The one-dimensional wave equation with initial-boundary values

The partial differential equations of mathematical physics are often solved conveniently by a method called *separation of variables.* For the one-dimensional wave equation let us consider the initial-boundary value problem

$$\begin{aligned}
\frac{\partial^2 D(x,t)}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2 D(x,t)}{\partial t^2} &= 0 && \text{for} \quad 0 < x < L, \quad t > 0, \\
D(x,0) &= f(x) && \text{for} \quad 0 \le x \le L, \\
\frac{\partial}{\partial t} D(x,0) &= 0 && \text{for} \quad 0 \le x \le L, \\
D(0,t) &= 0, \\
D(L,t) &= 0.
\end{aligned}$$

We seek solutions of the differential equation of the form $D(x,t) = X(x)T(t)$. Substituting this $D(x,t)$ in the wave equation, dividing by $D(x,t)$, and transposing, we find

$$\frac{T''(t)}{T(t)} = v^2 \frac{X''(x)}{X(x)}.$$

Because the left-hand side of this equation depends only upon t and the right-end side is independent of t, both sides of this equation must equal the same constant. We say that the wave equation is separable. Let us call this constant $-\omega^2$. The homogeneous initial and boundary conditions give

$$\begin{aligned}
v^2 X'' + \omega^2 X &= 0, \\
X(0) = X(L) &= 0,
\end{aligned}$$

$$\begin{aligned}
T'' + \omega^2 T &= 0, \\
T'(0) &= 0.
\end{aligned}$$

We now find the eigenvalues

$$\frac{\omega_n}{v} = k_n = \frac{n\pi}{L}$$

and the eigenfunctions $X_n = \sin\,(n\pi x/L)$. Using these values of ω, we have

$$T_n = \cos\,\frac{n\pi v}{L} t$$

so that we look for a solution of the form

$$D(x,t) = \sum_{n}^{\infty} a_n \sin \frac{n\pi x}{L} \cos \frac{n\pi v}{L} t.$$

Putting $t = 0$, we see that we wish to determine the constant a_1,a_2,... so that

$$f(x) = \sum_{n}^{\infty} a_n \sin\frac{n\pi x}{L}.$$

That is, we seek the *Fourier* (*sine*) *series* for $f(x)$.