Introduction to Condensed Matter

PHY 251/PHY 420

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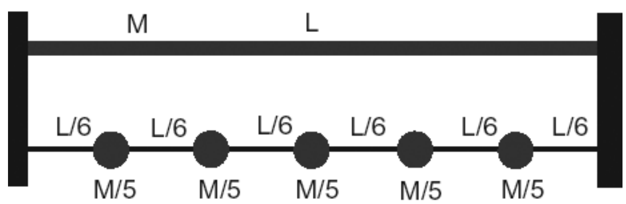
— ASSIGNMENT 2, 10/17/2011 — Due 10/24/2011

(*Undergraduate students do not have to answer the questions marked with the symbol* ■.)

1 Normal Modes of Discrete vs. Continuous Systems *(38 points)*

Referring to the figure below, you are given a uniform string of length L and total mass M that is stretched to a tension F_T . You are also given a set of 5 bodies, each of mass $M/5$, spaced at equal intervals on a massless string with tension F_T and total length L.



(a) Use boundary conditions to derive a general expression for the frequencies of the normal modes of oscillation of the string. Give the frequencies in terms of n, F_T, L, and M.

(b) Write down the frequencies of the five lowest normal modes of transverses oscillation of the string.

(c) Compare the numerical values of these normal mode frequencies with the normal mode frequencies of five beads on the massless string.

(d) Sketch the five lowest normal modes you found for the massive string. Sketch also the five normal modes of the massless-string-with-five-beads.

(e) In a sentence or two, discuss the differences, if any, in the normal modes of the two systems considered here.

2 Connection with the Theory of Elasticity *(30 points)*

We can derive the *continuous theory of elasticity* from the classical theory of the harmonic crystal, by considering only lattice deformations that are small on the scale of the interaction forces, that is, on the scale of the primitive cell. Using the one-dimensional monoatomic chain model,

(a) assuming that only neighbor ions interact, show that normal modes of long wavelengths $ka \ll 1$ are described by an equation of motion that reduces to the continuum elastic wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 u}{\partial t^2}.$$

Find the relation for the speed of sound, v, and explain the connection with the relation $v = \sqrt{F_\mathrm{T}/\mu}$, where F_T is the string tension and μ the mass for unit length.

In metals the effective ion-ion interaction may be of quite long range because mediated by the conduction electron sea. Reexamine the theory of the one-dimensional monoatomic chain assuming an effective ion-ion interaction extending through p nearest ions and a force constant (spring constant) equal to C_p.

(b) Demonstrate that the dispersion relation must be generalized to

$$\omega^2 = \frac{2}{m}\sum_{p>0} C_p(1 - \cos pka)$$

(c) How does the relation for the speed of sound, v, found in (a) change?

3 Ion Vibrations in Metals *(12 points)*

Consider point ions of mass M and charge e immersed in a uniform sea of conduction electrons. The ions are imagined to be in stable equilibrium when at regular lattice points. If one ion is displaced of a small distance r from its equilibrium position, the restoring force is largely due to the electric charge within the sphere of radius r centered at the equilibrium position. Take the number density of ions (or of conduction electrons) as $3/4\,\pi R^3$, which defines R.

(a) Show that the frequency of a single ion set into oscillation is

$$\omega = \sqrt{\frac{e^2}{MR^3}}$$

(b) Estimate the value of this frequency for sodium, roughly.

(c) From (a), (b), and some common sense, estimate the order of magnitude of the speed of sound in the sodium metal. Compare with typical values in solids.

4 Wave Pulse Propagation *(20 points)*

Consider a wave pulse $u(\vec{r},t) = A(\vec{r},t)\; e^{-i(\vec{k}\cdot\vec{r}-\omega t)}$ consisting of a three dimensional plane wave modulated by a slowly varying envelope function, $A(\vec{r},t)$. (Such a wave function may represent sound, light, or matter waves as well.) At time $t = 0$, we assume that the wave pulse is centered at $\vec{r} = 0$ and has a Gaussian envelope function

$$A(\vec{r}, 0) = e^{-\alpha r^2}.$$

(a) ■ If the dispersion relation $\omega(\vec{k})$ can be approximated around $\vec{k} = \vec{k}'$ by truncating the expansion to the first order (linear term)

$$\omega(\vec{k}') \cong \omega(\vec{k}) + \nabla_{\vec{k}}\, \omega(\vec{k})\Big|_{k=k'} \cdot (\vec{k}' - \vec{k}) = \omega + \vec{v}_g \cdot (\vec{k}' - \vec{k}),$$

what happens to the wave pulse at $t > 0$?

(b) ■ In one-dimensional medium, determine the envelope function $A(x, t)$ when the expansion includes also the second order. What happens to the width of the wave packet over time?

5 Singularity in Density of States *(20 points)*

Given the dispersion relation $\omega(k)$ of a monoatomic linear lattice of N atoms with nearest-neighbor interactions,

(a) show that the density of modes, i.e., the number of states per unit frequency range, is

$$D(\omega) = \frac{2N}{\pi} \sqrt{\frac{1}{(\omega_m^2 - \omega^2)}},$$

where ω_m is the maximum frequency.

(b) Suppose that an optical phonon branch in three dimensions (volume equal to L^3) has the form

$$\omega(k) = \omega_0 - Ak^2$$

near $k = 0$. Show that

$$D(\omega) = \begin{cases} \left(\frac{L}{2\pi}\right)^3 \left(\frac{2\pi}{A^{3/2}}\right) \sqrt{\omega_0 - \omega} & \text{for} \quad \omega < \omega_0 \\ 0 & \text{for} \quad \omega > \omega_0 \end{cases}.$$