $h : A \to B\otimes\Omega$ is a morphism of double n-Poisson algebras, then $h_V : A_V \to B_V\otimes\Omega$ is a morphism of DG n-Poisson algebras. Thus, the analog of Theorem 10 (with $(-)_V$ and $\boldsymbol{L}(-)_V$ replacing $(-)_V^{\mathrm{GL}}$ and $\boldsymbol{L}(-)_V^{\mathrm{GL}}$) holds for double n-Poisson algebras.

5.2. Cyclic graded algebras and double Poisson brackets. In this subsection, we will avoid specifying exact signs that are determined by the Koszul rule. Instead, such signs will be denoted by the symbol $\pm$. This is done in order to simplify cumbersome formulas, especially in the proof of Theorem 15.

5.2.1. Let A be a finite dimensional (graded) associative algebra with a symmetric inner product of degree n such that

$$\langle a, bc\rangle = \pm\langle ca, b\rangle, \quad \text{for any } a, b, c \in A. \tag{27}$$

According to Kontsevich ([**Ko**]) and Getzler-Kapranov ([**GK**]), such an algebra is called a n-*cyclic* associative algebra. In addition, if A is finite dimensional, the dual space $C := \mathrm{Hom}(A, k)$ is a coalgebra equipped with a symmetric bilinear pairing of degree $-n$. By the non degeneracy of the inner product on A, Equation (27) is dual to the following identity:

$$\langle v', w\rangle \cdot v'' = \pm\langle v, w''\rangle \cdot w', \quad \text{for any } v, w \in C. \tag{28}$$

Hence, C acquires the structure of *cyclic* $(-n)$-coalgebra. More generally, a DG coalgebra C equipped with a symmetric bilinear pairing $\langle -, -\rangle$ of degree n is called *cyclic* if in addition to (28),

$$\langle du, v\rangle \pm \langle u, dv\rangle = 0 \tag{29}$$

for all $u, v \in C$.

5.2.2. *Constructing the* $(n+2)$-*double Poisson bracket.* Let C be a n-cyclic coassociative coalgebra and let $\Omega(C)$ denote the cobar construction of C. Define $\{\!\{-,-\}\!\} : \Omega(C) \otimes \Omega(C) \to \Omega(C) \otimes \Omega(C)$ by

$$\begin{aligned} \{\!\{v, w\}\!\} := & \sum_{\substack{i=1,\cdots,n \\ j=1,\cdots,m}} \pm\langle v_i, w_j\rangle \cdot (w_1, \cdots, w_{j-1}, v_{i+1}, \cdots, v_n) \\ & \otimes\ (v_1, \cdots, v_{i-1}, w_{j+1}, \cdots, w_m)\,, \end{aligned} \tag{30}$$

where $v = (v_1, v_2, \cdots, v_n)$ and $w = (w_1, w_2, \cdots, w_m)$. The next theorem is the main result of this section.

THEOREM 15. *Let C be a n-cyclic coassociative DG coalgebra. The bracket* (30) *gives a double $(n+2)$-Poisson structure on the DG algebra $\Omega(C)$.*

REMARK 1. *By construction, $\Omega(C)$ is a cofibrant DG algebra.*

PROOF. The proof consists of three steps.

Step 1. First, recall that $\Omega(C)$ has a natural differential graded algebra structure, with multiplication given by the tensor product. We show that $\{\!\{-,-\}\!\}$ is a derivation for the second argument. For $u = (u_1, u_2, \cdots, u_p)$, $v = (v_1, v_2, \cdots, v_q)$, $w = (w_1, w_2, \cdots, w_r)$, we have

$$\begin{aligned} \{\!\{u, v\cdot w\}\!\} &= \{\!\{(u_1, \cdots, u_p), (v_1, \cdots, v_q, w_1, \cdots, w_r)\}\!\} \\ &= \sum_{\substack{i=1,\cdots,p \\ j=1,\cdots,q}} \pm\langle u_i, v_j\rangle \cdot (v_1, \cdots, v_{j-1}, u_{i+1}, \cdots, u_p) \\ &\otimes\ (u_1, \cdots, u_{i-1}, v_{j+1}, \cdots, v_q, w_1, \cdots, w_r) \\ &+ \sum_{\substack{i=1,\cdots,p \\ k=1,\cdots,r}} \pm\langle u_i, w_k\rangle \cdot (v_1, \cdots, v_q, w_1, \cdots, w_{k-1}, u_{i+1}, \cdots, u_p) \\ &\otimes\ (u_1, \cdots, u_{i-1}, w_{k+1}, \cdots, w_r). \end{aligned}$$

Hence

$$\{\!\{u, v\cdot w\}\!\} = \{\!\{u, v\}\!\}' \otimes \{\!\{u, v\}\!\}'' \cdot w \pm v \cdot \{\!\{u, w\}\!\}' \otimes \{\!\{u, w\}\!\}''. \tag{31}$$

Step 2. Next, we show that $\{\!\{-,-\}\!\}$ is skew symmetric and satisfies the double Jacobi identity. The skew symmetricity follows directly from the definition (30) as the pairing on $C[-1]$ induced by $\langle -,-\rangle$ is skew-symmetric. Therefore, we only need to check the double Jacobi identity. For $u=(u_1,u_2,\cdots,u_p)$, $v=(v_1,v_2,\cdots,v_q)$, $w=(w_1,w_2,\cdots,w_r)$, we have

$$\begin{aligned}
\{\!\{u,v\}\!\} &= \sum_{i,j} \pm\langle u_i,v_j\rangle\cdot(v_1,\cdots,v_{j-1},u_{i+1},\cdots,u_p)\\
&\qquad\qquad\qquad \otimes(u_1,\cdots,u_{i-1},v_{j+1},\cdots,v_q)\ ,\\
\{\!\{v,w\}\!\} &= \sum_{j,k} \pm\langle v_j,w_k\rangle\cdot(w_1,\cdots,w_{k-1},v_{j+1},\cdots,v_q)\\
&\qquad\qquad\qquad \otimes(v_1,\cdots,v_{j-1},w_{k+1},\cdots,w_r)\ ,\\
\{\!\{w,u\}\!\} &= \sum_{k,i} \pm\langle w_k,u_i\rangle\cdot(u_1,\cdots,u_{i-1},w_{k+1},\cdots,w_r)\\
&\qquad\qquad\qquad \otimes(w_1,\cdots,w_{k-1},u_{i+1},\cdots,u_p)\ .
\end{aligned}$$

Therefore

$$\Big\{\!\!\Big\{u,\{\!\{v,w\}\!\}'\Big\}\!\!\Big\}\otimes\{\!\{v,w\}\!\}''=$$

$$\begin{aligned}
&\sum_{\substack{i,j,k\\1\le l\le k-1}} \pm\langle v_j,w_k\rangle\langle u_i,w_l\rangle\cdot(w_1,\cdots,w_{l-1},u_{i+1},\cdots,u_p)\\
&\qquad\otimes(u_1,\cdots,u_{i-1},w_{l+1},\cdots,w_{k-1},v_{j+1},\cdots,v_q)\\
&\qquad\otimes(v_1,\cdots,v_{j-1},w_{k+1},\cdots,w_r)+
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\sum_{\substack{i,j,k\\j+1\le m\le q}} \pm\langle v_j,w_k\rangle\langle u_i,v_m\rangle\cdot(w_1,\cdots,w_{k-1},v_{j+1},\cdots,v_{m-1},u_{i+1},\cdots,u_p)\\
&\qquad\otimes(u_1,\cdots,u_{i-1},v_{m+1},\cdots,v_q)\otimes(v_1,\cdots,v_{j-1},w_{k+1},\cdots,w_r),
\end{aligned}\tag{33}$$

$$\{\!\{w,u\}\!\}''\otimes\Big\{\!\!\Big\{v,\{\!\{w,u\}\!\}'\Big\}\!\!\Big\}=$$

$$\begin{aligned}
&\sum_{\substack{i,j,k\\1\le t\le i-1}} \pm\langle w_k,u_i\rangle\langle v_j,u_t\rangle\cdot(w_1,\cdots,w_{k-1},u_{i+1},\cdots,u_p)\\
&\qquad\otimes(u_1,\cdots,u_{t-1},v_{j+1},\cdots,v_q)\\
&\qquad\otimes(v_1,\cdots,v_{j-1},u_{t+1},\cdots,u_{i-1},w_{k+1},\cdots,w_r)+
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\sum_{\substack{i,j,k\\k+1\le s\le n}} \pm\langle w_k,u_i\rangle\langle v_j,w_s\rangle\cdot(w_1,\cdots,w_{k-1},u_{i+1},\cdots,u_p)\\
&\qquad\otimes(u_1,\cdots,u_{i-1},w_{k+1},\cdots,w_{s-1},v_{j+1},\cdots,v_q)\\
&\qquad\otimes(v_1,\cdots,v_{j-1},w_{s+1},\cdots,w_n),
\end{aligned}\tag{35}$$

$$\{\!\{w, \{\!\{u,v\}\!\}'\}\!\}'' \otimes \{\!\{u,v\}\!\}'' \otimes \{\!\{w, \{\!\{u,v\}\!\}'\}\!\}' =$$

$$\sum_{\substack{i,j,k \\ 1\le n\le j-1}} \pm\langle u_i, v_j\rangle\langle w_k, v_n\rangle \cdot (w_1,\cdots,w_{k-1},v_{n+1},\cdots,v_{j-1},u_{i+1},\ldots,u_p)$$

$$\otimes (u_1,\cdots,u_{i-1},v_{j+1},\cdots,v_q)\otimes(v_1,\cdots,v_{n-1},w_{k+1},\cdots,w_r)+ \tag{36}$$

$$\sum_{\substack{i,j,k \\ j+1\le m\le p}} \pm\langle u_i, v_j\rangle\langle w_k, u_m\rangle \cdot (w_1,\cdots,w_{k-1},u_{m+1},\cdots,u_p)$$

$$\otimes (u_1,\cdots,u_{i-1},v_{j+1},\cdots,v_q) \tag{37}$$

$$\otimes (v_1,\cdots,v_{j-1},u_{i+1},\cdots,u_{m-1},w_{k+1},\cdots,w_r)$$

In the above equations, the summand (32) cancels with (35), (33) cancels with (36), and (34) cancels with (37). So we get

$$\{\!\{u, \{\!\{v,w\}\!\}'\}\!\} \otimes \{\!\{v,w\}\!\}'' \pm \{\!\{w,u\}\!\}'' \otimes \{\!\{v, \{\!\{w,u\}\!\}'\}\!\} \pm \{\!\{w, \{\!\{u,v\}\!\}'\}\!\}'' \otimes \{\!\{u,v\}\!\}'' \otimes \{\!\{w, \{\!\{u,v\}\!\}'\}\!\}' = 0,$$

which proves the double Jacobi identity.

Step 3. Using equation (31), one verifies without difficulty that if $\partial\{\!\{u,v\}\!\} = \{\!\{\partial u, v\}\!\} \pm \{\!\{u, \partial v\}\!\}$ and if $\partial\{\!\{u,w\}\!\} = \{\!\{\partial u, w\}\!\} \pm \{\!\{u, \partial w\}\!\}$, then $\partial\{\!\{u,vw\}\!\} = \{\!\{\partial u, vw\}\!\} \pm \{\!\{u, \partial(vw)\}\!\}$. By this fact and the skew-symmetry of $\{\!\{-,-\}\!\}$, it suffices to verify that $\partial\{\!\{u,v\}\!\} = \{\!\{\partial u, v\}\!\} \pm \{\!\{u, \partial v\}\!\}$ for all $u, v \in C$. In this case, $\partial\{\!\{u,v\}\!\} = 0$. On the other hand, $\partial u = du \pm (u', u'')$ and $\partial v = dv \pm (v', v'')$. Hence,

$$\{\!\{\partial u, v\}\!\} \pm \{\!\{u, \partial v\}\!\} = (\langle du, v\rangle \pm \langle u, dv\rangle) + (\langle u', v\rangle u'' \pm \langle u, v''\rangle v' \pm \langle v, u''\rangle u' \pm \langle v', u\rangle v'')\,,$$

where the first parenthesis in the right hand side vanishes by (29) and the second by (28). This proves that $\partial\{\!\{u,v\}\!\} = \{\!\{\partial u, v\}\!\} \pm \{\!\{u, \partial v\}\!\}$ for arbitrary $u, v \in \Omega(C)$, completing the proof of the theorem. □

5.2.3. **Remark**. One say that two morphisms $f, g : C_1 \to C_2$ of n-cyclic coalgebras are *homotopic* if there exists a family $\phi_t : C_1 \to C_2$ of morphisms of n-cyclic coalgebras varying polynomially with t as well as degree 1 coderivations s_t with respect to ϕ_t such that

$$\phi_0 = f, \phi_1 = g \text{ and } \frac{d\phi_t}{dt} = [d, s_t]\,.$$

We further require that for all $u, v \in C_1$,

$$\langle s_t(u), \phi_t(v)\rangle \pm \langle \phi_t(u), s_t(v)\rangle = 0\,.$$

The above notion of homotopy is dual to the notion of a polynomial M-homotopy betwee two morphisms in $\mathtt{DGA}_k$ (see [**BKR**], Proposition B.2 and subsequent remarks). Extending Theorem 15, one can further show that if $f, g : C_1 \to C_2$ are homotopic as morphisms of n-cyclic coalgebras, $\Omega(f)$ is homotopic to $\Omega(g)$ as morphisms of $(n+2)$-double Poisson algebras. We leave the relevant details to the motivated reader.

5.3. Cyclic homology of coalgebras. We recall the definition of Hochschild and cyclic homology of coalgebras. Given a coalgebra C over k consider the following double complex which is obtained by reversing the arrows in the standard (Tsygan) double complex of an algebra:

$$
\begin{tikzcd}
{} & {} & {} & \\
C^{\otimes 3} \arrow[u, "b"] \arrow[r, "1-T"] & C^{\otimes 3} \arrow[u, "b'"] \arrow[r, "N"] & C^{\otimes 3} \arrow[u, "b"] \arrow[r, "1-T"] & {} \\
C^{\otimes 2} \arrow[u, "b"] \arrow[r, "1-T"] & C^{\otimes 2} \arrow[u, "b'"] \arrow[r, "N"] & C^{\otimes 2} \arrow[u, "b"] \arrow[r, "1-T"] & {} \\
C \arrow[u, "b"] \arrow[r, "1-T"] & C \arrow[u, "b'"] \arrow[r, "N"] & C \arrow[u, "b"] \arrow[r, "1-T"] & {} \\
0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

This double complex is 2-periodic in horizontal direction, with operators b, b', T and N given by

$$
\begin{aligned}
b'(c_1, \cdots, c_n) &= \sum_{i=1}^{n-1} (-1)^{i-1} (c_1, \cdots, c_i', c_{i+1}', \cdots, c_n) \ , \\
b(c_1, \cdots, c_n) &= b'(c_1, \cdots, c_n) + \sum (-1)^n (c_1'', c_2, \cdots, c_n, c_1') \ , \\
T(c_1, \cdots, c_n) &= (-1)^{n-1} (c_2, \cdots, c_n, c_1) \ , \\
N = \sum_{i=0}^{n-1} T^i \ .
\end{aligned}
$$

The b-column is called the Hochschild chain complex $\mathrm{C}_\bullet(C, C)$ of C: it defines the *Hochschild homology* $\mathrm{HH}_\bullet(C)$. The kernel of $1 - T$ from the b-complex to the b'-complex is called the cyclic complex $\mathrm{CC}_\bullet(C)$: by definitiion, its homology is the *cyclic homology* $\mathrm{HC}_\bullet(C)$ of C.

REMARK 2. *If C is a coalgebra, then the dual complex $A := \mathrm{Hom}(C, k)$ admits an algebra structure. If furthermore C is finite dimensional, then the Hochschild complex $\mathrm{C}_\bullet(C, C)$ (resp., cyclic complex $\mathrm{CC}_*(C)$) is isomorphic to the Hochschild cochain complex $\mathrm{C}^\bullet(A, k)$ (reps., cyclic cochain complex $\mathrm{CC}^\bullet(A)$). Here the Hochschild cochain complex $\mathrm{C}^\bullet(A, k)$ is the Hochschild cochain complex of A with values in k. Otherwise if C is infinite dimensional, then the Hochschild complex $\mathrm{C}_\bullet(C, C)$ (reps. cyclic complex $\mathrm{CC}_\bullet(C)$) is a sub complex of the Hochschild cochain complex $\mathrm{C}^\bullet(A, k)$ (reps. cyclic cochain complex $\mathrm{CC}^\bullet(A)$).*

We collect some facts about the cyclic complex from Quillen [**Q2**, §1.3]. Let A be an associative algebra. The commutator subspace of A is $[A, A]$, which is the image of $m - m\sigma : A \otimes A \to A$, where m is the product and σ is the switching operator, and the commutator quotient space is

$$A_\natural := A/[A, A] = \mathrm{Coker}\{m - m\sigma : A \otimes A \to A\}.$$

Dually, suppose C is a coassociative coalgebra, the cocommutator sub space of C is

$$C^\natural := \mathrm{Ker}\{\Delta - \sigma\Delta : C \to C \otimes C\}.$$

Recall that the bar construction $\mathbf{B}(A)$ of A (resp. cobar construction $\boldsymbol{\Omega}(C)$ of C) is a differential graded (DG) coalgebra (resp. DG algebra). The following lemma is [**Q2**, Lemma 1.2].

LEMMA 9. *The space $\mathbf{B}^\natural_n(A)$ is the kernel of $(1-T)$ acting on $A^{\otimes n}$.*

Dually, the space $(\Omega_\natural(C))_n$ is the cokernel of $(1-T)$ acting on $C^{\otimes n}$. And therefore, via the isomorphisms

$$\mathrm{CC}_\bullet(A) = \mathrm{Coker}(1-T) \overset{\cong}{\to} \mathrm{Ker}(1-T), \quad \mathrm{CC}_\bullet(C) = \mathrm{Ker}(1-T) \overset{\cong}{\to} \mathrm{Coker}(1-T),$$

one obtains the following lemma.

LEMMA 10. *As complexes of k-vector spaces,*

$$\Omega(C)_\natural \;\cong\; \mathrm{CC}_\bullet(C)\,.$$

All explicit examples of derived NC Poisson structures in this paper arise by applying Theorem 15 and Lemma 8 to a cofibrant resolution of an honest algebra of A that is of the form $\Omega(C)$ for some finite dimensional cyclic DG coalgebra C. This makes the following corollary of this paper relevant.

COROLLARY 2. *Let C be a (DG) coalgebra such that $\Omega(C) \overset{\sim}{\twoheadrightarrow} A$ in $\mathtt{DGA}_k$ for some $A \in \mathtt{Alg}_k$. Then,*

$$\mathrm{HC}_\bullet(A) \;\cong\; \mathrm{HC}_\bullet(C)\,.$$

Now, suppose that A is a finite dimensional graded k-algebra. By definition, $\mathrm{HC}^i(A) \cong \mathrm{HC}_i(A)^*$. It is concentrated in homological degree $-i$. By Theorem 15, Lemma 8, Lemma 10 and the proof of Theorem 10 (i),

COROLLARY 3. *For any n-cyclic (finite dimensional) graded algebra A, $\mathrm{HC}^\bullet(A)[2-n]$ has the structure of a graded Lie algebra. Moreover, for any finite dimensional k-vector space V,*

$$(\mathrm{Tr}_V)_\bullet \;:\; \boldsymbol{\Lambda}(\mathrm{HC}^\bullet(A)) \to \mathrm{H}_\bullet(\Omega(A^*), V)^{\mathrm{GL}}$$

is a morphism of graded $(2-n)$-Poisson algebras.

For example, when $n=2$ and A is 2-cyclic, then $\Omega(A^*)$ is a double Poisson algebra by Theorem 15. Lemma 8 implies that $\Omega(A^*)$ acquires a noncommutative Poisson structure from its double Poisson structure. Lemma 10 implies that $\mathrm{HC}^\bullet(A)$ has a graded Lie algebra structure and Theorem 10 (i) implies that

$$(\mathrm{Tr}_V)_\bullet \;:\; \boldsymbol{\Lambda}(\mathrm{HC}^\bullet(A)) \to \mathrm{H}_\bullet(\Omega(A^*), V)^{\mathrm{GL}}$$

is a morphism of graded Poisson algebras. There is no shortage of cyclic 2-algebras: the cohomology $\mathrm{H}^\bullet(M, \mathbb{C})$ of any compact smooth 2-manifold M is such an algebra.

5.4. Derived Poisson structures on $k[x,y]$. The construction in the previous subsection is interesting when $C := k.a \oplus k.b \oplus k.s$ with $|a| = |b| = 1$ and $|s| = 2$ with $\Delta(a) = \Delta(b) = 0$ and $\Delta(s) = a \otimes b - b \otimes a$. In this case, there is a natural isomorphism

$$\Omega(C) \overset{\cong}{\to} R\ , \quad s \mapsto t,\ a \mapsto x,\ b \mapsto y\ ,$$

where $R := k\langle x, y, t\rangle$, $|x| = |y| = 0$, $|t| = 1$ with differential given by $dt := [x, y]$. Note that R is a almost free resolution of $A := k[x, y]$ in $\mathtt{DGA}_k$.

One can check that there is exactly one cyclic structure of degree -2 on C (up to multiplication by scalars) with $\omega(a, a) = \omega(b, b) = \omega(-, s) = 0$ and $\omega(a, b) = 1$. Similarly, there is exactly one cyclic structure of degree -3 on C (up to multiplication by scalars) with $\tilde{\omega}(a, s) = \tilde{\omega}(b, s) = 1$. By Theorem 15,

LEMMA 11. *The cyclic structure ω (resp., $\tilde{\omega}$) on C induces a NC Poisson (resp., NC (-1)-Poisson) structure on R.*

Since R is an almost free resolution of $A := k[x, y]$ (see [**BKR**], Example 4.1), taking homology yields a graded Lie bracket of degree 0 on $\overline{\mathrm{HC}}_\bullet(A)$ induced by the cyclic structure ω on C:

$$\{-,-\}_\natural :\ \overline{\mathrm{HC}}_\bullet(A) \times \overline{\mathrm{HC}}_\bullet(A) \to \overline{\mathrm{HC}}_\bullet(A)\ , \tag{38}$$

which is thus an example of a derived Poisson structure on A.

This structure has a natural geometric interpretation. If we restrict (38) to $\overline{\mathrm{HC}}_0(A) = \bar{A}$, we get the usual Poisson bracket on polynomials associated to the symplectic form $dx \wedge dy$. The Lie algebra $(\bar{A}, \{-,-\}_\natural)$ is thus isomorphic to the Lie algebra of (polynomial) symplectic vector fields on k^2. Now, if we identify $\overline{\mathrm{HC}}_1(A) = \Omega^1(A)/dA$ as in [**BKR**], Example 4.1, then, for any $\bar{f} \in \bar{A}$ and $\bar{\alpha} \in \Omega^1(A)/dA$,

$$\{\bar{f}, \bar{\alpha}\}_\natural = \mathcal{L}_{\theta_f}(\alpha)\ ,$$

where $\mathcal{L}_{\theta_f}$ is the Lie derivative on 1-forms taken along the Hamiltonian vector field θ_f. For example, if $f = x^p$ and $\alpha = y^q dx$, then $\bar{\alpha}$ corresponds to the class of the 1-cycle $qy^{q-1}t$ in $R_\natural$ (see [**BKR**], Example 4.1). Note that

$$\{x^p, y^{q-1}t\}_\natural\ =\ \sum_{i=1}^{q-1} px^{p-1}y^{q-1-i}ty^{i-1}\,.$$

Again, by [**BKR**], Example 4.1, the image of the R.H.S of the above equation is identified with the class of the 1-form $x^{p-1}y^{q-1}dx$ in $\overline{\mathrm{HC}}_1(A)$. Thus,

$$\{x^p, y^q dx\}_\natural\ =\ pqx^{p-1}y^{q-1}dx\,.$$

On the other hand, $\theta_f = px^{p-1}\frac{\partial}{\partial y}$, and hence,

$$\mathcal{L}_{\theta_f}(\alpha)\ =\ pqx^{p-1}y^{q-1}dx\,.$$

In addition, the restriction of (38) to $\overline{\mathrm{HC}}_1(A)$ is zero (for degree reasons). Thus, the graded Lie algebra $\overline{\mathrm{HC}}_\bullet(A)$ is isomorphic to the semidirect product $\bar{A} \ltimes (\Omega^1 A/dA)$, where $\bar{A}$ is equipped with the standard Poisson bracket and $\Omega^1(A)/dA$ is a Lie module over $\bar{A}$ with action induced by the Lie derivative on $\Omega^1(A)$. The Lie bracket (38) extends to the graded symmetric algebra $\mathbf{\Lambda}[\overline{\mathrm{HC}}(A)]$ making it a Poisson algebra. Theorem 10 implies that $\mathrm{H}_\bullet(A, V)^{\mathrm{GL}}$ has a (unique) graded Poisson structure such that the trace map $(\mathrm{Tr}_V)_\bullet : \mathbf{\Lambda}[\overline{\mathrm{HC}}(A)] \to \mathrm{H}_\bullet(A, V)^{\mathrm{GL}}$ is a morphism of Poisson algebras.

Similarly, taking homology yields a (graded) Lie bracket $\{-,-\}_{\natural,\tilde{\omega}}$ on $\overline{\mathrm{HC}}_\bullet(A)[-1]$ induced by the cyclic structure $\tilde{\omega}$ on C. Of course, for degree reasons the restriction of $\{-,-\}_{\natural,\tilde{\omega}}$ to $\overline{\mathrm{HC}}_0(A)$ is trivial. However, for $f \in \overline{A}$ and $\alpha \in \overline{\mathrm{HC}}_1(A) \cong \Omega^1(A)/dA$, the geometric interpretation of $\{f, \alpha\}_{\natural,\tilde{\omega}}$ remains mysterious.

We expect that the DG resolutions of algebras that are n-Calabi-Yau in the sense of Ginzburg [**G1**] (see also [**Ke**]) have analogous noncommutative $(2-n)$-Poisson structures. In particular, Ginzburg 2-Calabi-Yau algebras are expected to have derived NC Poisson structures.

5.5. Remarks on string topology. Let M be a smooth compact oriented manifold. Denote by LM the free loop space of M. In [**ChS**], M. Chas and D. Sullivan have shown that the S^1-equivariant homology $H_\bullet^{S^1}(LM)$ of LM has a natural Lie algebra structure. Their construction uses (in an essential way) the transversal intersection product of two chains in a manifold. Since the intersection product is only defined for transversal chains, it is difficult to realize the Lie algebra $H_\bullet^{S^1}(LM)$ algebraically. This is the subject of string topology, which has become a very active area of research in recent years.

By a well-known theorem of K.-T. Chen [**C**] and J. D. S. Jones [**J**], if M is simply connected, there is a quasi-isomorphism of complexes

$$\mathrm{CC}_\bullet(A(M)) \xrightarrow{\simeq} C^\bullet_{S^1}(LM) \ ,$$

where $A(M)$ is any DG algebra model (de Rham, singular, PL forms etc.) for the cochain complex of M. Similarly, using the methods of [**C**] and [**J**], one can construct a quasi-isomorphism

$$C_\bullet^{S^1}(LM) \xrightarrow{\simeq} \mathrm{CC}_\bullet(C(M)) \ ,$$

where $C(M)$ is any DG coalgebra model for the chain complex of M. On the other hand, Lambrechts and Stanley [**LS**] have recently shown that for M simply connected, there is a finite-dimensional DG coalgebra $C(M)$ with a cyclically invariant nondegenerate pairing, that is quasi-isomorphic to the singular chain complex of M. Further, the nondegenerate pairing on $C(M)$ gives the intersection product pairing at the homology level. Combining these results with our Corollary 3, we thus obtain a Lie algebra structure on the cyclic homology of $C(M)$ that realizes the Lie algebra of Chas and Sullivan (*cf.* [**CEG**]).

Besides Sullivan and his school, Blumberg, Cohen and Teleman are carrying out a project that aims to systematically lift the interesting structures on LM to the path space PM of M (see [**BCT**]). More precisely, they associate to M a category where the objects are the points of M and the space of morphisms between two objects is a (for example, singular) chain complex of the space of paths connecting them. Our present paper has essentially the same starting point as [**BCT**]: we have shown that the Lie algebra of string topology on $H_\bullet^{S^1}(LM)$ arises from the NC Poisson structure of the path space on M. More precisely, the cyclic homology of $\Omega(C)$, which is exactly the cyclic homology of the above described category, is isomorphic to $\mathrm{HC}_\bullet(C)$ (see Corollary 2), and hence is isomorphic to $H_\bullet^{S^1}(LM)$. This clarifies the relation between the above mentioned theorem of Jones and a well-known theorem of Goodwillie (see [**Go**]).

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Renormalization by any means necessary

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Renormalization

The most accurate physical theory we know today, the standard model of elementary particle physics, is based on quantum field theory. It describes all known forces and particles except gravity: electricity, magnetism, light, nuclear forces, neutrinos etc. Apart from two deep mysteries (dark matter and dark energy) it fits all known experimental facts. Quantum Field Theory [**A**] is the quantum theory of relativistic systems. Such a system can create any number of particle-antiparticle pairs, limited only by the amount of energy available. Therefore it must allow for an unbounded number of particles: it must have an infinite number of degrees of freedom.

The quantum theory of systems with a finite number of particles (atoms or molecules) was put on a sound mathematical basis beginning with von Neumann and later by the pioneering work of Kato. These mathematical theorems did not contribute much to the physical understanding of atomic theory. By the time Kato proved the self-adjointness of the atomic hamiltonian, quantum chemists had already understood how atoms and molecules form. They could calculate bond energies with an accuracy only limited by the computing power available then.

Quantum Field Theory is a different matter entirely. Because of the infinite number of degrees of freedom and the short distance singularities of the Green's functions, every calculation in a realistic quantum field theory is at best done with divergences. The mass of the electron has an infinite correction due to the emission and re-absorption of photons. Its magnetic moment, the energy levels of hydrogen etc. are all corrected this way. Following an idea of Dirac, in the 1940s Bethe,Tomonaga, Schwinger, Feynman and Dyson devised an elaborate scheme **-renormalization-** to remove these divergences and deduce physically correct answers from Quantum Electrodynamics. Although the manipulations are unpleasant,

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and the details daunting, what emerged was a physical theory of unprecedented accuracy and beauty. The magnetic moment of the electron has been calculated to an accuracy of 15 decimal places, in agreement with experiments.

There still remained the puzzle to explain the strong and weak forces, which involved the constituents of the atomic nucleus. A theory of weak interactions due to Fermi was refined by Sudarshan and Marshak. But it had divergences that could not be removed by renormalization. Glashow, Salam and Weinberg found a quantum field theory that incorporates the earlier ideas and is renormalizable.

There are very few quantum field theories that are renormalizable in four space-time dimensions. They are

- The Yang-Mills theory, characterized by a compact Lie group; it describes particles that are of spin one. Particles of spin zero and one can be coupled to it through a unitary representation.
- The Yukawa theory of spin half particles interacting with a spin zero particle.
- The Higgs model of Spin zero particles whose interaction is described by a polynomial of degree no greater than four.

Amazingly, this very limited set of ingredients are enough to explain all forces and particles of nature (except gravity). Moreover, all the renormalizable theories do occur with some choices of groups and representations.

There is a much greater variety of non-renormalizable theories: for which the scheme to remove divergences will fail at some order in perturbation theory. A naive quantization of General Relativity is such an example. The quantum version of the wave map (nonlinear sigma model) which describes certain low energy phenomena as well (pi mesons) cannot be a fundamental theory because it is not renormalizable. Such theories are still useful as approximations ("effective field theories").

Thus renormalizability is a very strict condition on a quantum field theory that very few of them satisfy. For reasons we do not know yet, nature chooses precisely these.

The Analysis of Renormalization

Many physicists believe that renormalizable quantum field theories are merely a stepping stone on the road to some more fundamental theory which is finite: in the same way that the Fermi theory of weak interactions was an effective theory for the standard model. Perhaps such a "theory of everything" would be a string theory, which could even explain gravity. Indeed string theories do seem to be free of the troublesome divergences. But any such extrapolation involves pushing what we know by many orders of magnitude in energy.

It is possible that far from being a nuisance, the divergences are clues to a new formulation of QFT. Finding a correct mathematical description of quantum field theory would put the currently known physical theories on a solid foundation. In an earlier era, calculus was beset by infinities and infinitesimals as well. The resolution was not to look for a theory of mechanics independent of calculus. Instead, modern real analysis reformulated the notions of continuity and differentiability, avoiding the direct use of infinitesimals. This did not just lead to new mathematics. Differences between an integrable and a chaotic system became evident. The solution is real analytic in the former and not in the other. Without analysis, physicists of the nineteenth century would not have understood why most physical systems could not be solved analytically, although a solution seemed possible in principle.

Thus, a careful mathematical study of the divergences of quantum field theory, and developing a rigorous theory of renormalization is a central challenge. A movement in this direction ("Constructive Quantum Field Theory") led by Glimm and Jaffe seems to have lost some of its momentum in the last few years. Rigorous mathematical training is rare among physicists, and command of physics even rarer among mathematicians. String theory has lured many people away from a deeper study of renormalization. Some new ideas were injected by A. Connes and disciples, but the emphasis on non-commutative geometry does not seem to have a basis in current physical reality.

Renormalization is a problem in analysis. The heuristic methods of QFT are similar to the way Euler did calculus. What needs to be developed is the equivalent of the more rigorous analysis of Weierstrass, Cauchy, Lebesgue, Banach etc. and its application back to physics as done by Poincare', Kolmogorov, Arnold, Moser etc.

Before understanding a deep physical theory like Yang-Mills theory rigorously, we should look at some simpler examples to get a general idea of the subject. One such example is the Kondo problem, which has served as a test-bed for ideas in renormalization for many decades. Wilson solved this problem by a combination of numerical and analytical methods in the 1970s. We will re-examine it in some approximations to discover its analytical context. But let us begin with an even more elementary example, a "toy model".

Renormalization in Quantum Mechanics

Is there a way to understand the basic idea of renormalization within quantum mechanics? We will present such an example, well-known among physicists[**B**]. A typical quantum system has a Hamiltonian

$$H = \Delta + V$$

The laplacian Δ describes the kinetic energy while V is a function that describes potential energy: it describes the interactions. Physical observables of quantum mechanics are represented by self-adjoint operators on L^2. To begin with, Δ is defined only on twice differentiable functions. It can be extended to a domain $D \subset L^2$: but this extension is not unique. Thus even without a potential, some physical interactions could be hidden within this extension. Physicists tend to describe such interactions still in terms of potentials, which then would be singular (e.g., delta function). Singularities that can be removed correspond to certain domains of self-adjointness. Thus the von Neumann theory of such extensions can be thought of an early version of the rigorous mathematical theory of renormalization that we seek.

But there is more to it than that. Of all the infinite number of self-adjoint extensions possible, renormalization theory picks out just a one or two parameter family. So there is still something to be understood in isolating the subset of extensions chosen by renormalization.

One Dimensional Delta Function. Consider a particle moving in one dimension under the influence of a delta function potential:

$$H = -\frac{d^2}{dx^2} - \text{“}J\delta(x)\text{”}$$

We put the quotes to emphasize that the precise meaning of $J\delta(x)$ is yet to be prescribed. For convenience, assume periodicity: H acts on wave-functions that are periodic with period 2π. In terms of Fourier components the eigenvalue equation becomes

$$k^2\psi_k - J\sum_{m\in Z}\psi_m = E\psi_k$$

The solution is easy:

$$\psi_k = \frac{J\psi_\bullet}{k^2-E}, \quad \psi_\bullet = \sum_{m\in Z}\psi_m$$

Eliminating the constant $\psi_\bullet$, we get an equation for the spectrum

$$1 = J\sum_{m\in Z}\frac{1}{m^2-E}.$$

The sum is convergent and can be evaluated by elementary complex analysis:

$$\frac{1}{J} = \frac{-\pi\cot\pi\sqrt{E}}{\sqrt{E}}$$

Given one eigenvalue (e.g., the smallest or "ground state energy") we can fix J and then all the others are determined. This example requires no renormalization.

Two Dimensional Delta Function. Suppose we repeat the analysis above on a two dimensional torus. The range of summation will now be Z^2. The series $\sum_{m\in Z^2}\frac{1}{|m|^2-E}$ diverges logarithmically.

The idea of renormalization translated to this situation is

- Put a cut-off in the magnitude of the Fourier index (momentum): $|m| < \Lambda$.
- Ask how should J ("the coupling constant") depend on Λ in order that the lowest energy is some given value $-\mu^2$.
- Then show that if we eliminate J this way, all the remaining eigenvalues remain finite as $\Lambda\to\infty$.
- We don't care about J. We care about eigenvalues, which describe energies and eigenfunctions which determine probabilities.

In more detail,

$$J^{-1}(\Lambda) = \sum_{|m|<\Lambda}\frac{1}{m^2+\mu^2}.$$

We can rewrite this as

$$\sum_{|m|<\Lambda}\frac{1}{m^2+\mu^2} - \sum_{|m|<\Lambda}\frac{1}{m^2-E} = 0$$

Although each series is divergent, the combined series has a limit as $\Lambda\to\infty$:

$$\chi(z) = \sum_{m\in Z\times Z}\left\{\frac{1}{m^2+\mu^2} - \frac{1}{m^2-z}\right\}$$

$\chi(z)$ is an analytic function whose zeroes are the eigenvalues. They are all determined in terms of the lowest eigenvalue. The divergent quantity $J(\Lambda)$ has disappeared from the final answer. This is an example of renormalization.

If you are familiar with Euler's work you will see that such tricks are quite old. The developments of modern analysis justified Euler's heuristic arguments.

This gives us hope that the much more complicated renormalizations of realistic quantum field theories can be understood by a proper development of analysis.

The Resolvent Formula. We can reformulate the above example (and many others like it) in more satisfactory mathematical terms using the resolvent ("Green's function" to physicists.)

Let $\hat{H}_0$ be an operator on $L^2(R)$ self-adjoint in some domain D. In most interesting cases $\hat{H}_0$ is a second order ordinary differential operator (e.g., Laplacian).We will see how to modify the domain and get an operator with a different spectrum, but without changing the effect on differentiable wave-functions. Physically, this amounts to changing the boundary conditions on the wave equation, which can also be thought of as singular potentials.

The resolvent of $\hat{H}_0$ is defined to be, for λ not in the spectrum,

$$\hat{R}_0 = \frac{1}{\lambda - \hat{H}_0}$$

It satisfies the equation

$$-\frac{d}{d\lambda}\hat{R}_0(\lambda) = \hat{R}_0^2(\lambda).$$

Conversely, any operator-valued analytic function satisfying this condition is of the form $(\lambda - \hat{H})^{-1}$; i.e., it is the resolvent of some operator. Given one such solution, we will construct another, as follows.

Let $\hat{P}$ be a linear map $L^2(R) \to R$. For example, it could be the evaluation at the origin: $\hat{P}\psi = \psi(0)$.

Since

$$[\hat{H}_0 - \lambda]\hat{R}_0(\lambda) = -1$$

$R_0(x, y|\lambda)$ satisfies the same differential equation as the eigenfunctions of $\hat{H}_0$ except at $x = y$ where it has a discontinuous derivative. For $\lambda \neq \lambda_n$, $\hat{R}_0$ is a compact operator; the integral kernel is often square summable in each variable.

PROPOSITION 0.1. *For μ not in the spectrum of $\hat{H}_0$, the kernel*

$$R(y, x|\lambda) = R_0(y, x|\lambda) + R_0(y, 0|\lambda)\frac{1}{\chi(\lambda) - \chi(\mu)}R_0(0, x|\lambda)$$

also satisfies the resolvent equation $\frac{d}{d\lambda}R(y, x|\lambda) = -\int R(y, z|\lambda)R(z, x|\lambda)dz$ provided that

$$\frac{d}{d\lambda}\chi(\lambda) = -\frac{d}{d\lambda}R_0(0, 0|\lambda).$$

PROOF. Denote the evaluation at the origin by $\hat{P}$. Then

$$\hat{R} = \hat{R}_0 + \hat{R}_0\hat{P}\frac{1}{\chi(\lambda) - \chi(\mu)}\hat{P}\hat{R}_0$$

By straightforward calculation

$$\hat{R}^2 = \hat{R}_0^2 + \hat{R}_0^2\hat{P}\frac{1}{\chi(\lambda) - \chi(\mu)}\hat{P}\hat{R}_0 + \hat{R}_0\hat{P}\frac{1}{\chi(\lambda) - \chi(\mu)}\hat{P}\hat{R}_0^2$$
$$+ \hat{R}_0\hat{P}\frac{1}{\chi(\lambda) - \chi(\mu)}\hat{P}\hat{R}_0^2\hat{P}\frac{1}{\chi(\lambda) - \chi(\mu)}\hat{P}\hat{R}_0$$

while

$$-\frac{d}{d\lambda}\hat{R} = \hat{R}_0^2 + \hat{R}_0^2\hat{P}\frac{1}{\chi(\lambda)-\chi(\mu)}\hat{P}\hat{R}_0 + \hat{R}_0\hat{P}\frac{1}{\chi(\lambda)-\chi(\mu)}\hat{P}\hat{R}_0^2$$
$$+ \hat{R}_0\hat{P}\left[-\frac{d}{d\lambda}\frac{1}{\chi(\lambda)-\chi(\mu)}\right]\hat{P}\hat{R}_0$$

so that

$$\frac{d}{d\lambda}\chi = \hat{P}\hat{R}_0^2\hat{P}$$
$$= -\frac{d}{d\lambda}\hat{P}\hat{R}_0\hat{P}$$

as needed. □

Thus $\hat{R}$ is the resolvent of some operator

$$\hat{R}(\lambda) = \frac{1}{\lambda - \hat{H}}$$

with the same form as $\hat{H}_0$ on smooth functions, but with a different domain of self-adjointness in $L^2(R)$.

Its spectrum is given by the equation

$$\chi(\lambda) = \chi(\mu)$$

In particular, μ itself is an eigenvalue, which determines the rest of the spectrum $\lambda_n(\mu)$ from the above equation. The orthonormal eigenfunctions of H are

$$\psi_n(x) = \frac{R_0(x,0|\lambda_n(\mu))}{\sqrt{\chi'(\lambda_n(\mu))}}$$

In the example of the last subsection,

$$R_0(x,y|\lambda) = \sum_{m\in Z^2}\frac{e^{im\cdot(x-y)}}{\lambda - |m|^2}$$

Although $R_0(0,0|\lambda) = \sum_{m\in Z^2}\frac{1}{\lambda-|m|^2}$ is divergent, its derivative $-\frac{d}{d\lambda}R_0(0,0|\lambda) = \sum_{m\in Z^2}\frac{1}{[\lambda-|m|^2]^2}$ converges. Hence $\chi(\lambda)-\chi(\mu) = \sum_{m\in Z^2}\left\{\frac{1}{[\lambda-|m|^2]} - \frac{1}{[\mu-|m|^2]}\right\}$ is convergent. $\chi(\mu)$ is simply the constant of integration in solving the equation $\frac{d}{d\lambda}\chi(\lambda) = -\frac{d}{d\lambda}R_0(0,0|\lambda)$.

The Kondo Problem

The simplest experimentally accessible case of renormalization is the Kondo problem. Electrons in a metal move more or less like free particles, occasionally scattered by the ions which oscillate around their equilibrium positions by thermal fluctuations. As temperature decreases, the ions move less and the resistance should decrease. It does, except that at some low temperature ($T \sim 10K$) the resistance starts to increase again, rising to a **finite** value as $T \to 0$. This "resistance minimum" was an important puzzle in the theory of metals in the 1950s.

Kondo gave an explanation to this phenomenon. Metals can have magnetic impurities (e.g., Iron atoms embedded within a Copper lattice) with ordered magnetic moments at low temperatures. These little magnets can scatter electrons too. At high temperatures, the direction of the atomic magnetic moments vary randomly, so their effect averages out to zero. At low temperatures, they get ordered, so there is a new way that electrons can lose their energy at low temperatures: by scattering by magnetic impurities. This is why the resistance increases at small temperatures.

The size of an atomic impurity is very small compared to the wavelength of the electron: it is much like a $\delta-$function interaction. Kondo calculated the magnetic contribution to resistance and found it to be proportional to $\log T^{-1}$. The good news is that it explains why resistance grows at low temperatures. The bad news is that it predicts **infinite** resistance at zero temperature. The Kondo problem is how to avoid this divergence.

Wilson's solution [**C**] of the Kondo problem in the 1970s was a major breakthrough: the first illustration of the power of his "renormalization group" (in fact only semi-group) method. His methods were a mix of numerical and analytical arguments. Much can be gained by clarifying and simplifying his arguments. We will reformulate this problem in terms of a Lie algebra, which allows for a solution based on representation theory of the infinite dimensional unitary group. The divergences arise if we do not take into account the infinite dimensionality of the group properly.

Only a brief summary is presented below. A much more self contained description can be found in this paper[**E**]. We will postpone to a later publication a re-examination of Wilson's reduction of the problem to a one dimensional quantum field theory. We will take as the hamiltonian of the Kondo problem

$$H = \sum_k \omega_k \Phi_k^k - J\Phi_d^\bullet \Phi_\bullet^d \equiv H_0 - JH_1$$

where

$$\Phi_L^K = \frac{1}{N} a^{\dagger K\sigma} a_{L\sigma}, \quad \Phi_d^\bullet = \sum_k \Phi_d^k, \quad \Phi_\bullet^d = \sum_k \Phi_d^k$$

The creation -annihilation operators of electrons are defined by the anti-commutation relations

$$[a^{\dagger K\sigma}, a_{L\sigma'}]_+ = \delta_L^K \delta_{\sigma\sigma'}, \quad [a^{\dagger K\sigma}, a^{\dagger L\sigma'}]_+ = 0 = [a_{K\sigma}, a_{L\sigma'}]_+$$

Here, σ is a discrete label taking values $1, \cdots, N$. It describes the spin of the electron. The physical value of N is two (a spin half particle can have two spin states); but it is useful to consider general values. Indeed, the limit $N \to \infty$ is simpler and surprisingly, can be a good approximation. The variable K describes the remaining degrees of freedom of the electron. Wilson showed that all of them except two types can be ignored:

(1) K takes some special value (we denote it by a dot $\bullet$) for an impurity electron that is at rest
(2) $K = -\Lambda, \cdots, -1, 1, \cdots \Lambda$ for the conduction band electrons. These numbers are proportional to the radial momentum measured w.r.t. the position of the impurity. Λ is a cut-off, which will eventually be sent to infinity.

When we sum over only the values $-\Lambda, \cdots, -1, 1, \cdots \Lambda$ (excluding the impurity) we will denote the index by a lower case letter k.

Thus, the hamiltonian of the Kondo problem consists of the kinetic energy of the conduction band electrons plus an interaction with the impurity. ω_k are the energies of the conduction band e.g., $\omega_k = \Omega[k - \frac{1}{2}]$ near Fermi surface for some constant Ω.

The impurity having zero size means that all the radial momenta couple with the same strength J. (The Fourier transform of the delta function is a constant.) The hamiltonian can be written entirely in terms of the bilinears Φ. They satisfy the commutation relation

$$\left[\Phi^K_L, \Phi^M_N\right] = \frac{1}{N}\left[\delta^M_L \Phi^K_N - \delta^K_N \Phi^M_L\right].$$

This is the statement that they form a representation of the Lie algebra $U(2N+1)$ or $A_{2N} \oplus R$ in Cartan's notation. In the limit of large N, these tend to a classical theory. That is, the commutation relations get replaced Poisson brackets.

$$-i\{\Phi^K_L, \Phi^M_N\ \} = \delta^M_L \Phi^K_N - \delta^K_N \Phi^M_L$$

In other words, for any function of $F(\Phi)$

$$-i\{F(\Phi), \Phi\} = \left[\frac{\partial F}{\partial \Phi}, \Phi\right]$$

where the r.h.s. is the matrix commutator. In particular for the hamiltonian

$$-i\frac{d\Phi}{dt} = [h(\Phi), \Phi], \qquad h(\Phi) = \frac{\partial H}{\partial \Phi}$$

$$h^m_n(\Phi) = \omega_m \delta^m_n, \quad h^d_m = -g^*(\Phi), \quad h^m_d = -g(\Phi).$$

with

$$g(\Phi) = J(\Lambda)\sum_{n=-\Lambda}^{\Lambda} \Phi^n_d.$$

All the divergences go away if we keep g fixed as $\Lambda \to \infty$.Diagonalization of $h(\Phi)$ is elementary: just solve

$$(\omega_k - \nu)U^k - gU^d = 0, \qquad -g^* U^\bullet = \nu U^d$$

The energies are roots of the characteristic function

$$X(\nu) = \nu - \sum_k \frac{|g|^2}{\nu - \omega_k} = \nu\left[1 + \sum_{k=1}^{\Lambda} \frac{2|g|^2}{\omega_k^2 - \nu^2}\right].$$

If we hold g fixed as the cutoff is removed, this sum is convergent. $(\omega_k \sim k)$.

The most difficult part of any Quantum Field Theory is knowing its ground state. In the large N limit it is given by the static solution of least energy. The static solution satisfies $[h(\Psi), \Psi] = 0$, so that $\Psi = U\ \mathrm{diag}(\nu)U^\dagger$ and by filling the negative energy states

$$n_\alpha = \begin{cases} 1, & \nu_\alpha < 0 \\ n_0, & \nu_\alpha = 0 \\ 0, & \nu_\alpha > 0 \end{cases}$$

Here n_0 is the number of electrons divided by N modulo 1. To keep g fixed in the UV limit, we must have $\lim_{\Lambda\to\infty} J(\Lambda)\sum_{k=1}^{\Lambda} \frac{1}{\omega_k} = 1$: asymptotic freedom. g sets the scale of energy: it is traded for J. It makes sense to pass to new variables $\phi = \Phi - \Psi$ that measure departure from this ground state. They satisfy the Poisson brackets

$$-i\left\{\phi^\alpha_\beta, \phi^\gamma_\delta\right\} = \delta^\gamma_\beta \phi^\alpha_\delta - \delta^\alpha_\delta \phi^\gamma_\beta + (n_\alpha - n_\gamma)\,\delta^\alpha_\delta \delta^\gamma_\beta$$

This is the central extension of the unitary Lie algebra, defined for example, in the book by Pressley-Segal[**D**]. If only a finite number of the ϕ^α_β are non-zero, we can supplement this with an element describing time evolution

$$-i\left\{h, \phi^\alpha_\beta\right\} = [\nu_\alpha - \nu_\beta]\phi^\alpha_\beta.$$

All the effects of the impurity-electron interaction are contained in the shift of the energies from ω_k to ν_α and in the occupation numbers n_α.

Now we have commutation relations rather than Poisson brackets:

$$\left[\hat{\phi}^\alpha_\beta, \hat{\phi}^\gamma_\delta\right] = \frac{1}{N}\left(\delta^\gamma_\beta\hat{\phi}^\alpha_\delta - \delta^\alpha_\delta\hat{\phi}^\gamma_\beta + [n_\alpha - n_\gamma]\,\delta^\alpha_\delta\delta^\gamma_\beta\right)$$

The representation of interest is

$$\hat{\phi}^\alpha_\beta = \frac{1}{N} : a^{\dagger\alpha\sigma}a_{\beta\sigma} : .$$

where the normal ordering is with respect to the Dirac vacuum of the energies e_α:

$$a^{\dagger\alpha} \mid 0\rangle = 0, \quad \nu_\alpha < 0, \qquad a_\alpha \mid 0\rangle = 0, \quad \nu_\alpha > 0.$$

The hamiltonian just describes quasi-particles with these energies:

$$\hat{H} = \frac{1}{N}\sum_\alpha \nu_\alpha : a^{\dagger\alpha\sigma}a_{\alpha\sigma} :$$

In real metals there are UV finite interactions which we ignored: a Fermi liquid.

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Berezin-Toeplitz quantization and star products for compact Kähler manifolds

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Abstract. For compact quantizable Kähler manifolds certain naturally defined star products and their constructions are reviewed. The presentation centers around the Berezin-Toeplitz quantization scheme which is explained. As star products the Berezin-Toeplitz, Berezin, and star product of geometric quantization are treated in detail. It is shown that all three are equivalent. A prominent role is played by the Berezin transform and its asymptotic expansion. A few ideas on two general constructions of star products of separation of variables type by Karabegov and by Bordemann–Waldmann respectively are given. Some of the results presented is work of the author partly joint with Martin Bordemann, Eckhard Meinrenken and Alexander Karabegov. At the end some works which make use of graphs in the construction and calculation of these star products are sketched.

1. Introduction

Without any doubts the concepts of quantization is of fundamental importance in modern physics. These concepts are equally influential in mathematics. The problems appearing in the physical treatments give a whole variety of questions to be solved by mathematicians. Even more, quantization challenges mathematicians to develop corresponding mathematical concepts with necessary rigor. Not only that they are inspiring in the sense that we mathematician provide solutions, but these developments will help to advance our mathematical disciplines. It is not the place here to try to give some precise definition what is quantization. I only mention that one mathematical aspect of quantization is to pass from the classical "commutative" world to the quantum "non-commutative" world. There are many possible aspects of this passage. One way is to replace the algebra of classical physical observables (functions depending locally on "position" and "momenta"), i.e. the commutative algebra of functions on the phase-space manifold, by a non-commutative algebra of operators acting on a certain Hilbert space. Another way

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is to "deform" the pointwise product in the algebra of functions into some non-commutative product $\star$. The first method is called operator quantization, the second deformation quantization and the product $\star$ is called a star product. In both cases by some limiting process the classical situation should be recovered. I did not touch the question whether it is possible at all to obtain such objects if one poses certain desirable conditions. For example, the desired properties for the star product (to be explained in the article further down) does not allow to deform the product inside of the function algebra for all functions. One is forced to pass to the algebra of formal power series over the functions and deform there. The resulting object will be a formal deformation quantization.

A special case of the operator method is geometric quantization. One chooses a complex hermitian (pre)quantum line bundle on the phase space manifold. The operators act on the space of global sections of the bundle or on suitable subspaces. In the that we can endow our phase-space manifold with the structure of a Kähler manifold (and only this case we are considering here) we have a more rigid situation. Our quantum line bundle should carry a holomorphic structure, if the bundle exists at all. The passage to the classical limit will be obtained by considering higher and higher tensor powers of the quantum line bundle. The sections of the bundle are the candidates of the quantum states. But they depend on too many independent variables. In the Kähler setting there is the naturally defined subspace of holomorphic sections. These sections are constant in anti-holomorphic directions. They will be the quantum states. This selection is sometimes called Kähler polarization.

In this review we will mainly deal with another type of operators on the space of holomorphic sections of the bundle. These will be the Toeplitz operators. They are naturally defined for Kähler manifolds. The assignment defines the Berezin-Toeplitz (BT) quantization scheme. Berezin himself considered it for certain special manifold [**11**], [**15**].

Being a quantum line bundle means that the curvature of the holomorphic hermitian line bundle is essentially equal to the Kähler form. See Section 2 for the precise formulation. A Kähler manifold is called quantizable if it admits a quantum line bundle. We will explain below that this is really a condition which not always can be fulfilled.

The author in joint work with Martin Bordemann and Eckhard Meinrenken [**18**] showed that at least in the compact quantizable Kähler case the BT-quantization has the correct semi-classical limit behavior, hence it is a quantization, see Theorem 3.3. In the compact Kähler case the operator of geometric quantization is asymptotically related to the Toeplitz operator, see (3.11). The details are presented in Section 3.

The special feature of the Berezin-Toeplitz quantization approach is that it does not only provide an operator quantization but also an intimately related star product, the Berezin-Toeplitz star product $\star_{BT}$. It is obtained by "asymptotic expansion" of the product of the two Toeplitz operators associated to the two functions to be $\star$-multiplied, see (4.4). After recalling the definition of a star product in Section 4.1, the results about existence and the properties of $\star_{BT}$ are given in Section 4.2. These are results of the author partly in joint work with Bordemann, Meinrenken, and Karabegov. The star product is a star product of separation of variables type (in the sense of Karabegov) or equivalently of Wick type (in the sense of Bordemann and Waldmann). We recall Karabegov's construction of star products of this type. In particular, we discuss his formal Berezin transform.

In Section 5 we introduce the disc bundle associated to the quantum line bundle and introduce the global Toeplitz operators. The individual Toeplitz operators for each tensor power of the line bundle correspond to its modes. The symbol calculus of generalized Toeplitz operators due to Boutet de Monvel and Guillemin [**21**] is used to prove some parts of the above mentioned results. In Section 5.3 as an illustration we explain how $\star_{BT}$ is constructed.

Other important techniques which we use in this context are Berezin-Rawnsley's coherent states, co- and contra-variant symbols [**24**] [**25**] [**26**] [**27**]. Starting from a function on M, assigning to it its Toeplitz operator and then calculating the co-variant symbol of the operator will yield another function. The corresponding map on the space of function is called Berezin transform I, see Section 7. The map will depend on the chosen tensor power m of the line bundle. Theorem 7.2, obtained jointly with Karabegov, shows that it has a complete asymptotic expansion. One of the ingredients of the proof is the off-diagonal expansion of the Bergman kernel in the neighborhood of the diagonal [**57**].

With the help of the Berezin transform I the Berezin star product can be defined

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)).$$

In Karabegov's terminology both star products are dual and opposite to each other.

In Section 8.3 a summary of the naturally defined star products are given. These are $\star_{BT}$, $\star_B$, $\star_{GQ}$ (the star product of geometric quantization), $\star_{BW}$ (the star product of Bordemann and Waldmann constructed in a manner à la Fedosov, see Section 9.1). The star products $\star_{BT}, \star_{BW}$ are of separation of variables type, $\star_B$ also but with the role of holomorphic and antiholomorphic variables switched, $\star_{GQ}$ is neither nor. The first three star products are equivalent.

How the knowledge of the asymptotic expansion of the Berezin transform will allow to calculate the coefficients of the Berezin star product and recursively of the Berezin-Toeplitz star product is explained in Section 8.4.

In the Section 9 we consider the Bordemann-Waldmann star product [**19**] and make some remarks how graphs are of help in expressing the star product in a convenient form. The work of Reshetikhin and Takhtajan [**77**], Gammelgaard [**48**], and Huo Xu [**92**], [**93**] are sketched.

In an excursion we describe Kontsevich's construction [**59**] of a star product for arbitrary Poisson structures on $\mathbb{R}^n$.

The closing Section 11 gives some applications of the Berezin-Toeplitz quantization scheme.

This review is based on a talk which I gave in the frame of the *Thematic Program on Quantization, Spring 2011*, at the University of Notre Dame, USA. Some of the material was added on the basis of the questions and the discussions of the audience. I am grateful to the organizers Sam Evens, Michael Gekhtman, Brian Hall, and Xiaobo Liu, and to the audience. All of them made this activity such a pleasant and successful event. In its present version the review supplements and updates [**85**],[**86**]. Other properties, like the properties of the coherent state embedding, more about Berezin symbols, traces and examples can be found there. In particular, [**85**] contains a more complete list of related works of other authors.

2. The geometric setup

In the following let (M, ω) be a Kähler manifold. This means M is a complex manifold (of complex dimension n) and ω, the Kähler form, is a non-degenerate closed positive $(1,1)$-form. In the interpretation of physics M will be the phase-space manifold. (But besides the jargon we will use nothing from physics here.) Further down we will assume that M is compact.

Denote by $C^\infty(M)$ the algebra of complex-valued (arbitrary often) differentiable functions with associative product given by point-wise multiplication. After forgetting the complex structure of M, our form ω will become a symplectic form and we introduce on $C^\infty(M)$ a Lie algebra structure, the *Poisson bracket* $\{.,.\}$, in the following way. First we assign to every $f \in C^\infty(M)$ its *Hamiltonian vector field* X_f, and then to every pair of functions f and g the *Poisson bracket* $\{.,.\}$ via

$$\omega(X_f, \cdot) = df(\cdot), \qquad \{f, g\} := \omega(X_f, X_g) \ . \tag{2.1}$$

In this way $C^\infty(M)$ becomes a *Poisson algebra*, i.e. we have the compatibility

$$\{h, f \cdot g\} = \{h, f\} \cdot g + f \cdot \{h, g\}, \qquad f, g, h \in C^\infty(M). \tag{2.2}$$

The next step in the geometric set-up is the choice of a quantum line bundle. In the Kähler case a *quantum line bundle* for (M, ω) is a triple (L, h, ∇), where L is a holomorphic line bundle, h a Hermitian metric on L, and ∇ a connection compatible with the metric h and the complex structure, such that the (pre)quantum condition

$$\begin{gathered}\operatorname{curv}_{L,\nabla}(X,Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = -\mathrm{i}\,\omega(X,Y), \\ \text{in other words} \quad \operatorname{curv}_{L,\nabla} = -\mathrm{i}\,\omega\end{gathered} \tag{2.3}$$

is fulfilled. By the compatibility requirement ∇ is uniquely fixed. With respect to a local holomorphic frame of the bundle the metric h will be represented by a function $\hat{h}$. Then the curvature with respect to the compatible connection is given by $\overline{\partial}\partial \log \hat{h}$. Hence, the quantum condition reads as

$$\mathrm{i}\,\overline{\partial}\partial \log \hat{h} = \omega \ . \tag{2.4}$$

If there exists such a quantum line bundle for (M, ω) then M is called quantizable. Sometimes the pair manifold and quantum line bundle is called quantized Kähler manifold.

Remark 2.1. Not all Kähler manifolds are quantizable. In the compact Kähler case from (2.3) it follows that the curvature is a positive form, hence L is a positive line bundle. By the Kodaira embedding theorem [**83**] there exists a positive tensor power $L^{\otimes m_0}$ which has enough global holomorphic sections to embed the complex manifold M via these sections into projective space $\mathbb{P}^N(\mathbb{C})$ of suitable dimension N. By Chow's theorem [**83**] it is a smooth projective variety. The line bundle $L^{\otimes m_0}$ which gives an embedding is called very ample. This implies for example, that only those higher dimensional complex tori are quantizable which admit "enough theta functions", i.e. which are abelian varieties.

A warning is in order, let $\phi : M \mapsto \mathbb{P}^N(\mathbb{C})$ be the above mentioned embedding as complex manifolds. This embedding is in general not a Kähler embedding, i.e. $\phi^*(\omega_{FS}) \neq \omega$, where ω_{FS} is the standard Fubini-Study Kähler form for $\mathbb{P}^N(\mathbb{C})$. Hence, we cannot restrict our attention only on Kähler submanifolds of projective space.

For compact Kähler manifolds we will always assume that the quantum bundle L itself is already very ample. This is not a restriction as $L^{\otimes m_0}$ will be a quantum line bundle for the rescaled Kähler form $m_0\omega$ for the same complex manifold M.

Next, we consider all positive tensor powers of the quantum line bundle: $(L^m, h^{(m)}, \nabla^{(m)})$, here $L^m := L^{\otimes m}$ and $h^{(m)}$ and $\nabla^{(m)}$ are naturally extended. We introduce a product on the space of sections. First we take the Liouville form $\Omega = \frac{1}{n!}\omega^{\wedge n}$ as volume form on M and then set for the product and the norm on the space $\Gamma_\infty(M, L^m)$ of global C^∞-sections (if they are finite)

$$\langle\varphi, \psi\rangle := \int_M h^{(m)}(\varphi, \psi)\,\Omega\;, \qquad ||\varphi|| := \sqrt{\langle\varphi, \varphi\rangle}\;. \tag{2.5}$$

Let $\mathrm{L}^2(M, L^m)$ be the L^2-completed space of bounded sections with respect to this norm. Furthermore, let $\Gamma^b_{hol}(M, L^m)$ be the space of global holomorphic sections of L^m which are bounded. It can be identified with a closed subspace of $\mathrm{L}^2(M, L^m)$. Denote by

$$\Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma^b_{hol}(M, L^m) \tag{2.6}$$

the orthogonal projection.

If the manifold M is compact "being bounded" is of course no restriction. Furthermore, $\Gamma_{hol}(M, L^m) = \Gamma^b_{hol}(M, L^m)$ and this space is finite-dimensional. Its dimension $N(m) := \dim \Gamma_{hol}(M, L^m)$ will be given by the Hirzebruch-Riemann-Roch Theorem [**83**]. Our projection will be

$$\Pi^{(m)} : \mathrm{L}^2(M, L^m) \to \Gamma_{hol}(M, L^m)\;. \tag{2.7}$$

If we fix an orthonormal basis $s^{(m)}_l, l = 1, \ldots, N(m)$ of $\Gamma_{hol}(M, L^m)$ then[1]

$$\Pi^{(m)}(\psi) = \sum_{l=1}^{N(m)} \langle s^{(m)}_l, \psi\rangle \cdot s^{(m)}_l. \tag{2.8}$$

3. Berezin-Toeplitz operator quantization

Let us start with the compact Kähler manifold case. I will make some remarks at the end of this section on the general setting. In the interpretation of physics, our manifold M is a phase-space. Classical observables are (real-valued) functions on the phase space. Their values are the physical values to be found by experiments. The classical observables commute under pointwise multiplication. One of the aspects of quantization is to replace the classical observable by something which is non-commutative. One approach is to replace the functions by operators on a certain Hilbert space (and the physical values to be measured should correspond to eigenvalues of them). In the Berezin-Toeplitz (BT) operator quantization this is done as follows.

Definition 3.1. For a function $f \in C^\infty(M)$ the associated *Toeplitz operator* $T^{(m)}_f$ *(of level m)* is defined as

$$T^{(m)}_f := \Pi^{(m)}(f\cdot) : \quad \Gamma_{hol}(M, L^m) \to \Gamma_{hol}(M, L^m)\;. \tag{3.1}$$

[1]In my convention the scalar product is anti-linear in the first argument.

In words: One takes a holomorphic section s and multiplies it with the differentiable function f. The resulting section $f \cdot s$ will only be differentiable. To obtain a holomorphic section, one has to project it back on the subspace of holomorphic sections.

With respect to the explicit representation (2.8) we obtain

$$T_f^{(m)}(s) := \sum_{l=1}^{N(m)} \langle s_l^{(m)}, f\, s\rangle\; s_l^{(m)}. \tag{3.2}$$

After expressing the scalar product (2.5) we get a representation of $T_f^{(m)}$ as an integral

$$T_f^{(m)}(s)(x) = \int_M f(y) \left(\sum_{l=1}^{N(m)} h^{(m)}(s_l^{(m)}, s)(y)\, s_l^{(m)}(x) \right) \Omega(y). \tag{3.3}$$

The space $\Gamma_{hol}(M, L^m)$ is the *quantum space (of level m)*. The linear map

$$T^{(m)} : C^\infty(M) \to \operatorname{End}\big(\Gamma_{hol}(M, L^m)\big), \quad f \to T_f^{(m)} = \Pi^{(m)}(f\cdot)\ , m \in \mathbb{N}_0 \tag{3.4}$$

is the *Toeplitz* or *Berezin-Toeplitz quantization map (of level m)*. It will neither be a Lie algebra homomorphism nor an associative algebra homomorphism as in general

$$T_f^{(m)}\, T_g^{(m)} = \Pi^{(m)}\,(f\cdot)\, \Pi^{(m)}\,(g\cdot)\, \Pi^{(m)} \neq \Pi^{(m)}\,(fg\cdot)\, \Pi = T_{fg}^{(m)}.$$

For M a compact Kähler manifold it was already mentioned that the space $\Gamma_{hol}(M, L^m)$ is finite-dimensional. On a fixed level m the BT quantization is a map from the infinite dimensional commutative algebra of functions to a noncommutative finite-dimensional (matrix) algebra. A lot of classical information will get lost. To recover this information one has to consider not just a single level m but all levels together as done in the

DEFINITION 3.2. The Berezin-Toeplitz (BT) quantization is the map

$$C^\infty(M) \to \prod_{m\in\mathbb{N}_0} \operatorname{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to (T_f^{(m)})_{m\in\mathbb{N}_0}. \tag{3.5}$$

In this way a family of finite-dimensional (matrix) algebras and a family of maps are obtained, which in the classical limit should "converges" to the algebra $C^\infty(M)$. That this is indeed the case and what "convergency" means will be made precise in the following.

Set for $f \in C^\infty(M)$ by $|f|_\infty$ the sup-norm of f on M and by

$$||T_f^{(m)}|| := \sup_{\substack{s\in\Gamma_{hol}(M,L^m)\\ s\neq 0}} \frac{||T_f^{(m)} s||}{||s||} \tag{3.6}$$

the operator norm with respect to the norm (2.5) on $\Gamma_{hol}(M, L^m)$.

That the BT quantization is indeed a quantization in the sense that it has the correct semi-classical limit, or that it is a strict quantization in the sense of Rieffel, is the content of the following theorem from 1994.

THEOREM 3.3. *[Bordemann, Meinrenken, Schlichenmaier]* [**18**]
(a) For every $f \in C^\infty(M)$ *there exists a* $C > 0$ *such that*

$$|f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{3.7}$$

In particular, $\lim_{m\to\infty} ||T_f^{(m)}|| = |f|_\infty$.
(b) For every $f, g \in C^\infty(M)$

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T_{\{f,g\}}^{(m)}|| \quad = \quad O(\frac{1}{m}) \ . \tag{3.8}$$

(c) For every $f, g \in C^\infty(M)$

$$||T_f^{(m)} T_g^{(m)} - T_{f\cdot g}^{(m)}|| \quad = \quad O(\frac{1}{m}) \ . \tag{3.9}$$

The original proof uses the machinery of generalized Toeplitz structures and operators as developed by Boutet de Monvel and Guillemin [**21**]. We will give a sketch of some parts of the proof in Section 5 and Section 7.3. In the meantime there also exists other proofs on the basis of Toeplitz kernels, Bergman kernels, Berezin transform etc. Each of them give very useful additional insights.

We will need in the following from [**18**]

PROPOSITION 3.4. *On every level* m *the Toeplitz map*

$$C^\infty(M) \to \mathrm{End}(\Gamma_{hol}(M, L^{(m)})), \qquad f \to T_f^{(m)},$$

is surjective.

Let us mention that for real-valued f the Toeplitz operator $T_f^{(m)}$ will be self-adjoint. Hence, they have real-valued eigenvalues.

REMARK 3.5. (Geometric Quantization.) Kostant and Souriau introduced the operators of geometric quantization in this geometric setting. In a first step the prequantum operator associated to the bundle L^m (and acting on its sections) for the function $f \in C^\infty(M)$ is defined as $P_f^{(m)} := \nabla^{(m)}_{X_f^{(m)}} + \mathrm{i}\, f \cdot id$. Here $X_f^{(m)}$ is the Hamiltonian vector field of f with respect to the Kähler form $\omega^{(m)} = m \cdot \omega$ and $\nabla^{(m)}_{X_f^{(m)}}$ is the covariant derivative. In the context of geometric quantization one has to choose a polarization. This corresponds to the fact that the "quantum states", i.e. the sections of the quantum line bundle, should only depend on "half of the variables" of the phase-space manifold M. In general, such a polarization will not be unique. But in our complex situation there is a canonical one by taking the subspace of holomorphic sections. This polarization is called *Kähler polarization*. This means that we only take those sections which are constant in anti-holomorphic directions. The operator of geometric quantization with Kähler polarization is defined as

$$Q_f^{(m)} := \Pi^{(m)} P_f^{(m)}. \tag{3.10}$$

By the surjectivity of the Toeplitz map there exists a function f_m, depending on the level m, such that $Q_f^{(m)} = T_{f_m}^{(m)}$. The Tuynman lemma [**89**] gives

$$Q_f^{(m)} = \mathrm{i} \cdot T^{(m)}_{f - \frac{1}{2m}\Delta f}, \tag{3.11}$$

where Δ is the Laplacian with respect to the Kähler metric given by ω. It should be noted that for (3.11) the compactness of M is essential.

As a consequence, which will be used later, the operators $Q_f^{(m)}$ and the $T_f^{(m)}$ have the same asymptotic behavior for $m \to \infty$.

REMARK 3.6. (The non-compact situation.) If our Kähler manifold is not necessarily compact then in a first step we consider as quantum space the space of bounded holomorphic sections $\Gamma^b_{hol}(M, L^m)$. Next we have to restrict the space of quantizable functions to a subspace of $C^\infty(M)$ such that the quantization map (3.5) (now restricted) will be well-defined. One possible choice is the subalgebra of functions with compact support. After these restrictions the Berezin-Toeplitz operators are defined as above. In the case of M compact, everything reduces to the already given objects. Unfortunately, there is no general result like Theorem 3.3 valid for arbitrary quantizable Kähler manifolds (e.g. for non-compact ones). There are corresponding results for special important examples. But they are more or less shown by case by case studies of the type of examples using tools exactly adapted to this situation. See [**85**] for references in this respect.

REMARK 3.7. (Auxiliary vector bundle.) We return to the compact manifold case. It is also possible to generalize the situation by considering an additional auxiliary hermitian holomorphic line bundle E. The sequence of quantum spaces is now the space of holomorphic sections of the bundles $E \otimes L^m$. For the case that E is a line bundle this was done, e.g. by Hawkins [**51**], for the general case by Ma and Marinescu, see [**64**] for the details. See also Charles [**32**]. By the hermitian structure of E we have a scalar product and a corresponding projection operator from the space of all sections to the space of holomorphic sections. The Toeplitz operator $T_f^{(m)}$ is defined for $f \in C^\infty(M, End(E))$. The situation considered in this review is that E equals the trivial line bundle. But similar results can be obtained in the more general set-up. This is also true with respect to the star product discussed in Section 4[2]. Of special importance, beside the trivial bundle case, is the case when the auxiliary vector bundle is a square root L_0 of the canonical line bundle K_M, i.e. $L_0^{\otimes 2} = K_M$ (if the square root exists). Recall that $K_M = \bigwedge^n \Omega_M$, where $n = \dim_{\mathbb{C}} M$ and Ω_M is the rang n vector bundle of holomorphic 1-differentials. The corresponding quantization is called quantization with metaplectic corrections. It turns out that with the metaplectic correction the quantization behaves better under natural constructions. An example is the *Quantization Commutes with Reduction* problem in the case that we have a well-defined action of a group G on the compact (quantizable) Kähler manifold with G-equivariant quantum line bundle. Under suitable conditions on the action we have a linear isomorphy of the G-invariant subspace of the quantum spaces $\mathrm{H}^0(M, L^m)^G$ with the quantum spaces $\mathrm{H}^0(M//G, (L//G)^m)$. This was shown by Guillemin and Sternberg [**49**]. But this isomorphy is not unitary. If one uses the quantum spaces with respect to the metaplectic correction then at least it is asymptotically (i.e. $m \to \infty$) unitary. This was shown independently[3] and with slightly different aspects by Ma and Zhang [**66**] (partly based on work of Zhang [**96**]) and by Hall and Kirwin [**50**]. See also [**63**].

[2]For E not a line bundle the Berezin-Toeplitz star product is a star product in $C^\infty(X, End(E))[[\nu]]$. This might be considered as a quantization with additional internal degrees of freedom, see [**64**, Remark 2.27].

[3]I am grateful to Xiaonan Ma for pointing this out to me.

For interesting details about these approaches see also the article of Kirwin [**58**] explaining some of the relations. For the general singular situation, see Li [**60**].

Another case when the quantization with metaplectic correction is more functorial is if one considers families of Kähler manifolds as they show up e.g. in the context of deforming complex structures on a given symplectic manifold. See work by Charles [**33**] and Andersen, Gammelgaard and Lauridsen [**6**].

4. Deformation quantization – star products

4.1. General definitions. There is another approach to quantization. One deforms the commutative algebra of functions "into non-commutative directions given by the Poisson bracket". It turns out that this can only be done on the formal level. One obtains a deformation quantization, also called star product. This notion was around quite a long time. See e.g. Berezin [**13**],[**15**], Moyal [**69**], Weyl [**91**], etc. Finally, the notion was formalized in [**9**]. See [**36**] for some historical remarks.

For a given Poisson algebra $(C^\infty(M), \cdot, \{\,,\,\})$ of smooth functions on a manifold M, a *star product* for M is an associative product $\star$ on $\mathcal{A} := C^\infty(M)[[\nu]]$, the space of formal power series with coefficients from $C^\infty(M)$, such that for $f, g \in C^\infty(M)$

(1) $f \star g = f \cdot g \mod \nu$,
(2) $(f \star g - g \star f)/\nu = -\mathrm{i}\{f, g\} \mod \nu$.

The star product of two functions f and g can be expressed as

$$f \star g = \sum_{k=0}^{\infty} \nu^k C_k(f, g), \qquad C_k(f, g) \in C^\infty(M), \tag{4.1}$$

and is extended $\mathbb{C}[[\nu]]$-bilinearly. It is called differential (or local) if the $C_k(\,,\,)$ are bidifferential operators with respect to their entries. If nothing else is said one requires $1 \star f = f \star 1 = f$, which is also called "null on constants".

Remark 4.1. (Existence) Given a Poisson bracket, is there always a star product? In the usual setting of deformation theory there always exists a trivial deformation. This is not the case here, as the trivial deformation of $C^\infty(M)$ to $\mathcal{A}$ extending the point-wise product trivially to the power series, is not allowed as it does not fulfill the second condition for the commutator of being a star product (at least not if the Poisson bracket is non-trivial). In fact the existence problem is highly non-trivial. In the symplectic case different existence proofs, from different perspectives, were given by DeWilde-Lecomte [**34**], Omori-Maeda-Yoshioka [**71**], and Fedosov [**44**]. The general Poisson case was settled by Kontsevich [**59**]. For more historical information see the review [**36**].

Two star products $\star$ and $\star'$ for the same Poisson structure are called *equivalent* if and only if there exists a formal series of linear operators

$$B = \sum_{i=0}^{\infty} B_i \nu^i, \qquad B_i : C^\infty(M) \to C^\infty(M),$$

with $B_0 = id$ such that $\quad B(f) \star' B(g) = B(f \star g)$.

To every equivalence class of a differential star product its *Deligne-Fedosov class* can be assigned. It is a formal de Rham class of the form

$$cl(\star) \in \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \mathrm{H}^2_{dR}(M,\mathbb{C})[[\nu]]). \tag{4.2}$$

This assignment gives a 1:1 correspondence between equivalence classes of star products and such formal forms.

In the Kähler case we might look for star products adapted to the complex structure. Karabegov [**52**] introduced the notion of star products with *separation of variables type* for differential star products. The star product is of this type if in $C_k(.,.)$ for $k \geq 1$ the first argument is only differentiated in holomorphic and the second argument in anti-holomorphic directions. Bordemann and Waldmann in their construction [**19**] used the name *star product of Wick type.*[4] All such star products $\star$ are uniquely given (not only up to equivalence) by their Karabegov form $kf(\star)$ which is a formal closed $(1,1)$ form. We will return to it in Section 4.3

4.2. The Berezin-Toeplitz deformation quantization.

THEOREM 4.2. [**18**],[**78**],[**80**],[**81**],[**57**] *There exists a unique differential star product*

$$f \star_{BT} g = \sum \nu^k C_k(f,g) \tag{4.3}$$

such that

$$T_f^{(m)} T_g^{(m)} \sim \sum_{k=0}^{\infty} \left(\frac{1}{m}\right)^k T^{(m)}_{C_k(f,g)}. \tag{4.4}$$

This star product is of separation of variables type with classifying Deligne-Fedosov class cl and Karabegov form kf

$$cl(\star_{BT}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}), \qquad kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \tag{4.5}$$

First, the asymptotic expansion in (4.4) has to be understood in a strong operator norm sense. For $f, g \in C^\infty(M)$ and for every $N \in \mathbb{N}$ we have with suitable constants $K_N(f,g)$ for all m

$$||T_f^{(m)} T_g^{(m)} - \sum_{0 \leq j < N} \left(\frac{1}{m}\right)^j T^{(m)}_{C_j(f,g)}|| \leq K_N(f,g) \left(\frac{1}{m}\right)^N . \tag{4.6}$$

Second, the used forms, resp. classes are defined as follows. Let K_M be the canonical line bundle of M, i.e. the n^{th} exterior power of the holomorphic bundle of 1-differentials. The canonical class δ is the first Chern class of this line bundle, i.e. $\delta := c_1(K_M)$. If we take in K_M the fiber metric coming from the Liouville form Ω then this defines a unique connection and further a unique curvature $(1,1)$-form ω_{can}. In our sign conventions we have $\delta = [\omega_{can}]$. The Karabegov form will be introduced in Section 4.3.

[4] In Karabegov's original approach the role of holomorphic and antiholomorphic variables are switched, i.e. in the approach of Bordemann-Waldmann they are of anti-Wick type. Unfortunately we cannot simply retreat to one these conventions, as we really have to deal in the following with naturally defined star products and relations between them, which are of separation of variables type of both conventions.

REMARK 4.3. Using Theorem 3.3 and the Tuynman relation (3.11) one can show that there exists a star product $\star_{GQ}$ given by asymptotic expansion of the product of geometric quantization operators. The star product $\star_{GQ}$ is equivalent to $\star_{BT}$, via the equivalence $B(f) := (id - \nu\frac{\Delta}{2})f$. In particular, it has the same Deligne-Fedosov class. But it is not of separation of variables type, see [**81**].

4.3. Star product of separation of variables type. In [**52, 53**] Karabegov not only gave the notion of *separation of variables type*, but also a proof of existence of such formal star products for any Kähler manifold, whether compact, non-compact, quantizable, or non-quantizable. Moreover, he classified them completely as individual star product not only up to equivalence.

In this set-up it is quite useful to consider more generally pseudo-Kähler manifolds (M, ω_{-1}), i.e. complex manifolds with a non-degenerate closed $(1,1)$-form ω_{-1} not necessarily positive. (In this context it is convenient to denote by ω_{-1} the ω we use at other places of the article.)

A formal form

$$\widehat{\omega} = (1/\nu)\omega_{-1} + \omega_0 + \nu\omega_1 + \dots \tag{4.7}$$

is called a formal deformation of the form $(1/\nu)\omega_{-1}$ if the forms $\omega_r,\ r \geq 0$, are closed but not necessarily nondegenerate (1,1)-forms on M. Karabegov showed that to every such $\widehat{\omega}$ there exists a star product $\star$. Moreover he showed that all deformation quantizations with separation of variables on the pseudo-Kähler manifold (M, ω_{-1}) are bijectively parameterized by the formal deformations of the form $(1/\nu)\omega_{-1}$. By definition the *Karabegov form* $kf(\star) := \widehat{\omega}$, i.e. it is taken to be the $\widehat{\omega}$ defining $\star$.

Let us indicate the principal idea of the construction. First, assume that we have such a star product $(\mathcal{A} := C^\infty(M)[[\nu]], \star)$. Then for $f, g \in \mathcal{A}$ the operators of left and right multiplication L_f, R_g are given by $L_f g = f \star g = R_g f$. The associativity of the star-product $\star$ is equivalent to the fact that L_f commutes with R_g for all $f, g \in \mathcal{A}$. If a star product is differential then L_f, R_g are formal differential operators. Now Karabegov constructs his star product associated to the deformation $\widehat{\omega}$ in the following way. First he chooses on every contractible coordinate chart $U \subset M$ (with holomorphic coordinates $\{z_k\}$) its formal potential

$$\widehat{\Phi} = (1/\nu)\Phi_{-1} + \Phi_0 + \nu\Phi_1 + \dots, \qquad \widehat{\omega} = i\partial\bar{\partial}\widehat{\Phi}. \tag{4.8}$$

Then the construction is done in such a way that the left (right) multiplication operators $L_{\partial\widehat{\Phi}/\partial z_k}$ ($R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$) on U are realized as formal differential operators

$$L_{\partial\widehat{\Phi}/\partial z_k} = \partial\widehat{\Phi}/\partial z_k + \partial/\partial z_k, \quad \text{and} \quad R_{\partial\widehat{\Phi}/\partial \bar{z}_l} = \partial\widehat{\Phi}/\partial \bar{z}_l + \partial/\partial \bar{z}_l. \tag{4.9}$$

The set $\mathcal{L}(U)$ of all left multiplication operators on U is completely described as the set of all formal differential operators commuting with the point-wise multiplication operators by antiholomorphic coordinates $R_{\bar{z}_l} = \bar{z}_l$ and the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. From the knowledge of $\mathcal{L}(U)$ the star product on U can be reconstructed. This follows from the simple fact that $L_g(1) = g$ and $L_f(L_g)(1) = f \star g$. The operator corresponding to the left multiplication with the (formal) function g can recursively (in the ν-degree) be calculated from the fact that it commutes with the operators $R_{\partial\widehat{\Phi}/\partial \bar{z}_l}$. The local star-products agree on the intersections of the charts and define the global star-product $\star$ on M. See the original work of Karabegov [**52**] for these statements.

We have to mention that this original construction of Karabegov will yield a star product of separation of variables type but with the role of holomorphic and antiholomorphic variables switched. This says for any open subset $U \subset M$ and any holomorphic function a and antiholomorphic function b on U the operators L_a and R_b are the operators of point-wise multiplication by a and b respectively, i.e., $L_a = a$ and $R_b = b$.

The construction of Karabegov is on one side very universal without any restriction on the (pseudo) Kähler manifold. But it does not establish any connection to an operator representation. The existence of such an operator representation is related in a vague sense to the quantization condition. The BT deformation quantization has such a relation and singles out a unique star product. Modulo switching the role of holomorphic and anti-holomorphic variable $\star_{BT}$ corresponds to a unique Karabegov form. This form is given in (4.5). The identification is done in Section 8.1 further down. That the form starts with $(-1/\nu)\omega$ is due to the fact that the role of the variables have to be switched to end up in Karabegov's classification.

4.4. Karabegov's formal Berezin transform. Given a pseudo-Kähler manifold (M, ω_{-1}). In the frame of his construction and classification Karabegov assigned to each star products $\star$ with the separation of variables property the formal *Berezin transform* $I_\star$. It is as the unique formal differential operator on M such that for any open subset $U \subset M$, antiholomorphic functions a and holomorphic functions b on U the relation

$$a \star b = I(b \cdot a) = I(b \star a), \tag{4.10}$$

holds true. The last equality is automatic and is due to the fact, that by the separation of variables property $b \star a$ is the point-wise product $b \cdot a$. He shows

$$I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \quad I_0 = id, \quad I_1 = \Delta. \tag{4.11}$$

Let us summarize. Karabegov's classification gives for a fixed pseudo-Kähler manifold a 1:1 correspondence between
(1) the set of star products with separation of variables type in Karabegov convention and
(2) the set of formal deformations (4.7) of ω_{-1}.
Moreover, the formal Berezin transform $I_\star$ determines the $\star$ uniquely.

We will introduce further down a Berezin transform in the set-up of the BT quantization. In [**57**] it is shown that its asymptotic expansion gives a formal Berezin transform in the sense of Karabegov, associated to a star product related to $\star_{BT}$ explained as follows.

4.5. Dual and opposite star products. Given for the pseudo-Kähler manifold (M, ω_{-1}) a star product $\star$ of separation of variables type (in Karabegov convention) Karabegov defined with the help of $I = I_\star$ the following associated star products. First the *dual* star-product $\tilde{\star}$ on M is defined for $f, g \in \mathcal{A}$ by the formula

$$f \,\tilde{\star}\, g = I^{-1}(I(g) \star I(f)). \tag{4.12}$$

It is a star-product with separation of variables but now on the pseudo-Kähler manifold $(M, -\omega_{-1})$. Denote by $\tilde{\omega} = -(1/\nu)\omega_{-1} + \tilde{\omega}_0 + \nu\tilde{\omega}_1 + \ldots$ the formal form parameterizing the star-product $\tilde{\star}$. By definition $\tilde{\omega} = kf(\tilde{\star})$. Its formal Berezin transform equals I^{-1}, and thus the dual to $\tilde{\star}$ is again $\star$.

Given a star product, the opposite star product is obtained by switching the arguments. Of course the sign of the Poisson bracket is changed. Now we take the opposite of the dual star-product, $\star' = \tilde{\star}^{op}$, given by

$$f \star' g = g \,\tilde{\star}\, f = I^{-1}(I(f) \star I(g)). \tag{4.13}$$

It defines a deformation quantization with separation of variables on M, but with the roles of holomorphic and antiholomorphic variables swapped - in contrast to $\star$. But now the pseudo-Kähler manifold will be (M, ω_{-1}). Indeed the formal Berezin transform I establishes an equivalence of the deformation quantizations $(\mathcal{A}, \star)$ and $(\mathcal{A}, \star')$.

How is the relation to the Berezin-Toeplitz star product $\star_{BT}$ of Theorem 4.2? There exists a certain formal deformation $\widehat{\omega}$ of the form $(1/\nu)\omega$ which yields a star product $\star$ in the Karabegov sense [**57**]. The opposite of its dual will be equal to the Berezin-Toeplitz star product, i.e.

$$\star_{BT} \;=\; \tilde{\star}^{op} \;=\; \star' \,. \tag{4.14}$$

The classifying Karabegov form $kf(\tilde{\star})$ will be the form (4.5). Here we fix the convention that we take for determining the Karabegov form of the BT star product the Karabegov form of the opposite one to adjust to Karabegov's original convention, i.e.

$$kf(\star_{BT}) := kf(\star_{BT}^{op}) = kf(\tilde{\star}). \tag{4.15}$$

As $\tilde{\star}$ is a star product for the pseudo-Kähler manifold $(M, -\omega)$ the $kf(\star_{BT})$ starts with $(-1/\nu)\omega$.

The formula (4.13) gives an equivalence between $\star$ and $\star_{BT}$ via I. Hence, we have for the Deligne-Fedosov class $cl(\star) = cl(\star_{BT})$, see the formula (4.5). We will identify $\widehat{\omega} = kf(\star)$ in Section 8.1.

5. Global Toeplitz operators

In this section we will indicate some parts of the proofs of Theorem 4.2 and Theorem 3.3. For this goal we consider the bundles L^m over the compact Kähler manifold M as associated line bundles of one unique S^1-bundle over M. The Toeplitz operator will appear as "modes" of a global Toeplitz operator. Moreover, we will need the same set-up to discuss coherent states, Berezin symbols, and the Berezin transform in the next sections.

5.1. The disc bundle. Recall that our quantum line bundle L was assumed to be already very ample. We pass to its dual line bundle $(U, k) := (L^*, h^{-1})$ with dual metric k. In the example of the projective space, the quantum line bundle is the hyperplane section bundle and its dual is the tautological line bundle. Inside the total space U, we consider the circle bundle

$$Q := \{\lambda \in U \mid k(\lambda, \lambda) = 1\},$$

and denote by $\tau : Q \to M$ (or $\tau : U \to M$) the projections to the base manifold M.

The bundle Q is a contact manifold, i.e. there is a 1-form ν such that $\mu = \frac{1}{2\pi}\tau^*\Omega \wedge \nu$ is a volume form on Q. Moreover,

$$\int_Q (\tau^* f)\mu = \int_M f\,\Omega, \qquad \forall f \in C^\infty(M). \tag{5.1}$$

Denote by $\mathrm{L}^2(Q,\mu)$ the corresponding L^2-space on Q. Let $\mathcal{H}$ be the space of (differentiable) functions on Q which can be extended to holomorphic functions on the disc bundle (i.e. to the "interior" of the circle bundle), and $\mathcal{H}^{(m)}$ the subspace of $\mathcal{H}$ consisting of m-homogeneous functions on Q. Here m-homogeneous means $\psi(c\lambda) = c^m\psi(\lambda)$. For further reference let us introduce the following (orthogonal) projectors: the *Szegö projector*

$$\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}, \tag{5.2}$$

and its components the *Bergman projectors*

$$\hat{\Pi}^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}. \tag{5.3}$$

The bundle Q is a S^1-bundle, and the L^m are associated line bundles. The sections of $L^m = U^{-m}$ are identified with those functions ψ on Q which are homogeneous of degree m. This identification is given on the level of the L^2 spaces by the map

$$\gamma_m : \mathrm{L}^2(M, L^m) \to \mathrm{L}^2(Q,\mu), \quad s \mapsto \psi_s \quad \text{where} \tag{5.4}$$

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))). \tag{5.5}$$

Restricted to the holomorphic sections we obtain the unitary isomorphism

$$\gamma_m : \Gamma_{hol}(M, L^m) \cong \mathcal{H}^{(m)}. \tag{5.6}$$

5.2. Toeplitz structure. Boutet de Monvel and Guillemin introduced the notion of a Toeplitz structure (Π, Σ) and associated generalized Toeplitz operators [**21**]. If we specialize this to our situation then Π is the Szegö projector (5.2) and Σ is the submanifold

$$\Sigma := \{\ t\nu(\lambda) \mid \lambda \in Q,\ t > 0\ \} \ \subset\ T^*Q \setminus 0 \tag{5.7}$$

of the tangent bundle of Q defined with the help of the 1-form ν. It turns out that Σ is a symplectic submanifold, a symplectic cone.

A (generalized) *Toeplitz operator* of order k is an operator $A : \mathcal{H} \to \mathcal{H}$ of the form $A = \Pi \cdot R \cdot \Pi$ where R is a pseudo-differential operator (ΨDO) of order k on Q. The Toeplitz operators constitute a ring. The symbol of A is the restriction of the principal symbol of R (which lives on T^*Q) to Σ. Note that R is not fixed by A, but Boutet de Monvel and Guillemin showed that the symbols are well-defined and that they obey the same rules as the symbols of ΨDOs. In particular, the following relations are valid:

$$\sigma(A_1A_2) = \sigma(A_1)\sigma(A_2), \qquad \sigma([A_1, A_2]) = \mathrm{i}\,\{\sigma(A_1), \sigma(A_2)\}_\Sigma. \tag{5.8}$$

Here $\{.,.\}_\Sigma$ is the restriction of the canonical Poisson structure of T^*Q to Σ coming from the canonical symplectic form on T^*Q. Furthermore, a Toeplitz operator of order k with vanishing symbol is a Toeplitz operator of order $k-1$.

We will need the following two generalized Toeplitz operators:

(1) The generator of the circle action gives the operator $D_\varphi = \frac{1}{\mathrm{i}}\frac{\partial}{\partial\varphi}$, where φ is the angular variable. It is an operator of order 1 with symbol t. It operates on $\mathcal{H}^{(m)}$ as multiplication by m.

(2) For $f \in C^\infty(M)$ let M_f be the operator on $\mathrm{L}^2(Q,\mu)$ corresponding to multiplication with $\tau^* f$. We set

$$T_f = \Pi \cdot M_f \cdot \Pi : \quad \mathcal{H} \to \mathcal{H}\ . \tag{5.9}$$

As M_f is constant along the fibers of τ, the operator T_f commutes with the circle action. Hence we can decompose

$$T_f = \prod_{m=0}^{\infty} T_f^{(m)} \ , \tag{5.10}$$

where $T_f^{(m)}$ denotes the restriction of T_f to $\mathcal{H}^{(m)}$. After the identification of $\mathcal{H}^{(m)}$ with $\Gamma_{hol}(M, L^m)$ we see that these $T_f^{(m)}$ are exactly the Toeplitz operators $T_f^{(m)}$ introduced in Section 3. We call T_f the global Toeplitz operator and the $T_f^{(m)}$ the local Toeplitz operators. The operator T_f is of order 0. Let us denote by $\tau_\Sigma : \Sigma \subseteq T^*Q \to Q \to M$ the composition then we obtain for its symbol $\sigma(T_f) = \tau_\Sigma^*(f)$.

5.3. The construction of the BT star product. To give a sketch of the proof of Theorem 4.2 we will need the statements of Theorem 3.3. The part (a) of this theorem we will show with the help of the asymptotic expansion of the Berezin transform in Section 7.3. The other parts will be sketched here, too. Full proofs of Theorem 4.2 can be found in [**81**], [**80**]. Full proofs of Theorem 3.3 in [**18**].

Let the notation be as in the last subsection. In particular, let T_f be the Toeplitz operator, D_φ the operator of rotation, and $T_f^{(m)}$, resp. $(m\cdot)$ their projections on the eigenspaces $\mathcal{H}^{(m)} \cong \Gamma_{hol}(M, L^m)$.

(a) The definition of the $C_j(f,g) \in C^\infty(M)$
The construction is done inductively in such a way that

$$A_N = D_\varphi^N T_f T_g - \sum_{j=0}^{N-1} D_\varphi^{N-j} T_{C_j(f,g)} \tag{5.11}$$

is always a Toeplitz operator of order zero. The operator A_N is S^1-invariant, i.e. $D_\varphi \cdot A_N = A_N \cdot D_\varphi$. As it is of order zero his symbol is a function on Q. By the S^1-invariance the symbol is even given by (the pull-back of) a function on M. We take this function as next element $C_N(f,g)$ in the star product. By construction, the operator $A_N - T_{C_N(f,g)}$ is of order -1 and $A_{N+1} = D_\varphi(A_N - T_{C_N(f,g)})$ is of order 0 and exactly of the form given in (5.11).

The induction starts with

$$A_0 = T_f T_g, \qquad \text{and} \tag{5.12}$$

$$\sigma(A_0) = \sigma(T_f)\sigma(T_g) = \tau_\Sigma^*(f) \cdot \tau_\Sigma^*(g) = \tau_\Sigma^*(f \cdot g) \ . \tag{5.13}$$

Hence, $C_0(f,g) = f \cdot g$ as required.
It remains to show statement (4.6) about the asymptotics. As an operator of order zero on a compact manifold A_N is bounded (ΨDOs of order 0 on compact manifolds are bounded). By the S^1-invariance we can write $A = \prod_{m=0}^{\infty} A^{(m)}$ where $A^{(m)}$ is the restriction of A on the orthogonal subspace $\mathcal{H}^{(m)}$. For the norms we get $||A^{(m)}|| \le ||A||$. If we calculate the restrictions we obtain

$$||m^N T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{N-1} m^{N-j} T_{C_j(f,g)}^{(m)}|| = ||A_N^{(m)}|| \le ||A_N|| \ . \tag{5.14}$$

After dividing by m^N Equation (4.6) follows. Bilinearity is clear. For $N = 1$ we obtain (3.9) and Theorem 3.3, Part (c).

(b) The Poisson structure

First we sketch the proof for (3.8). For a fixed $t > 0$

$$\Sigma_t := \{t \cdot \nu(\lambda) \mid \lambda \in Q\} \quad \subseteq \Sigma. \tag{5.15}$$

It turns out that $\omega_{\Sigma|\Sigma_t} = -t\tau_\Sigma^*\omega$. The commutator $[T_f, T_g]$ is a Toeplitz operator of order -1. From the above we obtain with (5.8) for the symbol of the commutator

$$\sigma([T_f, T_g])(t\nu(\lambda)) = \mathrm{i}\,\{\tau_\Sigma^* f, \tau_\Sigma^* g\}_\Sigma(t\nu(\lambda)) = -\,\mathrm{i}\, t^{-1}\{f, g\}_M(\tau(\lambda)) \ . \tag{5.16}$$

We consider the Toeplitz operator

$$A := D_\varphi^2\,[T_f, T_g] + \mathrm{i}\, D_\varphi\, T_{\{f,g\}} \ . \tag{5.17}$$

Formally this is an operator of order 1. Using $\sigma(T_{\{f,g\}}) = \tau_\Sigma^*\{f, g\}$ and $\sigma(D_\varphi) = t$ we see that its principal symbol vanishes. Hence it is an operator of order 0. Arguing as above we consider its components $A^{(m)}$ and get $||A^{(m)}|| \le ||A||$. Moreover,

$$A^{(m)} = A_{|\mathcal{H}^{(m)}} = m^2[T_f^{(m)}, T_g^{(m)}] + \mathrm{i}\, m T_{\{f,g\}}^{(m)}. \tag{5.18}$$

Taking the norm bound and dividing it by m we get part (b) of Theorem 3.3. Using (5.6) the norms involved indeed coincide.

For the star product we have to show that $C_1(f, g) - C_1(g, f) = -\,\mathrm{i}\,\{f, g\}$. We write explicitly (5.14) for $N = 2$ and the pair of functions (f, g):

$$||m^2 T_f^{(m)} T_g^{(m)} - m^2 T_{f\cdot g}^{(m)} - m T_{C_1(f,g)}^{(m)}|| \le K \ . \tag{5.19}$$

A corresponding expression is obtained for the pair (g, f). If we subtract both operators inside of the norm we obtain (with a suitable K')

$$||m^2(T_f^{(m)} T_g^{(m)} - T_g^{(m)} T_f^{(m)}) - m(T_{C_1(f,g)}^{(m)} - T_{C_1(g,f)}^{(m)})|| \le K' \ . \tag{5.20}$$

Dividing by m and multiplying with i we obtain

$$||m\,\mathrm{i}\,[T_f^{(m)}, T_g^{(m)}] - T^{(m)}_{\mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}|| = O(\frac{1}{m}) \ . \tag{5.21}$$

Using the asymptotics given by Theorem 3.3(b) for the commutator we get

$$||T^{(m)}_{\{f,g\} - \mathrm{i}\left(C_1(f,g) - C_1(g,f)\right)}|| = O(\frac{1}{m}) \ . \tag{5.22}$$

Taking the limit for $m \to \infty$ and using Theorem 3.3(a) we get

$$||\{f, g\} - \mathrm{i}\,(C_1(f, g) - C_1(g, f))||_\infty = 0 \ . \tag{5.23}$$

Hence indeed, $\{f, g\} = \mathrm{i}\,(C_1(f, g) - C_1(g, f))$. For the associativity and further results, see [**81**].

Within this approach the calculation of the coefficient functions $C_k(f, g)$ is recursively and not really constructive. In Section 8.4 we will show another way how to calculate the coefficients. It is based on the asymptotic expansion of the Berezin transform, which itself is obtained via the off-diagonal expansion of the Bergman kernel.

In fact the Toeplitz operators again can be expressed via kernel functions also related to the Bergman kernel. In this way certain extensions of the presented results are possible. See in particular work by Ma and Marinescu for compact symplectic manifolds and orbifolds. One might consult the review [**64**] for results and further references.

For another approach (still symbol oriented) to Berezin Toeplitz operator and star product quantization see Charles [**31**], [**30**].

6. Coherent states and symbols

Berezin constructed for an important but limited classes of Kähler manifolds a star product. The construction was based on his covariant symbols given for domains in $\mathbb{C}^n$. In the following we will present their definition for arbitrary compact quantizable Kähler manifolds.

6.1. Coherent states. We look again at the relation (5.5)

$$\psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))),$$

but now from the point of view of the linear evaluation functional. This means, we fix $\alpha \in U \setminus 0$ and vary the sections s.

The *coherent vector (of level m)* associated to the point $\alpha \in U \setminus 0$ is the element $e_\alpha^{(m)}$ of $\Gamma_{hol}(M, L^m)$ with

$$\langle e_\alpha^{(m)}, s\rangle = \psi_s(\alpha) = \alpha^{\otimes m}(s(\tau(\alpha))) \tag{6.1}$$

for all $s \in \Gamma_{hol}(M, L^m)$. A direct verification shows $e_{c\alpha}^{(m)} = \bar{c}^m \cdot e_\alpha^{(m)}$ for $c \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$. Moreover, as the bundle is very ample we get $e_\alpha^{(m)} \neq 0$.

This allows the following definition.

Definition 6.1. The *coherent state (of level m)* associated to $x \in M$ is the projective class

$$\mathrm{e}_x^{(m)} := [e_\alpha^{(m)}] \in \mathbb{P}(\Gamma_{hol}(M, L^m)), \qquad \alpha \in \tau^{-1}(x), \alpha \neq 0. \tag{6.2}$$

The *coherent state embedding* is the antiholomorphic embedding

$$M \quad \to \quad \mathbb{P}(\Gamma_{hol}(M, L^m)) \;\cong\; \mathbb{P}^N(\mathbb{C}), \qquad x \mapsto [e_{\tau^{-1}(x)}^{(m)}]. \tag{6.3}$$

See [**10**] for some geometric properties of the coherent state embedding.

Remark 6.2. A coordinate independent version of Berezin's original definition and extensions to line bundles were given by Rawnsley [**76**]. It plays an important role in the work of Cahen, Gutt, and Rawnsley on the quantization of Kähler manifolds [**24, 25, 26, 27**], via Berezin's covariant symbols. In these works the coherent vectors are parameterized by the elements of $L \setminus 0$. The definition here uses the points of the total space of the dual bundle U. It has the advantage that one can consider all tensor powers of L together on an equal footing.

6.2. Covariant Berezin symbol.

Definition 6.3. For an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ its *covariant Berezin symbol* $\sigma^{(m)}(A)$ *(of level m)* is defined as the function

$$\sigma^{(m)}(A) : M \to \mathbb{C}, \quad x \mapsto \sigma^{(m)}(A)(x) := \frac{\langle e_\alpha^{(m)}, A e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \quad \alpha \in \tau^{-1}(x) \setminus \{0\}. \tag{6.4}$$

Using the *coherent projectors* (with the convenient bra-ket notation)

$$P_x^{(m)} = \frac{|e_\alpha^{(m)}\rangle\langle e_\alpha^{(m)}|}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \qquad \alpha \in \tau^{-1}(x) \tag{6.5}$$

it can be rewritten as $\sigma^{(m)}(A) = \mathrm{Tr}(AP_x^{(m)})$. In abuse of notation $\alpha \in \tau^{-1}(x)$ should always mean $\alpha \neq 0$.

6.3. Contravariant Symbols. We need Rawnsley's epsilon function $\epsilon^{(m)}$ [**76**] to introduce contravariant symbols in the general Kähler manifold setting. It is defined as

$$\epsilon^{(m)}: M \to C^\infty(M), \quad x \mapsto \epsilon^{(m)}(x) := \frac{h^{(m)}(e_\alpha^{(m)}, e_\alpha^{(m)})(x)}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}, \ \alpha \in \tau^{-1}(x). \tag{6.6}$$

As $\epsilon^{(m)} > 0$ we can introduce the modified measure $\Omega_\epsilon^{(m)}(x) := \epsilon^{(m)}(x)\Omega(x)$ on the space of functions on M. If M is a homogeneous manifold under a transitive group action and everything is invariant, $\epsilon^{(m)}$ will be constant. This was the case considered by Berezin.

Definition 6.4. Given an operator $A \in \mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ then **a** contravariant Berezin symbol $\check{\sigma}^{(m)}(A) \in C^\infty(M)$ of A is defined by the representation of the operator A as an integral

$$A = \int_M \check{\sigma}^{(m)}(A)(x) P_x^{(m)}\, \Omega_\epsilon^{(m)}(x), \tag{6.7}$$

if such a representation exists.

We quote from [**85**, Prop. 6.8] that the Toeplitz operator $T_f^{(m)}$ admits such a representation with $\check{\sigma}^{(m)}(T_f^{(m)}) = f$. This says, the function f itself is a contravariant symbol of the Toeplitz operator $T_f^{(m)}$. Note that the contravariant symbol is not uniquely fixed by the operator. As an immediate consequence from the surjectivity of the Toeplitz map it follows that every operator A has a contravariant symbol, i.e. every operator A has a representation (6.7). For this we have to keep in mind, that our Kähler manifolds are compact.

Now we introduce on $\mathrm{End}(\Gamma_{hol}(M, L^{(m)}))$ the Hilbert-Schmidt norm $\langle A, C\rangle_{HS} = Tr(A^* \cdot C)$. In [**79**] (see also [**86**]), we showed that

$$\langle A, T_f^{(m)}\rangle_{HS} = \langle \sigma^{(m)}(A), f\rangle_\epsilon^{(m)} \ . \tag{6.8}$$

This says that the Toeplitz map $f \to T_f^{(m)}$ and the covariant symbol map $A \to \sigma^{(m)}(A)$ are adjoint. By the adjointness property from the surjectivity of the Toeplitz map the following follows.

Proposition 6.5. *The covariant symbol map is injective.*

Other results following from the adjointness are

$$\mathrm{tr}(T_f^{(m)}) = \int_M f\, \Omega_\epsilon^{(m)} = \int_M \sigma^{(m)}(T_f^{(m)})\, \Omega_\epsilon^{(m)}. \tag{6.9}$$

$$\dim \Gamma_{hol}(M, L^m) = \int_M \Omega_\epsilon^{(m)} = \int_M \epsilon^{(m)}(x)\, \Omega. \tag{6.10}$$

In particular, in the special case that $\epsilon^{(m)}(x) = const$ then

$$\epsilon^{(m)} = \frac{\dim \Gamma_{hol}(M, L^m)}{vol_\Omega(M)}. \tag{6.11}$$

6.4. The original Berezin star product. Under very restrictive conditions on the manifold it is possible to construct the *Berezin star product* with the help of the covariant symbol map. This was done by Berezin himself [**13**],[**14**] and later by Cahen, Gutt, and Rawnsley [**24**][**25**][**26**][**27**] for more examples. We will indicate this in the following.

Denote by $\mathcal{A}^{(m)} \le C^\infty(M)$, the subspace of functions which appear as level m covariant symbols of operators. By Proposition 6.5 for the two symbols $\sigma^{(m)}(A)$ and $\sigma^{(m)}(B)$ the operators A and B are uniquely fixed. Hence, it is possible to define the deformed product by

$$\sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B) := \sigma^{(m)}(A \cdot B). \tag{6.12}$$

Now $\star_{(m)}$ defines on $\mathcal{A}^{(m)}$ an associative and noncommutative product.

It is even possible to give an expression for the resulting symbol. For this we introduce the *two-point function*

$$\psi^{(m)}(x,y) = \frac{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle\langle e_\beta^{(m)}, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle\langle e_\beta^{(m)}, e_\beta^{(m)}\rangle} \tag{6.13}$$

with $\alpha = \tau^{-1}(x)$ and $\beta = \tau^{-1}(y)$. This function is well-defined on $M \times M$. Furthermore, we have the *two-point symbol*

$$\sigma^{(m)}(A)(x,y) = \frac{\langle e_\alpha^{(m)}, Ae_\beta^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\beta^{(m)}\rangle}. \tag{6.14}$$

It is the analytic extension of the real-analytic covariant symbol. It is well-defined on an open dense subset of $M \times M$ containing the diagonal. Then

$$\begin{aligned} \sigma^{(m)}(A) \star_{(m)} \sigma^{(m)}(B)(x) &= \sigma^{(m)}(A \cdot B)(x) = \frac{\langle e_\alpha^{(m)}, A \cdot B\, e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \\ &= \frac{1}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle}\int_M \langle e_\alpha^{(m)}, Ae_\beta^{(m)}\rangle\langle e_\beta^{(m)}, Be_\alpha^{(m)}\rangle\frac{\Omega_\epsilon^{(m)}(y)}{\langle e_\beta^{(m)}, e_\beta^{(m)}\rangle} \\ &= \int_M \sigma^{(m)}(A)(x,y)\cdot\sigma^{(m)}(B)(y,x)\cdot\psi^{(m)}(x,y)\cdot\Omega_\epsilon^{(m)}(y)\ . \end{aligned} \tag{6.15}$$

The crucial problem is how to relate different levels m to define for all possible symbols a unique product not depending on m. In certain special situations like those studied by Berezin, and Cahen, Gutt and Rawnsley the subspaces are nested into each other and the union $\mathcal{A} = \bigcup_{m\in\mathbb{N}} \mathcal{A}^{(m)}$ is a dense subalgebra of $C^\infty(M)$. This is the case if the manifold is a homogeneous manifold and the epsilon function $\epsilon^{(m)}$ is a constant. A detailed analysis shows that in this case a star product is given.

For related results see also work of Moreno and Ortega-Navarro [**68**], [**67**]. In particular, also the work of Engliš [**42, 41, 40, 39**]. Reshetikhin and Takhtajan [**77**] gave a construction of a (formal) star product using formal integrals (and associated Feynman graphs) in the spirit of the Berezin's covariant symbol construction, see Section 9.2

In Section 8.2 using the Berezin transform and its properties discussed in the next section (at least in the case of quantizable compact Kähler manifolds) we will introduce a star product dual to the by Theorem 4.2 existing $\star_{BT}$. It will generalizes the above star product.

7. The Berezin transform and Bergman kernels

7.1. Definition and asymptotic expansion of the Berezin transform.

DEFINITION 7.1. The map

$$I^{(m)} : C^\infty(M) \to C^\infty(M), \qquad f \mapsto I^{(m)}(f) := \sigma^{(m)}(T_f^{(m)}), \tag{7.1}$$

obtained by starting with a function $f \in C^\infty(M)$, taking its Toeplitz operator $T_f^{(m)}$, and then calculating the covariant symbol is called the *Berezin transform (of level m)*.

To distinguish it from the formal Berezin transforms introduced by Karabegov for any of his star products sometimes we will call the above the geometric Berezin transform. Note that it is uniquely fixed by the geometric setup of the quantized Kähler manifold.

From the point of view of Berezin's approach the operator $T_f^{(m)}$ has as a contravariant symbol f. Hence $I^{(m)}$ gives a correspondence between contravariant symbols and covariant symbols of operators. The Berezin transform was introduced and studied by Berezin [**14**] for certain classical symmetric domains in $\mathbb{C}^n$. These results where extended by Unterberger and Upmeier [**90**], see also Engliš [**40, 41, 42**] and Engliš and Peetre [**43**]. Obviously, the Berezin transform makes perfect sense in the compact Kähler case which we consider here.

THEOREM 7.2. [**57**] *Given $x \in M$ then the Berezin transform $I^{(m)}(f)$ has a complete asymptotic expansion in powers of $1/m$ as $m \to \infty$*

$$I^{(m)}(f)(x) \quad \sim \quad \sum_{i=0}^{\infty} I_i(f)(x) \frac{1}{m^i}\ , \tag{7.2}$$

where $I_i : C^\infty(M) \to C^\infty(M)$ are linear maps given by differential operators, uniformly defined for all $x \in M$. Furthermore, $I_0(f) = f, \quad I_1(f) = \Delta f$.

Here Δ is the Laplacian with respect to the metric given by the Kähler form ω. By *complete asymptotic expansion* the following is understood. Given $f \in C^\infty(M)$, $x \in M$ and an $N \in \mathbb{N}$ then there exists a positive constant A such that

$$\left| I^{(m)}(f)(x) - \sum_{i=0}^{N-1} I_i(f)(x) \frac{1}{m^i} \right|_\infty \quad \le \quad \frac{A}{m^N}\ .$$

The proof of this theorem is quite involved. An important intermediate step of independent interest is the off-diagonal asymptotic expansion of the Bergman kernel function in the neighborhood of the diagonal, see [**57**]. We will discuss this in the next subsection.

7.2. Bergman kernel. Recall from Section 5 the definition of the Szegö projectors $\Pi : \mathrm{L}^2(Q,\mu) \to \mathcal{H}$ and its components $\hat\Pi^{(m)} : \mathrm{L}^2(Q,\mu) \to \mathcal{H}^{(m)}$, the Bergman projectors. The Bergman projectors have smooth integral kernels, the *Bergman kernels* $\mathcal{B}_m(\alpha,\beta)$ defined on $Q \times Q$, i.e.

$$\hat\Pi^{(m)}(\psi)(\alpha) = \int_Q \mathcal{B}_m(\alpha,\beta)\psi(\beta)\mu(\beta). \tag{7.3}$$

The Bergman kernels can be expressed with the help of the coherent vectors.

PROPOSITION 7.3.

$$\mathcal{B}_m(\alpha,\beta) = \psi_{e_\beta^{(m)}}(\alpha) = \overline{\psi_{e_\alpha^{(m)}}(\beta)} = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle. \tag{7.4}$$

For the proofs of this and the following propositions see [**57**], or [**82**].

Let $x, y \in M$ and choose $\alpha, \beta \in Q$ with $\tau(\alpha) = x$ and $\tau(\beta) = y$ then the functions

$$u_m(x) := \mathcal{B}_m(\alpha,\alpha) = \langle e_\alpha^{(m)}, e_\alpha^{(m)} \rangle, \tag{7.5}$$

$$v_m(x,y) := \mathcal{B}_m(\alpha,\beta) \cdot \mathcal{B}_m(\beta,\alpha) = \langle e_\alpha^{(m)}, e_\beta^{(m)} \rangle \cdot \langle e_\beta^{(m)}, e_\alpha^{(m)} \rangle \tag{7.6}$$

are well-defined on M and on $M \times M$ respectively. The following proposition gives an integral representation of the Berezin transform.

PROPOSITION 7.4.

$$\begin{aligned} \left(I^{(m)}(f)\right)(x) &= \frac{1}{\mathcal{B}_m(\alpha,\alpha)} \int_Q \mathcal{B}_m(\alpha,\beta)\mathcal{B}_m(\beta,\alpha)\tau^* f(\beta)\mu(\beta) \\ &= \frac{1}{u_m(x)} \int_M v_m(x,y) f(y) \Omega(y)\ . \end{aligned} \tag{7.7}$$

Typically, asymptotic expansions can be obtained using stationary phase integrals. But for such an asymptotic expansion of the integral representation of the Berezin transform we will not only need an asymptotic expansion of the Bergman kernel along the diagonal (which is well-known) but in a neighborhood of it. This is one of the key results obtained in [**57**]. It is based on works of Boutet de Monvel and Sjöstrand [**23**] on the Szegö kernel and in generalization of a result of Zelditch [**95**] on the Bergman kernel on the diagonal. The integral representation is used then to prove the existence of the asymptotic expansion of the Berezin transform. See [**82**] for a sketch of the proof.

Having such an asymptotic expansion it still remains to identify its terms. As it was explained in Section 4.3, Karabegov assigns to every formal deformation quantizations with the "separation of variables" property a *formal Berezin transform I*. In [**57**] it is shown that there is an explicitely specified star product $\star$ (see Theorem 5.9 in [**57**]) with associated formal Berezin transform such that if we replace $\frac{1}{m}$ by the formal variable ν in the asymptotic expansion of the Berezin transform $I^{(m)}f(x)$ we obtain $I(f)(x)$. This will finally prove Theorem 7.2. We will exhibit the star product $\star$ in Section 8.1.

Of course, for certain restricted but important non-compact cases the Berezin transform was already introduced and calculated by Berezin. It was a basic tool in his approach to quantization [**12**]. For other types of non-compact manifolds similar results on the asymptotic expansion of the Berezin transform are also known. See the extensive work of Engliš, e.g. [**40**].

REMARK 7.5. More recently, direct approaches to the asymptotic expansion of the Bergman kernel (outside the diagonal) were given, some of them yielding low order coefficients of the expansion. As examples, let me mention Berman, Berndtsson, and Sjöstrand, [**16**], Ma and Marinescu [**63**], Dai. Lui, and Ma [**35**]. See also Engliš [**39**].

7.3. Proof of norm property of Toeplitz operators. In [**79**] I conjectured (7.2) (which we later proved in joint work with Karabegov) and showed how such an asymptotic expansion supplies a different proof of Theorem 3.3, Part (a). For completeness I reproduce the proof here.

PROPOSITION 7.6.

$$|I^{(m)}(f)|_\infty = |\sigma^{(m)}(T_f^{(m)})|_\infty \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{7.8}$$

PROOF. Using Cauchy-Schwarz inequality we calculate ($x = \tau(\alpha)$)

$$|\sigma^{(m)}(T_f^{(m)})(x)|^2 = \frac{|\langle e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle|^2}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle^2} \le \frac{\langle T_f^{(m)} e_\alpha^{(m)}, T_f^{(m)} e_\alpha^{(m)}\rangle}{\langle e_\alpha^{(m)}, e_\alpha^{(m)}\rangle} \le ||T_f^{(m)}||^2 \ . \tag{7.9}$$

Here the last inequality follows from the definition of the operator norm. This shows the first inequality in (7.8). For the second inequality introduce the multiplication operator $M_f^{(m)}$ on $\Gamma_\infty(M, L^m)$. Then $||T_f^{(m)}|| = ||\Pi^{(m)}\, M_f^{(m)}\, \Pi^{(m)}|| \le ||M_f^{(m)}||$ and for $\varphi \in \Gamma_\infty(M, L^m)$, $\varphi \ne 0$

$$\frac{||M_f^{(m)}\varphi||^2}{||\varphi||^2} = \frac{\int_M h^{(m)}(f\varphi, f\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} = \frac{\int_M f(z)\overline{f(z)} h^{(m)}(\varphi,\varphi)\Omega}{\int_M h^{(m)}(\varphi,\varphi)\Omega} \le |f|_\infty^2 \ . \tag{7.10}$$

Hence,

$$||T_f^{(m)}|| \le ||M_f^{(m)}|| = \sup_{\varphi\ne 0} \frac{||M_f^{(m)}\varphi||}{||\varphi||} \le |f|_\infty. \tag{7.11}$$

□

PROOF. (Theorem 3.3 Part (a).) Choose as $x_e \in M$ a point with $|f(x_e)| = |f|_\infty$. From the fact that the Berezin transform has as leading term the identity it follows that $|(I^{(m)}f)(x_e) - f(x_e)| \le C/m$ with a suitable constant C. Hence, $\big||f(x_e)| - |(I^{(m)}f)(x_e)|\big| \le C/m$ and

$$|f|_\infty - \frac{C}{m} = |f(x_e)| - \frac{C}{m} \quad \le \quad |(I^{(m)}f)(x_e)| \quad \le \quad |I^{(m)}f|_\infty \ . \tag{7.12}$$

Putting (7.8) and (7.12) together we obtain

$$|f|_\infty - \frac{C}{m} \quad \le \quad ||T_f^{(m)}|| \quad \le \quad |f|_\infty \ . \tag{7.13}$$

□

8. Berezin transform and star products

8.1. Identification of the BT star product. In [**57**] there was another object introduced, the *twisted product*

$$R^{(m)}(f,g) := \sigma^{(m)}(T_f^{(m)} \cdot T_g^{(m)}) \ . \tag{8.1}$$

Also for it the existence of a complete asymptotic expansion was shown. It was identified with a twisted formal product. This allowed relating the BT star product with a special star product within the classification of Karabegov. From this the properties of Theorem 4.2 of locality, separation of variables type, and the calculation to the classifying forms and classes for the BT star product follows.

As already announced in Section 4.3, the BT star product $\star_{BT}$ is the opposite of the dual star product of a certain star product $\star$. To identify $\star$ we will give its classifying Karabegov form $\widehat{\omega}$. As already mentioned above, Zelditch [**95**] proved that the function u_m (7.5) has a complete asymptotic expansion in powers of $1/m$. In detail he showed

$$u_m(x) \sim m^n \sum_{k=0}^{\infty} \frac{1}{m^k}\, b_k(x), \quad b_0 = 1. \tag{8.2}$$

If we replace in the expansion $1/m$ by the formal variable ν we obtain a formal function s defined by

$$e^s(x) = \sum_{k=0}^{\infty} \nu^k\, b_k(x). \tag{8.3}$$

Now take as formal potential (4.8)

$$\widehat{\Phi} = \frac{1}{\nu}\Phi_{-1} + s,$$

where Φ_{-1} is the local Kähler potential of the Kähler form $\omega = \omega_{-1}$. Then $\widehat{\omega} = \mathrm{i}\,\partial\bar{\partial}\widehat{\Phi}$. It might also be written in the form

$$\widehat{\omega} = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\bar{\partial}\log \mathcal{B}_m(\alpha,\alpha)). \tag{8.4}$$

Here we denote the replacement of $1/m$ by the formal variable ν by the symbol $\mathbb{F}$.

8.2. The Berezin star products for arbitrary Kähler manifolds. We will introduce for general quantizable compact Kähler manifolds the Berezin star product. We extract from the asymptotic expansion of the Berezin transform (7.2) the formal expression

$$I = \sum_{i=0}^{\infty} I_i\, \nu^i, \quad I_i : C^\infty(M) \to C^\infty(M), \tag{8.5}$$

as a *formal Berezin transform*, and set

$$f \star_B g := I(I^{-1}(f) \star_{BT} I^{-1}(g)). \tag{8.6}$$

As $I_0 = id$ this $\star_B$ is a star product for our Kähler manifold, which we call the *Berezin star product*. Obviously, the formal map I gives the equivalence transformation to $\star_{BT}$. Hence, the Deligne-Fedosov classes will be the same. It will be of separation of variables type but with the role of the variables switched. We showed in [**57**] that $I = I_\star$ with star product given by the form (8.4). We can rewrite (8.6) as

$$f \star_{BT} g := I^{-1}(I(f) \star_B I(g)). \tag{8.7}$$

and get exactly the relation (4.13). Hence, $\star = \star_B$ and both star products $\star_B$ and $\star_{BT}$ are dual and opposite to each other.

When the definition with the covariant symbol works (explained in Section 6.4) $\star_B$ will coincide with the star product defined there.

8.3. Summary of naturally defined star products for compact Kähler manifolds. By the presented techniques we obtained for quantizable compact Kähler manifolds three different naturally defined star products $\star_{BT}$, $\star_{GQ}$, and $\star_B$. All three are equivalent and have classifying Deligne-Fedosov class

$$cl(\star_{BT}) = cl(\star_B) = cl(\star_{GQ}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] - \frac{\delta}{2}). \tag{8.8}$$

But all three are distinct. In fact $\star_{BT}$ is of separation of variables type (Wick-type), $\star_B$ is of separation of variables type with the role of the variables switched (anti-Wick-type), and $\star_{GQ}$ neither. For their Karabegov forms we obtain (see [**57**],[**85**])

$$kf(\star_{BT}) = \frac{-1}{\nu}\omega + \omega_{can}. \qquad kf(\star_B) = \frac{1}{\nu}\omega + \mathbb{F}(\mathrm{i}\,\partial\overline{\partial}\log u_m). \tag{8.9}$$

The function u_m was introduced above as the Bergman kernel evaluated along the diagonal in $Q \times Q$.

REMARK 8.1. Based on Fedosov's method Bordemann and Waldmann [**19**] constructed also a unique star product $\star_{BW}$ which is of Wick type, see Section 9.1. The opposite star product has Karabegov form $kf(\star_{BW}^{opp}) = -(1/\nu)\,\omega$ and it has Deligne-Fedosov class $cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$ [**54**]. It will be equivalent to $\star_{BT}$ if the canonical class is trivial.

More precisely, in [**54**] Karabegov considered the "anti-Wick" variant of the Bordemann-Waldmann construction. This yields a star product with separation of variables in the convention of Karabegov. It has Karabegov form $(1/\nu)\omega$ and the same Deligne-Fedosov class as (8.8). Hence, it is equvialent to $\star_{BT}$. Recently, in [**55**], [**56**] Karabegov gave a more direct construction of the star product with Karabegov form $(1/\nu)\omega$. Karabegov calls this star product *standard star product*.

8.4. Application: Calculation of the coefficients of the star products. The proof of Theorem 4.2 gives a recursive definition of the coefficients $C_k(f,g)$. Unfortunately, it is not very constructive. For their calculation the Berezin transform will also be of help. Theorem 7.2 shows for quantizable compact Kähler manifolds the existence of the asymptotic expansion of the Berezin transform (7.2). We get the formal Berezin transform $I = \mathbb{F}(I^{(m)})$, see (8.5), which is the formal Berezin transform of the star product $\star_B$

$$I = \sum_{i=0}^{\infty} I_i\,\nu^i, \quad I_i : C^\infty(M) \to C^\infty(M).$$

We will show that if we know I explicitely we obtain explicitly $\star_B$ by giving the coefficients $C_k^B(f,g)$ of $\star_B$. For this the knowledge of the coefficients $C_k^{BT}(f,g)$ for $\star_{BT}$ will not be needed. All we need is the existence of $\star_{BT}$ to define $\star_B$. The operators I_i can be expressed (at least in principle) by the asymptotic expansion of expressions formulated in terms of the Bergman kernel.

As I is the formal Berezin transform in the sense of Karabegov assigned to $\star_B$ we get for local functions f, g , f anti-holomorphic, g holomorphic

$$f \star g = I(g \cdot f) = I(g \star f). \tag{8.10}$$

Expanding the formal series for $\star_B$ (4.1) and for I (8.5) we get for the coefficients

$$C_k^B(f,g) = I_k(g \cdot f). \tag{8.11}$$

Let us take local complex coordinates. As $\star_B$ is a differential star product, the C_k^B are bidifferential operators. As $\star_B$ is of separation of variables type, in C_k^B the first argument is is only differentiated with respect to anti-holomorphic coordinates, the second with respect to holomorphic coordinates. Moreover, it was shown by Karabegov that the C_k are bidifferential operators of order $(0,k)$ in the first argument and order $(k,0)$ in the second argument and that I_k is a differential operator of type (k,k).

As f is anti-holomorphic, in I_k it will only see the anti-holomorphic derivatives. The corresponding is true for the holomorphic g. By locality it is enough to consider the local functions z_i and $\overline{z}_i$ and we get that C_k^B can be obtained by "polarizing" I_k.

In detail, if we write I_k as summation over multi-indices (i) and (j) we get

$$I_k = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(i)+(j)}}{\partial z_{(i)} \partial \overline{z}_{(j)}}, \quad a^k_{(i),(j)} \in C^\infty(M) \tag{8.12}$$

and obtain for the coefficient in the star product $\star_B$

$$C_k^B(f,g) = \sum_{(i),(j)} a^k_{(i),(j)} \frac{\partial^{(j)} f}{\partial \overline{z}_{(j)}} \frac{\partial^{(i)} g}{\partial z_{(i)}}, \tag{8.13}$$

where the summation is limited by the order condition. Hence, knowing the components I_k of the formal Berezin transform I gives us C_k^B.

From I we can recursively calculate the coefficients of the inverse I^{-1} as I starts with id. From $f \star_{BT} g = I^{-1}(I(f) \star_B I(g))$, which is the Relation (8.6) inverted, we can calculate (at least recursively) the coefficients C_k^{BT}. In practice, the recursive calculations turned out to become quite involved.

The chain of arguments presented above was based on the existence of the Berezin transform and its asymptotic expansion for every quantizable compact Kähler manifold. The asymptotic expansion of the Berezin transform itself is again based on the asymptotic off-diagonal expansion of the Bergman kernel. Indeed, the Toeplitz operator can also be expressed via the Bergman kernel. Based on this it is clear that the same procedure will work for those non-compact manifolds for which we have at least the same (suitably adapted) objects and corresponding results.

REMARK 8.2. In the purely formal star product setting studied by Karabegov [**52**] the set of star products of separation of variables type, the set of formal Berezin transforms, and the set of formal Karabegov forms are in 1:1 correspondence. Given $I_\star$ the star product $\star$ can be recovered via the correspondence (8.12) with (8.13). What generalizes $\star_{BT}$ in this more general setting is the dual and opposite of $\star$.

EXAMPLE 8.3. As a simple but nevertheless instructive case let us consider $k=1$. Recall that n is the complex dimension of M. Starting from our Kähler form ω expressed in local holomorphic coordinates z_i as $\omega = \mathrm{i} \sum_{i,j=1}^n g_{ij} dz_i \wedge d\overline{z}_j$ the Laplace-Beltrami operator is given by

$$\Delta = \sum_{i,j} g^{ij} \frac{\partial^2}{\partial z_i \partial \overline{z}_j}, \tag{8.14}$$

here (g^{ij}) is as usual the inverse matrix to (g_{ij}). [5] The Poisson bracket is given (up to ϵ which is a factor of signs, complex units, and factors of 1/2 due to preferred conventions) by

$$\{f,g\} = \epsilon \cdot \sum_{i,j} g^{ij} \left(\frac{\partial f}{\partial \overline{z}_i}\frac{\partial g}{\partial z_j} - \frac{\partial f}{\partial z_j}\frac{\partial g}{\partial \overline{z}_i} \right) \tag{8.15}$$

From $I_1 = \Delta$ we deduce immediately with (8.14)

$$C_1^B(f,g) = \sum_{i,j} g^{ij} \frac{\partial f}{\partial \overline{z}_i}\frac{\partial g}{\partial z_j}. \tag{8.16}$$

The inverse of I starts with $id - \Delta\nu + ...$ If we isolate using (8.7) from

$$(id - \Delta\nu)(((id + \Delta\nu)f) \star_B ((id + \Delta\nu)g)) \tag{8.17}$$

the terms of order one in ν we get

$$C_1^{BT}(f,g) = C_1^B(f,g) + (\Delta f)g + f(\Delta g) - \Delta(fg) = -\sum_{i,j} g^{ij} \frac{\partial f}{\partial z_i}\frac{\partial g}{\partial \overline{z}_j}. \tag{8.18}$$

This is of course not a surprise. We could have it deduced also directly. Our star products are of separation of variables type and the C_1 have to have a form like (8.16) (or (8.18)) with coefficients a^{ij} which a priori could be different from g^{ij} and $-g^{ij}$ respectively. From $C_1(f,g) - C_1(g,f) = -\mathrm{i}\,\{f,g\}$ it follows that they are equal.

Calculating the higher orders can become quite tedious. First of course the Berezin transform is only known in closed form for certain homogeneous spaces. For general (compact) manifolds by Proposition 7.4 its asymptotic expansion can be expressed in terms of asymptotic expansions of the Bergman kernel. The Bergman kernel can be expressed locally with respect to adapted coordinates via data associated to the Kähler metric. Hence the coefficients C_k^B and C_k^{BT} can be also expressed in these data. In case that the Berezin transform exists it was an important achievement of Mirek Engliš to exploit this in detail also in the noncompact case, under the condition that the Berezin transform exists [**39**], [**42**]. He calculated small order terms in the star products.

Later, Marinescu and Ma used also Bergman kernel techniques in a different way even in the case of compact symplectic manifolds and orbifolds and allowing an auxiliary vector bundles. In their approach they introduced Toeplitz kernels and calculated small order terms for the Berezin-Toeplitz star product [**65**]. A Berezin transform does not show up. See [**64**] for a review of their techniques, results and further reference to related literature. See also results of Charles [**30**], [**31**], [**32**], [**33**].

9. Other constructions of star products – Graphs

9.1. Bordemann and Waldmann. [**19**] Fedosov's proof of the existence of a star product for every symplectic manifold was geometric in its very nature [**44**]. He considers a certain infinite-dimensional bundle $\hat{W} \to M$ of formal series of symmetric and antisymmetric forms on the tangent bundle of M. For this bundle

[5] From the context it should be clear that g and g_{ij} are unrelated objects.

he defines the fiber-wise Weyl product. Denote by $\hat{\mathcal{W}}$ the sheaf of smooth sections of this bundle, with $\circ$ as induced product.

Starting from a symplectic torsion free connection he constructs recursively what is called the Fedosov derivation D for the sheaf of sections. It is flat, in the sense that $D^2 = 0$. The kernel of D is a $\circ$-subalgebra. Let $\mathcal{W}$ be the elements of $\hat{\mathcal{W}}$ for which the values have antisymmetric degree zero. The natural projection to the symmetric degree zero part gives a linear isomorphism from the $\circ$-subalgebra $\sigma : \mathcal{W}_D = \ker D \cap \mathcal{W} \to C^\infty(M)[[\nu]]$. The algebra structure of $\mathcal{W}_D$ gives the star product we were looking for, i.e. $f \star g := \sigma(\tau(f) \circ \tau(g))$ with τ the inverse of σ which recursively can by calculated.

In case that M is an arbitrary Kähler manifold, Bordemann and Waldmann [**19**] were able to modify the set-up by taking the fiber-wise Wick product. By a modified Fedosov connection a star product $\star_{BW}$ is obtained which is of Wick type, i.e. $C_k(.,.)$ for $k \geq 1$ has only holomorphic derivatives in the first argument and anti-holomorphic arguments in the second argument. Equivalently, it is of separation of variables type. As already remarked earlier, its Karabegov form is $-(1/\nu)\omega$ and it has Deligne-Fedosov class $cl(\star_{BW}) = \frac{1}{\mathrm{i}}(\frac{1}{\nu}[\omega] + \frac{\delta}{2})$. It will be equivalent to the BT star product if the canonical class is trivial.

Later Neumaier [**70**] was able to show that each star product of separation of variables type (i.e. the star products opposite to the Karabegov star product from Section 4.3) can be obtained by the Bordemann-Waldmann construction by adding a formal closed $(1, 1)$ form as parameter in the construction.

9.2. Reshetikhin and Takhtajan. [**77**] In the following subsections we will indicate certain relations between the question of existence and/or the calculation of coefficients of star products and their description by graphs. One of the problems in the context of star products is that the questions reduce often to rather intricate combinatorics of derivatives of the involved functions and other “internal” geometrical data coming from the manifold, like Poisson form, Kähler form, etc. One has to keep track of multiple derivations of many products and sums involving tensors related to the Poisson structure, metric, etc. and the functions f and g. In this respect graphs are usually a very helpful tool to control the combinatorics and to find “closed expressions” in terms of graphs.

Berezin in his approach to define a star product for complex domains in $\mathbb{C}^n$ used analytic integrals depending on a real parameter $\hbar$. Compare this to (6.15) where due to compactness we have a discrete parameter $1/m$. In these integrals scalar products of coherent states show up. Similar to Proposition 7.3 they are identical to the Bergman kernel. Under the condition that the Kähler form is real-analytic its Kähler potential Φ admits an analytic continuation $\Phi(z, \overline{w})$ on $\mathbb{C}^n \times \mathbb{C}^n$. [6] The Bergman kernel can be rewritten with a suitable complementary factor $e_\hbar(z, \overline{w})$ as

$$\mathcal{B}_\hbar(z, \overline{w}) = \mathrm{e}^{\Phi(v,\overline{w})} e_\hbar(z, \overline{w}). \tag{9.1}$$

Moreover, one considers Calabi's diastatic function

$$\Phi(z, \overline{z}, w, \overline{w}) = \Phi(z, \overline{w}) + \Phi(w, \overline{z}) - \Phi(z, \overline{z}) - \Phi(w, \overline{w}). \tag{9.2}$$

[6] In this subsection for the formalism of analytic continuation, it is convenient to write $f(z, \overline{z})$ for a function f on M to indicate its dependence on holomorphic and anti-holomorphic directions.

The corresponding integral rewrites as

$$(f \star_\hbar g)(z,\overline{z}) = \int_{\mathbb{C}^n} f(z,\overline{w})g(w,\overline{z})\frac{e_\hbar(z,\overline{w})e_\hbar(w,\overline{z})}{e_\hbar(z,\overline{z})}e^{(\Phi(z,\overline{z},w,\overline{w})/\hbar}\,\Omega_\hbar, \tag{9.3}$$

where $\Omega_\hbar$ is the $\hbar$ normalized Liouville form. To show that the integral gives indeed a star product Berezin needs the crucial assumption $e_\hbar(z,\overline{w})$ is constant. The desired results are obtained via the Laplace method.

Reshetikhin and Takhtajan consider now such type of integrals (still ignoring the $e_\hbar(z,\overline{w})$) as formal integrals and make a formal Laplace expansion to obtain a "star" product, which we denote for the moment by $\bullet$. The coefficients of the expansion for $f \bullet g$ can be expressed with the help of partition functions of a restricted set $\mathcal{G}$ of locally oriented graphs (Feynman diagrams) fulfilling some additional conditions and equipped with additional data. In particular, each $\Gamma \in \mathcal{G}$ contains two special vertices, a vertex R with only incoming edges and and a vertex L with only outgoing edges. Furthermore, the other vertices are divided into two sets, the solid and the hollow vertices. The "star" product for $\mathbb{C}^n$ as formal power series in ν can be written as

$$f \bullet g = \sum_{\Gamma\in\mathcal{G}} \frac{\nu^{\chi(\Gamma)}}{|Aut(\Gamma)|}D_\Gamma(f,g). \tag{9.4}$$

Here $Aut(\Gamma)$ is the subgroup of automorphism of the graph Γ respecting the special structure, $\chi(\Gamma)$ is the number of edges of Γ minus the number of "solid" vertices. The crucial part is $D_\Gamma(f,g)$ *the partition function* of the graph Γ equipped with certain additional data. It encodes the information from the formal expansion of the integral associated to this graph. The special vertex L is responsible for differentiating f with respect to anti-holomorphic coordinates and R for differentiating g with respect to holomorphic coordinates. It is sketched that the product $\bullet$ is "functorial" with respect to holomorphic changes of coordinates and that it defines a formal deformation quantization for any arbitrary complex manifold M with Kähler form ω. But as in general $1 \bullet f \neq f \neq f \bullet 1$, i.e. it is not null on constants. Essentially this is due to the fact, that the complementary factors $e_\hbar(z,\overline{w})$ (9.1) were not taken into account. But the obtained algebra contains a unit element $e_\nu(z,\overline{z})$ which is invertible. This unit is used to twist $\bullet$

$$(f \star g)(z,\overline{z}) = e_\nu^{-1}(z,\overline{z})((f\cdot e_\nu)\bullet(g\cdot e_\nu)) \tag{9.5}$$

to obtain a star product $\star$ which is null on constants. As the notation already indicates, the unit $e_\nu(z,\overline{z})$ is related to the formal Bergman kernel evaluated along the diagonal.

9.3. Gammelgaard. [**48**] His starting point is the formal deformation $\widehat{\omega}$ of the pseudo-Kähler form $\omega = \omega_{-1}$ given by (4.7). Let $\star$ be the unique star product of separation of variables type (in the convention of Karabegov) associated to $\widehat{\omega}$ which exists globally. Gammelgaard gives a local expression of this star product by

$$f \star g = \sum_{\Gamma\in\mathcal{A}_2} \frac{\nu^{W(\Gamma)}}{|Aut(\Gamma)|}D_\Gamma(f,g). \tag{9.6}$$

This looks similar to (9.4) but of the set of graphs to be considered are different. Also the partition functions will be different. Local means that he chooses for every point a contractible neighborhood such that $\widehat{\omega}$ has a formal potential (4.8). The set $\mathcal{A}_2$ is the set of isomorphy classes of directed acyclic graphs (parallel edges are allowed) which have exactly one vertex which is a sink (i.e. has only incoming edges) and one vertex which is a source (i.e. has only outgoing edges). These two vertices are called external vertices, the other internal. As usual we denote by E the set of edges and by V the set of vertices of the graph Γ. The graphs are weighted by assigning to every internal vertex v an integer $w(v) \geq -1$. Each internal vertex has at least one incoming and one outgoing edge. If $w(v) = -1$ then at least 3 edges are connected with v. The total weight $W(\Gamma)$ of the graph Γ is defined as the sum $W(\Gamma) := |E| + \sum_{v \text{ internal}} w(v)$. Isomorphism are required to respect the structure. Also in this sense $|Aut(\Gamma)|$ has to be understood.

To each such graph a certain bidifferential operator is assigned. It involves the geometric data and the functions f and g. The function f corresponds to the external vertex which is a source and g to the sink. The internal vertices of weight k involve $-\Phi_k$ from (4.8). Incoming edges correspond to taking derivatives with respect to holomorphic coordinates, outgoing with respect to anti-holomorphic coordinates. Hence f is only differentiated with respect to anti-holomorphic and g with respect to holomorphic. The partition function is now obtained by contracting the tensors with the help of the Kähler metric.

In the main part of the paper [**48**] Gammelgaard shows that this definition is indeed associative and defines locally a star product with the (global) Karabegov form $\widehat{\omega}$ he started with. Hence it is the local restriction of $\star$.

The formula is particularly nice if there are not so many higher order terms in $\widehat{\omega}$. For example for $\widehat{\omega} = (1/\nu)\omega_{-1}$, i.e. the "standard star product" only those graphs contribute for which all vertices have weight -1. For the Berezin star product we will have in general higher degree contributions, see (8.9). But the opposite of the Berezin-Toeplitz star product has Karabegov form $-(1/\nu)\omega + \omega_{can}$, hence only graphs which have only vertices of weight -1 or 0 will contribute. As Gammelgaard remarks this allows to give explicit formulas for the coefficients of the BT star product. Recall that for the opposite star product only the role of f and g is switched.

As an example let me derive the "trivial coefficients". The only graph of weight zero is the one consisting on the two external vertices and no edge. Hence $C_0(f,g) = f \cdot g$ as required. The only graph of weight one consists of the two external vertices and a directed edge between them. Hence, we obtain for every $\widehat{\omega} = (1/\nu)\omega_{-1} + ...$ the expression (8.16), and for the Berezin-Toeplitz star product (8.18) (note that we have to take the pseudo-Kähler form $-\omega_{-1}$ and switch the role of f and g). Internal vertices will only show up for weights ≥ 2.

9.4. Huo Xu. [**92**],[**93**] His starting point is the Berezin transform. Let us assume it exists, which at least is true in the case of compact quantizable Kähler manifolds. As explained in Section 8.4 via the formula (8.13) the coefficients of the Berezin star product are given. Based on Engliš's work [**39**] Huo Xu found a very nice way to deal with the Bergman kernel [**92**] in terms of certain graphs. In [**93**] he applies the result to the Berezin transform and Berezin star product. His formula for the product is

$$f \star_B g = \sum_{\Gamma \in \mathcal{G}} \frac{\det(A(\Gamma_-) - Id)}{|\mathrm{Aut}'(\Gamma)|} \nu^{|E|-|V|} D_\Gamma(f,g) = \sum_{k=0}^{\infty} C_k^B(f,g)\nu^k. \tag{9.7}$$

Here $\mathcal{G}$ is a certain subset of pointed directed graphs (i.e. in technical terms it is the set of strongly connected pointed stable graphs – loops and cycles are allowed) consisting of the vertices $V \cup v$ (with v the distinguished vertex) and edges E. After erasing the vertex v the graph Γ_- is obtained. Now $A(\Gamma_-)$ is its adjacency matrix. $|\mathrm{Aut}'(\Gamma)|$ is the number of automorphisms of the pointed graph fixing the distinguished vector. The only object which is a function is again the *partition function* $D_\Gamma(f,g)$ of the graph defined like follows. Each such graph Γ encodes a "Weyl invariant" given in terms of partial derivatives and contractions of the metric. This defines the partition function, whereas the distinguished vertex is replaced by "f" and "g". All incoming edges are associated to f and correspond to $\frac{\partial}{\partial \bar z_i}$ derivatives and all outgoing are associated to g and correspond to $\frac{\partial}{\partial z_i}$. For the precise formulations of his results I refer to his work.

For small orders he classifies the graphs and calculates for k up to three the $C_k^B(f,g)$ and $C_k^{BT}(f,g)$ in terms of the metric data. But again the reformulation to explicit formulas tend to become quite involved with increasing k.

The approaches via graphs presented in Sections 9.2.,9.3, and 9.4 for sure are in some sense related as they center around the same objects. But the set of graphs considered are completely different. Further investigation is necessary to understand this relation. See in this direction the very recent preprint of Xu [**94**].

10. Excursion: The Kontsevich construction

Kontsevich showed in [**59**] the existence of a star product for every Poisson manifold $(M,\{.,.\})$. In fact he proves the more general formality conjecture which implies the existence. It is not my intention even to give a sketch of this here. Furthermore, in the Kähler case we are in the symplectic case and there are other existence and classification proofs obtained much earlier. Nevertheless, as we are dealing with graphs and star product in the previous section, it is very interesting to sketch his explicit formula for the star product in terms of Feynman diagrams.

He considers star products for open sets in $\mathbb{R}^d$ with arbitrary Poisson structure given by the Poisson bivector $\alpha = (\alpha^{ij})$. In local coordinates $\{x_i\}$ the Poisson bracket is given as

$$\{f,g\}(x) = \sum_{i,j=1}^{d} \alpha^{ij}(x)\partial_i f \partial_j g, \quad \partial_i := \frac{\partial}{\partial x_i}. \tag{10.1}$$

The star product is defined by

$$f \star g = f \cdot g + \sum_{n=1}^{\infty} \left(\frac{\mathrm{i}\,\nu}{2}\right)^n \sum_{\Gamma \in \mathcal{G}_n} w_\Gamma D_\Gamma(f,g). \tag{10.2}$$

Here $\mathcal{G}_n$ is a certain subset of graphs of order n, and the partition function D_Γ is a bidifferential operator involving the Poisson bivector α (of homogeneity n). The graph Γ encodes which derivatives have to be taken in D_Γ and w_Γ is a weight function.

More precisely, $\mathcal{G}_n$ consists of oriented graphs with $n+2$ vertices, labeled by $1,2,\ldots,n,L,R$, such that at each numbered vertex $[i]$, $i=1,\ldots,n$ exactly two edges $e_i^1 = (i, v_1(i))$ and $e_i^2 = (i, v_2(i))$ start and end at two different other vertices (including L and R) but not at $[i]$ itself. Each such graphs has $2n$ edges. Denote by E_Γ the set of edges. The number of graphs in G_n is $(n(n+1))^2$ for $n \geq 1$ and

1 for $n = 0$. The bidifferential operator is defined by

$$
\begin{aligned}
D_\Gamma(f,g) := \sum_{I:E_\Gamma \to \{1,2,\ldots,d\}} &\left(\prod_{k=1}^{n} \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,k)}} \partial_{I(e)} \right) \alpha^{I(e_k^1)I(e_k^2)} \right) \times \\
&\times \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,L)}} \partial_{I(e)} \right) f \cdot \left(\prod_{\substack{e\in E_\Gamma \\ e=(*,R)}} \partial_{I(e)} \right) g.
\end{aligned} \tag{10.3}
$$

The summation can be considered as assigning to the $2n$ edges independent indices $1 \le i_1, i_2, \ldots, i_{2n} \le d$ as labels.

EXAMPLE 10.1. Let Γ be the graph with vertices $(1,2,L,R)$ and edges

$$e_1^1 = (1,2), \quad e_1^2 = (1,L), \quad e_2^1 = (2,L), \quad e_2^2 = (2,R).$$

Then

$$D_\Gamma(f,g) = \sum_{i_1,i_2,i_3,i_4=1}^{d} (\alpha^{i_1 i_2})(\partial_{i_1}\alpha^{i_3 i_4})(\partial_{i_2}\partial_{i_3} f)(\partial_{i_4} g).$$

The weights $w(\Gamma)$ are calculated by considering the upper half-plane $H := \{z \in \mathbb{C} \mid Im(z) > 0\}$ with the Poincare metric. Let $C_n(H) := \{u \in H^n \mid u_i \ne u_j, \text{ for } i \ne j\}$ be the configuration space of n ordered distinct points on H. For any two points z and w on H we denote by $\phi(z,w)$ the (counterclock-wise) angle between the geodesic connecting z and i∞ (which is a straight line) and the geodesic between z and w. Let $d\phi(z,w) = \frac{\partial}{\partial z}\phi(z,w)dz + \frac{\partial}{\partial w}\phi(z,w)dw$ be the differential. The weight is then defined as

$$w_\Gamma = \frac{1}{(2\pi)^{2n} n!} \int_{C_n(H)} \wedge_{i=1}^{n} d\phi(u_i, u_{v_1(i)}) \wedge d\phi(u_i, u_{v_2(i)}), \tag{10.4}$$

with the convention that for L and R the values at the boundary (of H) $u_L = 0$ and $u_R = 1$ are taken.

REMARK 10.2. In [**29**] Cattaneo and Felder gave a field-theoretical interpretation of the formula (10.2). They introduce a sigma model defined on the unit disc D (conformally equivalent to the upper half-plane) with values in the Poisson manifold M as target space. The model contains two bosonic fields: (1) X, which is function on the disc, and (2) η, which is a differential 1-form on D taking values in the pullback under X of the cotangent bundle of M, i.e. a section of $X^*(T^*M) \otimes T^*D$.

In local coordinates X is given by d functions $X_i(u)$ and η by d differential 1-forms $\eta_i(u) = \sum_\mu \eta_{i,\mu}(u)du^\mu$. The boundary condition for η is that for $u \in \partial D$, $\eta_i(u)$ vanishes on vectors tangent to ∂D. The action is defined as

$$S[X,\eta] = \int_D \sum_i \eta_i(u) \wedge dX^i(u) + \frac{1}{2}\sum_{i,j} \alpha^{ij}(X(u))\eta_i(u) \wedge \eta_j(u). \tag{10.5}$$

If $0, 1, \infty$ are any three cyclically ordered points on the boundary of the disc, the star product can be given (at least formally) as the semi-classical expansion of the path-integral

$$f \star g\ (x) = \int_{X(\infty)=x} f(X(1))g(X(0)) \exp(\frac{i}{\hbar} S[X,\eta]) dX d\eta\ . \tag{10.6}$$

To make sense of the expansion and to perform the quantization a gauge action has to be divided out. After this the same formula as by Kontsevich is obtained, except that in the sum over the graphs also graphs with loops (also called tadpoles) appear. The corresponding integrals which supply the weights associated to the graphs with loops are not absolutely convergent. These graphs are removed by a certain technique called finite renormalization. In this way Cattaneo and Felder give a very elucidating (partly heuristic) approach to Kontsevich formula for the star product.

How the Kontsevich construction is related to the other graph construction presented in Section 9 is unclear at the moment.

11. Some applications of the Berezin-Toeplitz operators

In this closing section we will give some references indicating some applications of the Berezin-Toeplitz quantization scheme. The interested reader is invited to check the quoted literature for full details, and more references. This list of applications and references is rather incomplete.

11.1. Pull-back of the Fubini-Study metric, extremal metrics, balanced embeddings. Let (M, ω) be a Kähler manifold with very ample quantum line bundle L. After choosing an orthonormal basis of the space $\Gamma_{hol}(M, L^m)$ we can use them to construct an embedding $\phi^{(m)} : M \to \mathbb{P}^{N(m)}$ of M into projective space of dimension $N(m)$, see Remark 2.1. On $\mathbb{P}^{N(m)}$ we have as standard Kähler form the Fubini-Study form ω_{FS} (and its associated metric). The pull-back $(\phi^{(m)})^* \omega_{FS}$ will define a Kähler form on M. It will not depend on the orthogonal basis chosen for the embedding. In general it will not coincide with a scalar multiple of the Kähler form ω we started with (see [**10**] for a thorough discussion of the situation).

It was shown by Zelditch [**95**], by generalizing a result of Tian [**88**] and Catlin [**28**], that $(\Phi^{(m)})^* \omega_{FS}$ admits a complete asymptotic expansion in powers of $\frac{1}{m}$ as $m \to \infty$.

In fact it is related to the asymptotic expansion of the Bergman kernel (7.5) along the diagonal. The pullback calculates as [**95**, Prop.9]

$$\left(\phi^{(m)}\right)^* \omega_{FS} = m\omega + \mathrm{i}\, \partial\bar\partial \log u_m(x)\ . \tag{11.1}$$

In our context of star products it is interesting to note that if in (11.1) we replace $1/m$ by ν we obtain the Karabegov form of the star product $\star_B$ (8.9)

$$\widehat{\omega} = \mathbb{F}(\left(\phi^{(m)}\right)^* \omega_{FS}). \tag{11.2}$$

The asymptotic expansion of $(\phi^{(m)})^* \omega_{FS}$ is called Tian-Yau-Zelditch expansion. Donaldson [**37**], [**38**] took it as the starting point to study the existence and uniqueness of constant scalar curvature Kähler metrics ω on compact manifolds. If they exists at all he approximates them by using so-called balanced metrics on sequences of powers of the line bundle L obtained by balanced embeddings. Balanced embeddings are embeddings fulfilling certain additional properties introduced by Luo [**62**]. They are related to stability of the embedded manifolds in the sense of classifications in algebraic geometry.

It should be remarked that the "balanced condition" is equivalent to the fact that Rawnsley's [**76**] epsilon function (6.6) is constant. See also [**85**, Prop.6.6]. This function was introduced in 1975 by Rawnsley in the context of quantization of Kähler manifolds and further developed by Cahen, Gutt, and Rawnsley [**24**]. In particular it will be constant if the quantization is "projectively induced", i.e. coming from the projective space of the coherent state embedding (6.3). See Section 6.4 for consequences about the possibility of Berezin's original construction of a star product.

Let me give beside the already mentioned a few more names related to the existence and uniqueness of constant scalar curvature Kähler metrics: Lu [**61**], Arezzo and Loi [**8**], Fine [**45**]. For sure much more should be mentioned, but space limitation do not allow.

11.2. Topological quantum field theory and mapping class groups. In the context of Topological Quantum Field Theory (TQFT) the moduli space M of gauge equivalence classes of flat $SU(n)$ connections (possibly with monodromy around a fixed point) over a compact Riemann surface Σ plays an important role. This moduli space carries a symplectic structure ω and a complex line bundle L. After choosing a complex structure on Σ this moduli space will be endowed with a complex structure, ω will become a Kähler form and L get a holomorphic structure. Moreover L will be a quantum line bundle in the sense discussed in this review. Hence, we can employ the Berezin-Toeplitz quantization procedure to it. The quantum space of level m will be as above the (finite-dimensional) space of holomorphic sections of the bundle L^m over M. If we vary the complex structure on Σ the differentiable (symplectic) data will stay the same, but the complex geometric data will vary. In particular, our family of quantum spaces will define a vector bundle over the Teichmüller space (which is the space of complex structures on Σ). This bundle is called the *Verlinde bundle* of level m. There is a canonical projectively flat connection for this bundle, the Axelrod-de la Pietra-Witten/Hitchin connection.

Via the projection to the subspace of holomorphic section, the Toeplitz operators will depend on the complex structure. For a fixed differentiable function f on the moduli space of connections the Toeplitz operators will define a section of the endomorphism bundle of the Verlinde bundle.

The mapping class group acts on the geometric situation. In particular, it acts on the space of projectively covariant constant sections of the Verlinde bundle. This yields a representation of the mapping class group. By general results about the order of the elements in the mapping class group it cannot act faithfully. But it was a conjecture of Tuarev, that at least it acts asymptotically faithful. This says that given a non-trivial element of the mapping class group there is a level m such that the element has a non-trivial action on the space of projectively covariant constant sections of the Verlinde bundle of level m.

This conjecture was shown by J. Andersen in a beautiful proof using Berezin-Toeplitz techniques. For an exact formulation of the statement see [**2**], resp. the overview by Andersen and Blaavand [**4**], and [**84**].

With similar techniques Andersen could show that the mapping class groups Γ_g for genus $g \geq 2$ do not have Property (T) [**3**]. Roughly speaking Property (T) means that the trivial representation is isolated (with respect to a certain topology) in the space of all unitary representations.

There are quite a number of other interesting results shown and techniques developed by Andersen using Berezin-Toeplitz quantization operators and star products, e.g. in the context of Abelian Chern-Simons Theory [**1**], modular functors (joint with K. Ueno) [**7**], and formal Hitchin connections [**5**].

11.3. Spectral theory – quantum chaos. The large tensor power behaviour of the sections of the quantum bundle and of the Toeplitz operators are of interest.

Shiffman and Zelditch considered in [**87**] the limit distribution of zeros of such sections. The results are related to models in *quantum chaos*. See also other publications of the same authors.

As mentioned in Section 3, the Toeplitz operators associated to real-valued functions are self-adjoint. Hence, they have a real spectrum. With respect to this the following result on the trace is of importance

$$\mathrm{Tr}^{(m)}\,(T_f^{(m)}) = m^n \left(\frac{1}{\mathrm{vol}(\mathbb{P}^n(\mathbb{C}))}\int_M f\,\Omega + O(m^{-1})\right) . \tag{11.3}$$

Here $n = \dim_{\mathbb{C}} M$ and $\mathrm{Tr}^{(m)}$ denotes the trace on $\mathrm{End}(\Gamma_{hol}(M, L^m))$. See [**18**], resp. [**81**] for a detailed proof.

On the spectral analysis of Toeplitz operators see e.g. articles by Paoletti [**72**], [**73**], [**74**]. For relation to index theory see e.g. work of Boutet de Monvel, Leichtnam, Tang, and Weinstein [**22**], and Bismut, Ma, and Zhang [**17**].

11.4. Automorphic forms. Another field where the set-up developed in this review shows up in a natural way is the theory of automorphic forms. For example, let $B^n = SU(n,1)/S(U(n)\times U(1))$ be the open unit ball and Γ a discrete, cocompact subgroup of $SU(n,1)$ then the quotient $X = \Gamma/B^n$ is a compact complex manifold. Moreover, the invariant Kähler form on B^n will descends to a Kähler form ω on the quotient. The canonical line bundle (i.e. the bundle of holomorphic n-forms) is a quantum line bundle for (X, ω).

By definition the sections of the tensor powers of this line bundle correspond to functions on B^n which are equivariant under the action of Γ with a certain factor of automorphy. In other words they are automorphic forms. The power of the factor of automorphy is related to the tensor power of the bundle. An important problem is to construct sections, resp. automorphic forms. For example, Poincaré series are obtained by an averaging procedure and give naturally such sections. But it is not clear that they are not identically zero. T. Foth [**46**] worked in the frame-work of Berezin-Toeplitz operators to show that at least for higher tensor powers there are non-vanishing Poincaré series. In this process she used techniques proposed by Borthwick, Paul, and Uribe [**20**] and assigns to Legendrian tori sections of the bundles. By asymptotic expansion the non-vanishing follows. See also [**47**].

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Commutation of geometric quantization and algebraic reduction

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Abstract. We discuss conditions under which geometric quantization of a symplectic manifold (P, ω) in terms of a polarization F commutes with algebraic reduction of a Hamiltonian action of a connected Lie group G. If F is a Kähler polarization of (P, ω), we show that geometric quantization commutes with algebraic reduction provided the zero level of the momentum map contains a Lagrangian submanifold of (P, ω). If G and P are compact, and the action is free, we recover the results of Guillemin and Sternberg.

We consider also an example of an improper action of G that preserves a real polarization. We show that, in this example, quantization of algebraic reduction gives a space naturally isomorphic to the space of generalized invariant vectors..

1. Introduction

Commutation of quantization and reduction was first established by Guillemin and Sternberg in the context of geometric quantization of a Hamiltonian action of a compact Lie group G on a compact symplectic manifold (P, ω) endowed with a Kähler polarization F [**5**]. They showed that the space of invariant vectors of the quantization representation of G has the same dimension as the space obtained by quantization of the Marsden-Weinstein reduction of the zero level of the momentum map $J : P \to \mathfrak{g}$ for the action, provided the action of G on $J^{-1}(0)$ is free. They extended this result to all quantizable coadjoint orbits of G in terms of the shifting trick established in [**6**].

Let $\mathcal{H}_F$ denote the Hilbert space of the quantization representation of G in the Kähler polarization F used by Guillemin and Sternberg. Since P is compact, $\mathcal{H}_F$ is finite dimensional. The compactness of G implies that all irreducible unitary representations α of G are finite dimensional, and can be obtained by geometric quantization of corresponding quantizable coadjoint orbits O_α [**9**]. Let $\mathcal{H}_\alpha$ denote the representation space of the irreducible unitary representation α. We have a direct sum decomposition

$$\mathcal{H}_F = \bigoplus\nolimits_\alpha m_\alpha \mathcal{H}_\alpha, \tag{1.1}$$

where m_α is the multiplicity of the representation α in the quantization representation. The results of Guillemin and Sternberg can be summarized by saying

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that the multiplicity m_α of the representation α in the quantization representation is determined by geometric quantization of the Marsden-Weinstein reduction at $J^{-1}(O_\alpha)$. Sjamaar has generalized the results of Guillemin and Sternberg to the case of non-free action of G on $J^{-1}(0)$ [**12**].

The aim of this paper is to describe commutation of quantization and algebraic reduction. The advantage of algebraic reduction is that it does not require freeness or properness of the action of G on (P, ω). Algebraic reduction was first introduced for the zero level of the momentum map by Śniatycki and Wenstein [**18**]. The generalization to non-zero coadjoint orbits is due to Kimura [**8**] and Wilbour [**20**]. The shifting trick for algebraic reduction was proved by Arms [**1**].

This presentation is based on an ArXive preprint [**14**] and papers [**3**], [**15**], [**16**]. An extended analysis of commutation of quantization and reduction for singular momentum maps will be given in [**17**].

2. Review of geometric quantization

In this section, we give a brief review of geometric quantization in order to establish the notation. In particular, we use a sign convention that might differ from the convention used in other papers in this volume.[1]

We consider a symplectic manifold (P, ω). For each $f \in C^\infty(P)$, the Hamiltonian vector field X_f is given by

$$X_f \lrcorner \omega = -df, \tag{2.1}$$

where $\lrcorner$ is the left interior product of a form and a vector field (contraction on the left). The Poisson bracket of functions f_1 and f_2 in $C^\infty(P)$ is given by

$$\{f_1, f_2\} = X_{f_2}(f_1). \tag{2.2}$$

The Poisson bracket (2.2) is bilinear, antisymmetric, acts as a derivation

$$\{f_1, f_2 f_3\} = f_2\{f_1, f_3\} + f_3\{f_1, f_2\}, \tag{2.3}$$

and satisfies the Jacobi identity

$$\{\{f_1, f_2\}, f_3\} + \{\{f_2, f_3\}, f_1\} + \{\{f_3, f_1\}, f_2\} = 0. \tag{2.4}$$

The map of the Poisson algebra of smooth functions on P to the Lie algebra of vector fields on P, associating to each function f its Hamiltonian vector field X_f, is a Lie algebra anti-homomorphism. In other words,

$$X_{\{f_1, f_2\}} = -[X_{f_1}, X_{f_2}] \tag{2.5}$$

for all $f_1, f_2 \in C^\infty(P)$.

Let

$$\Phi : G \times P \to P : (g, p) \longmapsto \Phi_g(p) \equiv gp \tag{2.6}$$

be a Hamiltonian action of a connected Lie group G on (P, ω) with an Ad^*_G-equivariant momentum map $J : P \to \mathfrak{g}^*$. For each $\xi \in \mathfrak{g}$, the action on P of the 1-parameter subgroup $\exp t\xi$ of G is given by translations along integral curves of the Hamiltonian vector field of J_ξ, where $J_\xi(p) = \langle J(p) \mid \xi \rangle$ for every $p \in P$. For every $\zeta, \xi \in \mathfrak{g}$,

$$\{J_\zeta, J_\xi\} = J_{[\zeta, \xi]}. \tag{2.7}$$

[1]The sign convention used here follows the tradition etablished in classical mechanics and theoretical physics; see [**19**], [**13**], [**21**].

Hence, the map $\xi \longmapsto J_\xi$ is a Lie algebra homomorphism of $\mathfrak{g}$ into $C^\infty(P)$.

According to Dirac, quantization assigns to each element f of a Poisson subalgebra $C_F^\infty(P)$ of $C^\infty(P)$, which will be specified later, a selfadjoint operator $\boldsymbol{Q}_f$ on a Hilbert space $\mathcal{H}_F$ in such a way that

$$[\boldsymbol{Q}_{f_1}, \boldsymbol{Q}_{f_2}] = -i\hbar \boldsymbol{Q}_{\{f_1,f_2\}} \tag{2.8}$$

for every $f_1, f_2 \in C_F^\infty(P)$ [**4**]. If $J_\xi \in C_F^\infty(P)$ for every $\xi \in \mathfrak{g}$, then the map $\xi \mapsto \frac{i}{\hbar}\boldsymbol{Q}_{J_\xi}$ is a representation of $\mathfrak{g}$ on $\mathcal{H}_F$. We are interested in the situation when this representation integrates to a unitary representation $\boldsymbol{U}$ of G on $\mathcal{H}_F$. The role of geometric quantization is to describe the subalgebra $C_F^\infty(P)$ and the Hilbert space $\mathcal{H}_F$ in geometric terms.

A polarization F of a symplectic manifold is an involutive Lagrangian subbundle of $T^{\mathbb{C}}P = TP \otimes \mathbb{C}$. We denote by $\overline{F}$ the complex conjugate of F and set

$$D = F \cap \overline{F} \cap TP, \text{ and } E = (F + \overline{F}) \cap TP.$$

We assume here that D and E are involutive distributions on P; the spaces P/D and P/E of integral manifolds of D and E, respectively, are quotient manifolds of P; and that the canonical projection $P/D \to P/E$ is a locally trivial fibration. Under these assumptions, the distribution F is locally spanned by Hamiltonian vector fields of smooth complex-valued functions on P. We refer to these properties of the polarization F by saying that F is strongly admissible.

Of special interest are two extremal cases described below.

Kähler Polarization: A polarization F is Kähler if $F \oplus \overline{F} = T^{\mathbb{C}}P$, and $i\omega(w, \bar{w}) > 0$ for all non-zero $w \in F$. These assumptions imply that there is a complex structure $\boldsymbol{J}$ on P such that F is the space of antiholomorphic directions. Moreover, P is a Kähler manifold such that $-\omega$ is the Kähler form on P.

Complete Real Polarization: A polarization F is real if $F = D^{\mathbb{C}} = D \otimes \mathbb{C}$, where D is an involutive Lagrangian distribution on P. The assumption that the space P/D of integral manifolds of D is a quotient manifold of P implies that leaves of D are affine manifolds. We assume here that leaves of D are simply connected complete affine manifolds. This implies that leaves of D are isomorphic to $\mathbb{R}^n$, where $n = \frac{1}{2}\dim P$.

Let $C^\infty(P)_F^0$ be the space of smooth complex valued functions on P that are constant along F; that is,

$$C^\infty(P)_F^0 = \{f \in C^\infty(P) \otimes \mathbb{C} \mid uf = 0 \text{ for all } u \in F\}.$$

Since F is a Lagrangian distribution on $T^{\mathbb{C}}P$, it follows that the Poisson bracket of every pair of functions $f_1, f_2 \in C^\infty(P)_F^0$ vanishes identically. Hence, $C^\infty(P)_F^0$ is an abelian Poisson subalgebra of $C^\infty(P)$. It is a classical analogue of Dirac's complete set of commuting observables; see Sec. 14 of reference [**4**].

We denote by $C_F^\infty(P)$ the space of functions on P whose Hamiltonian vector fields preserve F. In other words, $f \in C_F^\infty(P)$ if for every $h \in C^\infty(P)_F^0$, the Poisson bracket $\{f, h\} \in C^\infty(P)_F^0$. If $f_1, f_2 \in C_F^\infty(P)$ and $h \in C^\infty(P)_F^0$ then the Jacobi identity implies that

$$\{\{f_1, f_2\}, h\} = -\{f_2, \{f_1, h\}\} + \{f_1, \{f_2, h\}\} \in C^\infty(P)_F^0.$$

Hence, $C_F^\infty(P)$ is a Poisson subalgebra of $C^\infty(P)$. The Dirac quantization condition (2.8) applies to the Poisson algebra $C_F^\infty(P)$.

In order to construct the Hilbert space $\mathcal{H}_F$ of geometric quantization, consider first a complex line bundle $\lambda : L \to P$ with a connection ∇ and a connection invariant Hermitian form $\langle \cdot \mid \cdot \rangle$, such that

$$(\nabla_X \nabla_{X'} - \nabla_{X'} \nabla_X - \nabla_{[X,X']})\sigma = \frac{i}{\hbar}\omega(X, X')\sigma \tag{2.9}$$

for every pair X, X' of smooth vector fields on P and for each section σ of L, where $\hbar$ is Planck's constant divided by 2π. Such a line bundle exists if and only if the de Rham cohomology class $[(2\pi\hbar)^{-1}\omega]$ is in $H^2(\mathbb{Z})$. Let $S_0^\infty(L)$ denote the space of compactly supported smooth sections σ of L. It admits a Hermitian scalar product

$$(\sigma_1 \mid \sigma_2) = \int_P \langle \sigma_1 \mid \sigma_2 \rangle \omega^n, \tag{2.10}$$

where $n = \frac{1}{2}\dim P$. We denote by $\mathcal{H}$ the completion of $S_0^\infty(L)$ with respect to the scalar product (2.10). For each $f \in C^\infty(P)$, there exists a densely defined symmetric operator $\boldsymbol{P}_f$ on $\mathcal{H}$ such that

$$\boldsymbol{P}_f \sigma = (i\hbar\nabla_{X_f} + f)\sigma \tag{2.11}$$

for every $\sigma \in \mathcal{H}$. If the Hamiltonian vector field X_f is complete, then $\boldsymbol{P}_f$ is self-adjoint. For each $f_1, f_2 \in C^\infty(P)$ and $\sigma \in S^\infty(L)$

$$[\boldsymbol{P}_{f_1}, \boldsymbol{P}_{f_2}] = -i\hbar\boldsymbol{P}_{\{f_1,f_2\}}. \tag{2.12}$$

The map $f \mapsto \boldsymbol{P}_f$ given by equation (2.11) is called the prequantization of (P, ω). The restriction of the prequantization map to the Poisson algebra $\{J_\xi \mid \xi \in \mathfrak{g}\}$ gives rise to a representation $\xi \mapsto \frac{i}{\hbar}\boldsymbol{P}_{J_\xi}$ of the Lie algebra $\mathfrak{g}$ of G by skew adjoint operators. We assume that this representation of G integrates to a unitary representation $\boldsymbol{U}$ of G on $\mathcal{H}$, called the prequantization representation corresponding to the complex line bundle L.

Prequantization satisfies most conditions of the Dirac program. However, the interpretation of $(\sigma \mid \sigma)(p)$ as the probability density of localizing the state σ at a point $p \in P$ fails to satisfy the Heisenberg Uncertainty Principle. Moreover, prequantization representations of quantizable coadjoint orbits fail to be irreducible.

Let F be a polarization of (P, ω). We denote by $S_F^\infty(L)$ the space of smooth sections of L that are covariantly constant along F. In other words,

$$S_F^\infty(L) = \{\sigma \in S^\infty(L) \mid \nabla_u \sigma = 0 \ \forall\ u \in F\}.$$

For each $f \in C_F^\infty(P)$, the prequantization operator $\boldsymbol{P}_f$ maps $S_F(L)$ to itself. We denote by $\boldsymbol{Q}_f$ the linear operator on $S_F(L)$ obtained by restricting the prequantization operator $\boldsymbol{P}_f$ to domain $S_F(L)$ and codomain $S_F(L)$. Thus,

$$\boldsymbol{Q}_f \sigma = (i\hbar\nabla_{X_f} + f)\sigma \tag{2.13}$$

for every $f \in C_F^\infty(P)$ and $\sigma \in S_F(L)$.

In general, sections in $S_F(L)$ need not be square integrable over P. Hence, passing to $S_F(L)$, we may have lost the scalar product structure. In order to obtain a Hermitian scalar product in $S_F(L)$, we often need an additional structure; for example, a metaplectic structure. In the case of a Kähler polarization F, the line bundle L is holomorphic and $S_F(L)$ is the space of holomorphic sections of L. In this case, there exist non-zero square integrable holomorphic sections of L, and the representation space of geometric quantization is

$$\mathcal{H}_F = \mathcal{H} \cap S_F(L). \tag{2.14}$$

For a complete real polarization $F = D^{\mathbb{C}}$ on P, mentioned above, we may consider sections in $S_F(L)$ as sections of a complex line bundle over P/D, and introduce a scalar product in $S_F(L)$ by choosing a volume form on P/D.

3. Algebraic reduction

The action Φ of G on (P, ω) given in (2.6) has an Ad^*_G-equivariant momentum map $J : P \to \mathfrak{g}^*$. For $\mu \in \mathfrak{g}^*$, the isotropy group of μ is

$$G_\mu = \{g \in G \mid Ad^*_g \mu = \mu\}.$$

We denote by $\mathcal{J}_\mu$ the ideal in $C^\infty(P)$, generated by components of $J - \mu : P \to \mathfrak{g}^*$. Thus,

$$\mathcal{J}_\mu = \{\sum_{i=1}^{k} \langle J - \mu \mid \xi_i \rangle f_i \mid \xi_1, ..., \xi_k \in \mathfrak{g} \text{ and } f_1, ..., f_k \in C^\infty(P)\},$$

where $(\xi_1, ..., \xi_k)$ is a basis in $\mathfrak{g}$. Let $C^\infty(P)/\mathcal{J}_\mu$ be the quotient of $C^\infty(P)$ by $\mathcal{J}_\mu$. For each $f \in C^\infty(P)$, the class of f in $C^\infty(P)/\mathcal{J}_\mu$ is denoted by $[f]_\mu$. Since J is Ad^*-equivariant, it follows that for every $g \in G_\mu$,

$$\begin{aligned}
\Phi^*_g \left(\sum_{i=1}^{k} \langle J - \mu \mid \xi_i \rangle f_i \right) &= \sum_{i=1}^{k} \Phi^*_g \left(\langle J \mid \xi_i \rangle - \langle \mu \mid \xi_i \rangle \right) \Phi^*_g f_i \\
&= \sum_{i=1}^{k} (\langle \Phi^*_g J \mid \xi_i \rangle \Phi^*_g f_i - \langle \mu \mid \xi_i \rangle) \Phi^*_g f_i \\
&= \sum_{i=1}^{k} (\langle J \mid Ad_g \xi_i \rangle \Phi^*_g f_i - \langle \mu \mid Ad_g \xi_i \rangle) \Phi^*_g f_i \\
&= \sum_{i=1}^{k} \langle J - \mu \mid Ad_g \xi_i \rangle \Phi^*_g f_i.
\end{aligned}$$

Hence, $\mathcal{J}_\mu$ is G_μ-invariant. This implies that the action Φ of G on P induces an action

$$\widetilde{\Phi}^\mu : G_\mu \times (C^\infty(P)/\mathcal{J}_\mu) \to C^\infty(P)/\mathcal{J}_\mu : (g, [f]_\mu) \mapsto \widetilde{\Phi}^\mu_g [f]_\mu = [\Phi^*_{g^{-1}} f]_\mu$$

of G_μ on $C^\infty(P)/\mathcal{J}_\mu$. We denote by $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ the set of G_μ-invariant elements of $C^\infty(P)/\mathcal{J}_\mu$; that is,

$$(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu} = \{[f]_\mu \in C^\infty(P)/\mathcal{J}_\mu) \mid [\Phi^*_{g^{-1}} f]_\mu = [f]_\mu \ \forall \ g \in G_\mu\}.$$

It follows from the definition that

$$[f]_\mu \in (C^\infty(P)/\mathcal{J}_\mu)^{G_\mu} \iff \Phi^*_{g^{-1}} f - f \in \mathcal{J}_\mu \ \forall \ g \in G_\mu. \tag{3.1}$$

In particular,

$$[f]_\mu \in (C^\infty(P)/\mathcal{J}_\mu)^{G_\mu} \implies X_{J_\xi}(f) \in \mathcal{J}_\mu \ \forall \ \xi \in \mathfrak{g}_\mu,$$

where $\mathfrak{g}_\mu$ is the Lie algebra of G_μ. If G_μ is connected, then the reverse implication holds.

The Poisson algebra structure on $C^\infty(P)$ induces a Poisson algebra structure on $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$, with Poisson bracket $\{[f_1]_\mu, [f_2]_\mu\}$, such that

$$\{[f_1]_\mu, [f_2]_\mu\} = [\{f_1, f_2\}]_\mu. \tag{3.2}$$

We refer to $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ as the Poisson algebra of algebraic reduction of the action Φ on (P,ω) at $\mu \in \mathfrak{g}$. Note that $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ encodes not only information about the level set $J^{-1}(\mu)$, but also some information about its inclusion in (P,ω). If Φ is free and proper, then algebraic reduction is equivalent to the Marsden-Weinstein reduction [**11**]. If Φ is proper but not free, then algebraic reduction need not be equivalent to the singular reduction [**2**]. Unlike the Marsden-Weinstein reduction and the singular reduction, the algebraic reduction does not require the assumption that the action of G is proper.

The "shifting trick" of Guillemin and Sternberg, proved in [**6**] , establishes equivalence of the Marsden-Weinstein reduction at $\mu \in \mathfrak{g}^*$ and the Marsden-Weinstein reduction at $0 \in \mathfrak{g}^*$ of the action of G on the symplectic manifold $(\tilde{P},\tilde{\omega})$ constructed as follows. Suppose that the action of G on (P,ω) is free and proper. Let O_μ be the co-adjoint orbit through μ, and Ω_μ be the Kirillov-Kostant-Souriau form of O_μ. The product

$$\tilde{P} = P \times O_\mu,$$

carries a symplectic form

$$\tilde{\omega} = pr_1^*\omega \oplus (-pr_2^*\Omega_\mu), \tag{3.3}$$

where $pr_1 : \tilde{P} \to P$ and $pr_2 : \tilde{P} \to O_\mu$ are projections on the first and the second factor, respectively. The action of G on $\tilde{P}$, given by

$$\tilde{\Phi} : G \times \tilde{P} \to \tilde{P} : (g,(p,\lambda)) \mapsto \tilde{\Phi}_g(p,\lambda) = (\Phi_g(p), Ad_g^*\lambda),$$

corresponds to an Ad^*-equivariant momentum map

$$\tilde{J} = pr_1^*J - pr_2^*I,$$

where $I : O_\mu \to g^*$ is the inclusion map. This result was extended by Arms to [1] to algebraic reduction and an action of G on (P,ω) that need not be free or proper.

Theorem 3.1 (Arms [1]). *If G and G_μ are connected, and O_μ is an embedded submanifold of $\mathfrak{g}^*$, then the Poisson algebra $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ is naturally isomorphic to the Poisson algebra $(C^\infty(\tilde{P})/\tilde{\mathcal{J}}_0)^G$.*

Taking into account Theorem 3.1, we may restrict our attention to a quantization of the algebraic reduction at $0 \in \mathfrak{g}^*$, and for $\mu \neq 0$, we can interpret our results in terms of the natural isomorphism between $(C^\infty(P)/\mathcal{J}_\mu)^{G_\mu}$ and $(C^\infty(\tilde{P})/\tilde{\mathcal{J}}_0)^G$.

4. Quantization of algebraic reduction at $J = 0$

Suppose that we have a quantization of (P,ω) given in terms of a G-invariant polarization F and a prequantization line bundle L over P. This means that for each $\xi \in \mathfrak{g}$, the momentum J_ξ is quantizable and that the map $J_\xi \mapsto \frac{1}{i\hbar}\boldsymbol{Q}_{J_\xi}$ is a representation of $\mathfrak{g}$ on the space $S_F^\infty(L)$ of smooth sections of L that are covariantly constant along F. We assume that this representation of $\mathfrak{g}$ integrates to a linear representation $\boldsymbol{R}$ of G on $S_F^\infty(L)$. We denote by $\mathcal{H}_F$ the Hilbert space obtained by unitarization from $S_F^\infty(L)$, and by $\boldsymbol{U}$ the unitary representation of G on $\mathcal{H}_F$ obtained from the representation $\boldsymbol{R}$ on $S_F^\infty(L)$.

Our aim is to determine how much information about spaces $S_F^\infty(L)^G$ and $\mathcal{H}_F^G$ of invariant vectors of representations $\boldsymbol{R}$ and $\boldsymbol{U}$, respectively, is encoded in the reduced Poisson algebra, and how this information can be decoded in terms of quantization of $(C^\infty(P)/\mathcal{J}_0)^G$. Since geometric quantization depends on the choice of polarization and prequantization, we have to quantize $(C^\infty(P)/\mathcal{J}_0)^G$ in terms of the polarization F and the prequantization line bundle L that have been used to quantize (P,ω).

Functions that are quantizable in terms of the polarization F form a Poisson algebra $C^\infty_F(P)$. Therefore, we can expect that the set of elements of $(C^\infty(P)/\mathcal{J}_0)^G$ that are quantizable in terms of the polarization F is the Poisson subalgebra

$$(C^\infty_F(P)/\mathcal{J}_0)^G = (C^\infty_F(P)/(C^\infty_F(P)\cap\mathcal{J}_0))^G \tag{4.1}$$

of $(C^\infty(P)/\mathcal{J}_0)^G$. For each $f\in C^\infty_F(P)$, the equivalence class of f in $C^\infty_F(P)/\mathcal{J}_0$ is denoted by $[f]_0$. Since the group G is connected, the class $[f]_0$ is G-invariant if $X_{J_\xi}f\in\mathcal{J}_0$ for every $\xi\in\mathfrak{g}$.

In order to define the representation space of the geometric quantization of algebraic reduction at $J=0$, consider

$$\mathcal{J}_0S^\infty(L) = \operatorname{span}\{f\sigma \mid f\in\mathcal{J}_0 \text{ and } \sigma\in S^\infty(L)\},$$

where $S^\infty(L)$ denotes the space of smooth sections of L. For each $\sigma\in S^\infty(L)$, we denote by $[\sigma]\in S^\infty(L)/\mathcal{J}_0S^\infty(L)$ the equivalence class of σ.

Proposition 4.1. *For each $\xi\in\mathfrak{g}$ and $\sigma\in S^\infty(L)$, the class $[\boldsymbol{P}_{J_\xi}\sigma]\in S^\infty(L)/\mathcal{J}_0S^\infty(L)$ is independent of the choice of the representative σ of the class $[\sigma]\in S^\infty(L)/\mathcal{J}_0S^\infty(L)$.*

Proof. If $[\sigma]=[\sigma']$ then $\sigma=\sigma'+\sum_i f_iJ_{\xi_i}\sigma_i$. Since $\boldsymbol{P}_{J_\xi}=i\hbar\nabla_{X_{J_\xi}}+J_\xi$, it follows that

$$\boldsymbol{P}_{J_\xi}J_\zeta = J_\zeta\boldsymbol{P}_{J_\xi}+i\hbar X_{J_\xi}(J_\zeta) = J_\zeta\boldsymbol{P}_{J_\xi}+i\hbar J_{[\zeta,\xi]}.$$

Hence,

$$\begin{aligned}\boldsymbol{P}_{J_\xi}\sigma &= \boldsymbol{P}_{J_\xi}\sigma'+\boldsymbol{P}_{J_\xi}(\sum_i f_iJ_{\xi_i}\sigma_i) = \boldsymbol{P}_{J_\xi}\sigma'+\sum_i\boldsymbol{P}_{J_\xi}(J_{\xi_i}f_i\sigma_i)\\ &= \boldsymbol{P}_{J_\xi}\sigma'+\sum_i J_{\xi_i}\boldsymbol{P}_{J_\xi}(f_i\sigma_i)+i\hbar\sum_i X_{J_\xi}(J_{\xi_i})f_i\sigma_i\\ &= \boldsymbol{P}_{J_\xi}\sigma'+\sum_i J_{\xi_i}\boldsymbol{P}_{J_\xi}(f_i\sigma_i)+i\hbar\sum_i J_{[\xi_i,\xi]}f_i\sigma_i.\end{aligned}$$

Therefore, $[\boldsymbol{P}_{J_\xi}\sigma]=[\boldsymbol{P}_{J_\xi}\sigma']$. □

We have assumed that the action of $\mathfrak{g}$ on $S^\infty(L)$, given by $(\xi,\sigma)\mapsto\frac{i}{\hbar}\boldsymbol{P}_{J_\xi}\sigma$, integrates to a representation of G on $S^\infty(L)$. Hence, the action of $\mathfrak{g}$ on $S^\infty(L)/\mathcal{J}_0S^\infty(L)$, given by $(\xi,[\sigma])\mapsto\frac{i}{\hbar}[\boldsymbol{P}_{J_\xi}\sigma]$, integrates to a representation of G on $S^\infty(L)/\mathcal{J}_0S^\infty(L)$. We denote by $(S^\infty(L)/\mathcal{J}_0S^\infty(L))^G$ the space of G-invariant elements in $S^\infty(L)/\mathcal{J}_0S^\infty(L)$. Since G is connected, it follows that

$$[\sigma]\in(S^\infty(L)/\mathcal{J}_0S^\infty(L))^G \iff \boldsymbol{P}_{J_\xi}\sigma\in\mathcal{J}_0S^\infty(L) \text{ for all } \xi\in\mathfrak{g}.$$

Proposition 4.2. *The map:*

$$\begin{aligned}\boldsymbol{P} &: (C^\infty(P)/\mathcal{J}_0)^G\times(S^\infty(L)/\mathcal{J}_0S^\infty(L))^G\to(S^\infty(L)/\mathcal{J}_0S^\infty(L))^G\\ &: ([f],[\sigma])\mapsto\boldsymbol{P}_{[f]}[\sigma]=[\boldsymbol{P}_f\sigma]\end{aligned}$$

is well defined.

Proof. For $[f] \in (C^\infty(P)/\mathcal{J}_0)^G$, we have $X_{J_\xi}(f) \in \mathcal{J}_0$. Hence, for each $J_\xi\sigma \in \mathcal{J}S^\infty(L)$,

$$\begin{aligned}\boldsymbol{P}_f(J_\xi\sigma) &= (i\hbar\nabla_{X_f} + f)(J_\xi\sigma) = J_\xi(i\hbar\nabla_{X_f} + f)\sigma + i\hbar(X_f(J_\xi))\sigma \\ &= J_\xi(i\hbar\nabla_{X_f} + f)\sigma + i\hbar(X_{J_\xi}(f))\sigma \in \mathcal{J}_0 S^\infty(L).\end{aligned}$$

This implies that, for $[f] \in (C^\infty(P)/J)^G$, the operator $\boldsymbol{P}_f$ maps $\mathcal{J}_0 S^\infty(L)$ to itself. Hence, $[\boldsymbol{P}_f\sigma]$ does not depend on the representative σ of $[\sigma]$.

For $kJ_\xi \in \mathcal{J}_0$ and $[\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$, we have $X_{kJ_\xi} = kX_{J_\xi} + J_\xi X_k$. Hence,

$$\begin{aligned}\boldsymbol{P}_{kJ_\xi}\sigma &= (i\hbar\nabla_{X_{kJ_\xi}} + kJ_\xi)\sigma \\ &= (i\hbar k\nabla_{X_{kJ_\xi}} + i\hbar J_\xi\nabla_{X_k} + kJ_\xi)\sigma \\ &= i\hbar J_\xi\nabla_{X_k}\sigma + i\hbar k\boldsymbol{P}_{J_\xi}\sigma \in \mathcal{J}_0 S^\infty(L).\end{aligned}$$

Therefore, $[\boldsymbol{P}_f\sigma]$ does not depend on the representative f of $[f]$.

Combining these results, we obtain that an equivalence class $[\boldsymbol{P}_f\sigma] \in S^\infty(L)/\mathcal{J}_0 S^\infty(L)$ depends only on $[\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ and $[f] \in (C^\infty(P)/\mathcal{J}_0)^G$. It remains to show that $[\boldsymbol{P}_f\sigma]$ is G-invariant.

For $[f] \in (C^\infty(P)/J)^G$, $[\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ and $\xi \in \mathfrak{g}$,

$$\boldsymbol{P}_{J_\xi}\boldsymbol{P}_f\sigma = \boldsymbol{P}_f\boldsymbol{P}_{J_\xi}\sigma + [\boldsymbol{P}_{J_\xi}, \boldsymbol{P}_f]\sigma.$$

But $[\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ implies $\boldsymbol{P}_{J_\xi}\sigma \in \mathcal{J}_0 S^\infty(L)$ so that $\boldsymbol{P}_f\boldsymbol{P}_{J_\xi}\sigma \in \mathcal{J}_0 S^\infty(L)$ by the first part of the proof. On the other hand,

$$[\boldsymbol{P}_{J_\xi}, \boldsymbol{P}_f]\sigma = -i\hbar\boldsymbol{P}_{\{J_\xi,f\}}\sigma = -i\hbar(i\hbar\nabla_{X_{\{J_\xi,f\}}} + \{J_\xi, f\})\sigma.$$

But, $\{J_\xi, f\} = -X_{J_\xi}f \in \mathcal{J}_0$ because $[f]$ is G-invariant. Hence, $\{J_\xi, f\} = \sum_j f_j J_{\zeta_j}$ for some $f_j \in C^\infty(P)$ and $\zeta_j \in \mathfrak{g}$. Moreover, $X_{f_1 f_2} = f_1 X_{f_2} + f_2 X_{f_1}$ implies that

$$\sum_j \nabla_{X_{f_j J_{\zeta_j}}} = \sum_j (f_j\nabla_{X_{J_{\zeta_j}}} + J_{\zeta_j}\nabla_{X_{f_j}}).$$

Therefore,

$$\begin{aligned}(i\hbar\nabla_{X_{\{J_\xi,f\}}} + \{J_\xi, f\})\sigma &= \sum_j \Big(i\hbar(f_j\nabla_{X_{J_{\zeta_j}}} + J_{\zeta_j}\nabla_{X_{f_j}}) + f_j J_{\zeta_j}\Big)\sigma \\ &= \sum_j \Big((f_j\boldsymbol{P}_{J_{\zeta_j}} + i\hbar J_{\zeta_j}\nabla_{X_{f_j}}\Big)\sigma \in \mathcal{J}_0 S^\infty(P, L)\end{aligned}$$

because $[\sigma]$ is G-invariant. Therefore, $[\boldsymbol{P}_f\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$. □

Definition 4.3. The map associating to each $[f] \in (C^\infty(P)/J)^G$ an operator $\boldsymbol{P}_{[f]}$ on the space $(S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ is a prequantization of the reduced Poisson algebra.

Next, we take into account the polarization F of (P, ω). We assume that F is strongly admissible, which implies that it is locally spanned by Hamiltonian vector fields of functions that are constant along F. Quantization in terms of the polarization F assigns to each $f \in C_F^\infty(P)$ an operator $\boldsymbol{Q}_f$ on the space $\mathcal{S}_F^\infty(L) = \{\sigma \in \mathcal{S}^\infty(L) \mid \nabla_u\sigma = 0 \text{ for all } u \in F\}$. Moreover,

$$\boldsymbol{Q}_f\sigma = \boldsymbol{P}_f\sigma$$

for each $f \in C_F^\infty(P)$ and $\sigma \in \mathcal{S}_F^\infty(L)$.

Consider the space

$$(S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G = (S_F^\infty(L)/(S_F^\infty(L)\cap \mathcal{J}_0 S^\infty(L))^G \tag{4.2}$$

consisting of G-invariant $\mathcal{J}_0 S^\infty(L)$-equivalence classes of sections in $S_F^\infty(L)$. In other words,

$$[\sigma] \in (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G \iff Q_{J_\xi}\sigma \in \mathcal{J}_0 S^\infty(L) \text{ for all } \xi \in \mathfrak{g}.$$

The canonical projection map

$$\Pi : S_F^\infty(L)^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G : \sigma \mapsto [\sigma] \tag{4.3}$$

will play an essential role in the discussion of commutation of geometric quantization with algebraic reduction.

First, we have to define what we mean by geometric quantization of algebraic reduction. We begin with the following proposition.

Proposition 4.4. *The map*

$$\begin{aligned} \boldsymbol{Q} \;&:\; (C_F^\infty(P)/\mathcal{J}_0)^G \times (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G \\ &:\; ([f],[\sigma]) \mapsto \boldsymbol{Q}_{[f]}[\sigma] \end{aligned}$$

is well defined.

Proof. We know that, if $f \in C_F^\infty(P)$ and $\sigma \in S_F^\infty(L)$, then $\boldsymbol{P}_f\sigma \in S_F^\infty(L)$. In Proposition 4.2, we have shown that $[\boldsymbol{P}_f\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ is independent of the representatives f of $[f] \in (C^\infty(P)/\mathcal{J}_0)^G$ and σ of $[\sigma] \in (S^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$. Therefore, $[\boldsymbol{P}_f\sigma] \in (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$. Moreover, $[\boldsymbol{P}_f\sigma]$ is independent of the representative $f \in C_F^\infty(P)$ of $[f] \in (C_F^\infty(P)/\mathcal{J}_0)^G$. Similarly, $[\boldsymbol{P}_f\sigma]$ is independent of the representative $\sigma \in S_F^\infty(L)$ of $[\sigma] \in (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$. Hence the map $\boldsymbol{Q} : ([f],[\sigma]) \mapsto \boldsymbol{Q}_{[f]}[\sigma] = [\boldsymbol{P}_f\sigma]$ is well defined. □

Proposition 4.4 implies that we may adopt the following definition.

Definition 4.5. The map

$$\begin{aligned} \boldsymbol{Q} \;&:\; (C_F^\infty(P)/\mathcal{J}_0)^G \times (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G \\ &:\; ([f],[\sigma]) \mapsto \boldsymbol{Q}_{[f]}[\sigma] \end{aligned}$$

is the quantization, with respect to the polarization F, of the algebraic reduction at $J = 0$.

It follows from the definition above that algebraic reduction at $J = 0$ followed by quantization gives rise to the space $(S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ of quantum states of the theory. On the other hand, quantization followed by quantum reduction at $J = 0$ gives the space $S_F^\infty(L)^G$ of G-invariant sections in $S_F^\infty(L)$. We have a canonical projection map

$$\Pi : S_F^\infty(L)^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G : \sigma \mapsto [\sigma]. \tag{4.4}$$

Algebraic reduction at $J = 0$ commutes with quantization if Π is an isomorphism and

$$\Pi \circ \boldsymbol{Q}_f = \boldsymbol{Q}_{[f]} \circ \Pi.$$

for every $f \in C_F^\infty(P)^G$.

5. Kähler polarization

Consider now the case when F is a Kähler polarization of (P, ω).

LEMMA 5.1. *Let F be a Kähler polarization of (P, ω). If $J^{-1}(0)$ contains a Lagrangian submanifold of (P, ω), then $S_F^\infty(L) \cap \mathcal{J}_0 S^\infty(L) = 0$.*

PROOF. Let Λ be a Lagrangian submanifold of (P, ω) contained in $J^{-1}(0)$. Since F is a Kähler polarization and Λ is Lagrangian, it follows that

$$T_\Lambda^{\mathbb{C}} P = F_{|\Lambda} \oplus T^{\mathbb{C}}\Lambda.$$

Moreover, for every $\sigma \in S_F^\infty \cap \mathcal{J}_0 S^\infty(L)$, the restriction $\sigma_{|\Lambda}$ of σ to Λ vanishes identically because $\Lambda \subset J^{-1}(0)$. Hence, all derivatives of σ in directions in $T^{\mathbb{C}}\Lambda$ are zero. On the other hand, all derivatives of σ in directions in F are zero, because σ is holomorphic. Hence, all derivatives of σ vanish on $\Lambda \subset P$. Since σ is holomorphic, it follows that $\sigma = 0$. Therefore, $S_F^\infty(L) \cap \mathcal{J}_0 S^\infty(L) = 0$. □

THEOREM 5.2. *Let F be a Kähler polarization of (P, ω). If $J^{-1}(0)$ contains a Lagrangian submanifold of (P, ω), then the natural projection $\Pi : S_F^\infty(L)^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G : \sigma \mapsto [\sigma]$ is an isomorphism, and*

$$\Pi \circ \boldsymbol{Q}_f = \boldsymbol{Q}_{[f]} \circ \Pi.$$

for every $f \in C_F^\infty(P)^G$.

PROOF. Using identification (4.2) and Lemma 5.1, we get

$$S_F^\infty(L)/\mathcal{J}_0 S^\infty(L) = (S_F^\infty(L)/(S_F^\infty(L) \cap \mathcal{J}_0 S^\infty(L)) = S_F^\infty(L)/(0) = S_F^\infty(L).$$

Hence,

$$S_F^\infty(L)^G = (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G.$$

In other words, the projection map $\Pi : S_F^\infty(L)^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G : \sigma \mapsto [\sigma]$ is an isomorphism.

For every G-invariant function $f \in C_F^\infty(P)$, the class $[f]$ is in $f \in C_F^\infty(P)^G$. Moreover, for each $\sigma \in S_F^\infty(L)^G$, $\boldsymbol{Q}_f \sigma \in S_F^\infty(L)^G$ and $[\sigma] = \Pi(\sigma) \in (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$. Hence,

$$\boldsymbol{Q}_{[f]} \circ \Pi\sigma = \boldsymbol{Q}_{[f]}[\sigma] = [\boldsymbol{Q}_f \sigma] = \Pi \circ \boldsymbol{Q}_f \sigma,$$

which completes the proof. □

According to Theorem 5.2 geometric quantization in terms of a Kähler polarization commutes with algebraic reduction at $J = 0$ if $J^{-1}(0)$ contains a Lagrangian submanifold. If G is compact and its action on $J^{-1}(0)$ is free, we can show that $J^{-1}(0)$ contains a Lagrangian submanifold. Moreover, in this case, the algebraic reduction is equivalent to the Marsden-Weinstein reduction. Hence, if G and P are compact, and the action of G on $J^{-1}(0)$ is free, our results are equivalent to the results of Guillemin and Sternberg.

In the proof of Theorem 5.2 we have shown not only that the projection map $\Pi : S_F^\infty(L)^G \to (S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ is an isomorphism, but that the spaces $S_F^\infty(L)^G$ and $(S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ may be considered equal, provided the assumptions of the theorem are satisfied. This is a consequence of the identification

$$S_F^\infty(L)/\mathcal{J}_0 S^\infty(L) = S_F^\infty(L)/(S_F^\infty(L) \cap \mathcal{J}_0 S^\infty(L)$$

made in equation (4.2). This result is not as deep as it might appear. Observe that assumptions of Theorem 5.2 do not imply vanishing of $\mathcal{J}_0 S^\infty(L) = 0$. It follows from Lemma 5.1 that $S_F^\infty(L) \cap \mathcal{J}_0 S^\infty(L) = 0$, which implies that the projection map from $S^\infty(L)$ to $S^\infty(L)/\mathcal{J}_0 S^\infty(L)$ restricted to $S_F^\infty(L)$ is an isomorphism.

In reference [**12**], Sjamaar investigated the case when the action of G on $J^{-1}(0)$ is not free. In this case, Sjamaar showed that the dimension of $S_F^\infty(L)^G$ is equal to the dimension of the space of the space of holomorphic sections of $L_{|J^{-1}(0)}/G$ over $J^{-1}(0)/G$, and he interpreted this result as quantization of commutation and reduction. It should be noted that Sjamaar's results do not require the assumption that $J^{-1}(0)$ contains a Lagrangian manifold.

If P is compact, then $S_F^\infty(L)$ is finite dimensional, and it coincides with the Hilbert space $\mathcal{H}_F = \mathcal{H} \cap S_F^\infty(L)$. In this case, the linear representation $\boldsymbol{R}$ of G on $S_F^\infty(L)$, generated by the quantization of the momenta J_ξ, for $\xi \in \mathfrak{g}$, is the same as the unitary representation $\boldsymbol{U}$ of G on $\mathcal{H}$.

If P is not compact, then the linear representation $\boldsymbol{R}$ of G on $S_F^\infty(L)$, need not coincide with the unitary representation $\boldsymbol{U}$ on $\mathcal{H}_F$. In this case, we have a decomposition

$$\mathcal{H}_F = \bigoplus\nolimits_\alpha m_\alpha \mathcal{H}_\alpha \ \oplus \ \int_\beta \mathcal{H}_\beta d\mu_\beta, \tag{5.1}$$

where α labels the representations corresponding to the discrete part of the spectral measure and β labels the representations corresponding to the continuous part of the measure. In this case, the multiplicities m_α may be infinite.

The space $\mathcal{H}_F^G = \mathcal{H} \cap S_F^\infty(L)^G$ consists of G-invariant normalizable sections. Hence, it corresponds to a trivial representation α_0 of G. In other words, $\mathcal{H}_F^G = m_{\alpha_0}\mathcal{H}_{\alpha_0}$ in decomposition (5.1). If we could describe $\Pi(\mathcal{H}_F^G)$ in terms of the inclusion $J^{-1}(0) \hookrightarrow P$, we would have a characterization of the space $\mathcal{H}_F^G = \Pi^{-1}(\Pi(\mathcal{H}_F^G))$ in terms of the reduction data.[2] In other words, we would be able to determine from algebraic reduction at $J = 0$ the multiplicity m_{α_0} of the trivial representation of G contained in the quantization representation. For a representation α corresponding to a non-zero quantizable coadjoint orbit, we would get analogous results using the shifting trick of Guillemin and Sternberg.

If the quotient $S_F^\infty(L)^G/\mathcal{H}_F^G$ does not vanish, then the continuous part of the spectral measure contains the trivial representation in its support. The major challenge of the theory is to determine the continuous part $d\mu_\beta$ of the spectral measure, in terms of reduction at corresponding quantizable orbits.

6. Real polarization

In this section, we discuss an example in which the trivial representation corresponds to a point of the continuous spectrum in the decomposition of the quantization representation of G into its irreducible components. We show that under certain conditions, $(S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ is naturally isomorphic to the space of G-invariant generalized vectors of the representation.

We assume that the polarization F is real; that is $F = D \otimes \mathbb{C}$, where D is an involutive Lagrangian distribution on P. We also assume that the momentum map $J : P \to \mathfrak{g}^*$ is constant along D. In this case, for each $\xi \in \mathfrak{g}$ and $\sigma \in S_F^\infty(L)$, we have

$$\boldsymbol{Q}_{J_\xi}\sigma = J_\xi \sigma. \tag{6.1}$$

[2]Partial results in this direction have been obtained by Hall and Kirwan [**7**], and Li [**10**].

Hence, a section $\sigma \in S_F^\infty(L)$ is G-invariant only if $J\sigma = 0$. Thus, the support of σ is contained in $J^{-1}(0)$. Further, we assume that $J^{-1}(0)$ is nowhere dense in P. This implies that the equation $J\sigma = 0$ has only weak solutions. In order to discuss generalized sections of $S_F^\infty(L)$, we need to describe the topology on $S_F^\infty(L)$.

We make a few additional simplifying assumptions. First, we assume that the space Q of integral manifolds of the distribution D is a quotient manifold of P and that the leaves of D are complete affine spaces. Let $\vartheta : P \to Q$ be the map associating to each $p \in P$ the maximal integral manifold of D through p. We assume that ϑ is a submersion. Moreover, we assume that the prequantization bundle L is trivial; that is $L = P \times \mathbb{C}$, and that there exists a section $\sigma_0 \in S_F^\infty(L)$ such that $\sigma_0(p) = (p, 1)$. Under these assumptions,

$$S_F^\infty(L) = \{\vartheta^*(\psi)\sigma_0 \mid \psi \in \mathbb{C} \otimes C^\infty(Q)\}. \tag{6.2}$$

Let

$$\mathcal{D}_F = \{\vartheta^*(\psi)\sigma_0 \in S_F^\infty(L) \mid \psi \in \mathbb{C} \otimes C_0^\infty(Q)\},$$

where $\mathbb{C} \otimes C_0^\infty(Q)$ is the space of compactly supported complex-valued smooth functions on Q. We endow $\mathcal{D}_F$ with a topology of uniform convergence of all derivatives of functions $\psi \in \mathbb{C} \otimes C_0^\infty(Q)$ on compact sets. Let

$$(\sigma_1 \mid \sigma_2)_Q = (\vartheta^*(\psi_1)\sigma_0 \mid \vartheta^*(\psi_2)\sigma_0)_Q = \int_Q \bar{\psi}_1(q)\psi_2(q)d\mu(q) \tag{6.3}$$

be a scalar product on $\mathcal{D}_F$. We denote by $\mathcal{H}_F$ the completion of $\mathcal{D}_F$ with respect to norm given by the scalar product (2.10) and by $\mathcal{D}'_F$ the topological dual of $\mathcal{D}_F$. Then,

$$\mathcal{D}_F \subset \mathcal{H}_F \subset \mathcal{D}'_F,$$

$\mathcal{D}_F$ is dense in $\mathcal{H}_F$, and $\mathcal{H}_F$ is dense in $\mathcal{D}'_F$.

For each $\xi \in \mathfrak{g}$, the quantum operator $\boldsymbol{Q}_{J_\xi}$ on $S_F^\infty(L)$ preserves $\mathcal{D}_F$. Hence, it extends to a self-adjoint operator on $\mathcal{H}_F$ and gives rise to a dual operator $\boldsymbol{Q}'_{J_\xi}$ on $\mathcal{D}'_F$ such that, for every $\xi \in \mathfrak{g}$, $\varphi \in \mathcal{D}'_F$ and $\sigma \in \mathcal{D}_F$,

$$(\boldsymbol{Q}'_{J_\xi}\varphi \mid \sigma)_Q = (\varphi \mid \boldsymbol{Q}_{J_\xi}\sigma)_Q,$$

where $(\cdot \mid \cdot)_Q$ denotes the evaluation map corresponding to the scalar product (2.10). The space of generalized invariant vectors is

$$\ker \boldsymbol{Q}'_J = \{\varphi \in \mathcal{D}'_F \mid \boldsymbol{Q}'_{J_\xi}\varphi = 0 \text{ for all } \xi \in \mathfrak{g}\}.$$

On the other hand, the range of $\boldsymbol{Q}_J$ in $\mathcal{D}_F$ is

$$\operatorname{range} \boldsymbol{Q}_J = \{\boldsymbol{Q}_{J_{\xi_1}}\sigma_1 + \boldsymbol{Q}_{J_{\xi_2}}\sigma_2 + ... + \boldsymbol{Q}_{J_{\xi_k}}\sigma_k \mid \sigma_1, ..., \sigma_k \in \mathcal{D}_F\},$$

where $(\xi_1, ..., \xi_k)$ form a basis of $\mathfrak{g}$. There is a duality between $\ker \boldsymbol{Q}'_J$ and $\mathcal{D}_F/\operatorname{range} \boldsymbol{Q}_J$ such that for every $\varphi \in \ker \boldsymbol{Q}'_J$ and every class $[\sigma] \in \mathcal{D}_F/\operatorname{range} \boldsymbol{Q}_J$, we have

$$\langle \varphi \mid [\sigma] \rangle = \langle \varphi \mid \sigma \rangle.$$

Since each $\boldsymbol{Q}_{J_\xi}$ is a multiplication operator, it follows that

$$\operatorname{range} \boldsymbol{Q}_J = \{J_{\xi_1}\sigma_1 + J_{\xi_2}\sigma_2 + ... + J_{\xi_k}\sigma_k \mid \sigma_1, ..., \sigma_k \in \mathcal{D}\} = \mathcal{J}_0\mathcal{D}_F.$$

In the following discussion, we look for conditions under which $\mathcal{D}_F/\operatorname{range} \boldsymbol{Q}_J$ and $(S_F^\infty(L)/\mathcal{J}_0 S^\infty(L))^G$ are isomorphic. We begin with a simple lemma.

LEMMA 6.1. *The class $[\sigma] \in S^\infty(L)/\mathcal{J}_0 S^\infty(L)$ of $\sigma \in S^\infty(L)$ is uniquely determined by the restriction of σ to any open set containing $J^{-1}(0)$.*

PROOF. For $\sigma \in S^{\infty}(L)$, if (support $\sigma) \cap J^{-1}(0) = \emptyset$, then $[\sigma] = 0$. $\square$

THEOREM 6.2. *If Q is locally compact and $\vartheta(J^{-1}(0))$ is compact, then $\mathcal{D}_F/\mathcal{J}_0\mathcal{D}_F$ and $(S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L))^G$ are isomorphic.*

PROOF. For every $\sigma \in S^{\infty}_F(L)$, there exists an open set $V_\sigma \subset Q = P/D$ such that $V_\sigma \supseteq \vartheta(J^{-1}(0))$, $[\sigma]$ is uniquely determined by $\vartheta^{-1}(V_\sigma)$, and $\overline{V}_\sigma$ is compact. Hence, there exists $\sigma' \in \mathcal{D} = S^{\infty}_F(L) \cap \mathcal{H}$ such that $\sigma_{|\vartheta^{-1}(V_\sigma)} = \sigma'_{|\vartheta^{-1}(V_\sigma)}$ and $[\sigma] = [\sigma']$. Therefore,

$$S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L) = S^{\infty}_F(L)/(\mathcal{J}_0 S^{\infty}(L) \cap S^{\infty}_F(L)) = \mathcal{D}_F/(\mathcal{J}_0 S^{\infty}(L) \cap \mathcal{D}_F).$$

For $\sigma' \in \mathcal{D}_F$, the class $[\sigma] \in \mathcal{D}_F/(\mathcal{J}_0 S^{\infty}(L) \cap \mathcal{D}_F)$ is given by

$$[\sigma'] = \{\sigma' + \sum_i f_i J_{\xi_i} \sigma_i \mid f_i J_{\xi_i} \sigma_i \in \mathcal{D}_F\}.$$

But, $f_i J_{\xi_i} \sigma_i \in \mathcal{D}_F$ implies that $f_i J_{\xi_i} \sigma_i = \vartheta^*(\psi_i)\sigma_0$, where ψ_i has a compact support in Q. There exists a function $\chi_i \in \mathbb{C} \otimes C^{\infty}_0(Q)$ such that $\chi_i(q) = 1$ for every q in the support of ψ_i. Then, $\psi_i = \psi_i \chi_i$ and

$$f_i J_{\xi_i} \sigma_i = \vartheta^*(\psi_i)\sigma_0 = \vartheta^*(\psi_i \chi_i)\sigma_0 = \vartheta^*(\psi_i)\vartheta^*(\chi_i)\sigma_0 = f_i J_{\xi_i} \vartheta^*(\chi_i)\sigma_i \in \mathcal{J}_0\mathcal{D}_F.$$

This implies

$$[\sigma] = [\sigma'] \in \mathcal{D}_F/\mathcal{J}_0\mathcal{D}_F,$$

so that

$$S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L) = \mathcal{D}_F/\mathcal{J}_0\mathcal{D}_F. \tag{6.4}$$

By definition,

$$(S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L))^G = \{[\sigma] \in S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L) \mid [\boldsymbol{Q}_{J_\xi}\sigma] = 0 \text{ for all } \xi \in \mathfrak{g}\}.$$

But, $\boldsymbol{Q}_{J_\xi}\sigma = J_\xi \sigma$. Hence, $[\boldsymbol{Q}_{J_\xi}\sigma] = 0$ for all $\sigma \in S^{\infty}_F(L)$ and for all $\xi \in \mathfrak{g}$. Therefore,

$$(S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L))^G = S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L). \tag{6.5}$$

Equations (6.4) and (6.5) yield

$$(S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L))^G = \mathcal{D}_F/\mathcal{J}_0\mathcal{D}_F,$$

which completes the proof. $\square$

We have shown that in the case under consideration, the representation space $(S^{\infty}_F(L)/\mathcal{J}_0 S^{\infty}(L))^G$ of the quantization of the singular reduction at $J = 0$ is naturally isomorphic to the space $\mathcal{D}_F/\mathcal{J}_0\mathcal{D}_F$ of generalized invariant vectors of the geometric quantization of the original phase space (P, ω). Using the shifting trick of Guillemin and Sternberg, we can obtain an equivalent result for non-zero coadjoint orbits O of G. However, we have not been able to obtain the contribution of the trivial representation to the spectral measure $d\mu_\beta$ in the decomposition (5.1).

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