Introduction to Condensed Matter

PHY 251/PHY 420

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SOLUTIONS —- ASSIGNMENT 1

(*Undergraduate students do not have to answer the questions marked with the symbol* ■.)

1 Classical Theory of Conduction *(20 points)*

The Drude model of metal is based on a regular three-dimensional, fixed array of ions in presence of a large number of electrons that as a classical ideal gas are free to move throughout the entire metal.

1.1) Calculate the root mean square speed, v_{rms}, of free electrons at $T = 300$ K and in the absence of external fields.

Solution

The probability distribution function for the speeds of particles in a classical ideal gas is derived by the Boltzmann distribution and the result is the Maxwell distribution of molecular speed (*eq 2.1*).

The most probable speed v_{m}, the average speed $\langle v \rangle$ the rms speed v_{rms} are:

$$v_{\text{m}} = \sqrt{\frac{2KT}{m}} \qquad \left\langle v \right\rangle = \sqrt{\frac{8KT}{\Pi}} \qquad v_{\text{rms}} = \sqrt{\frac{3KT}{m}}$$

The *rms* speed can be computed from the equipartition theorem considering that the average is done over all directions.

$$\left\langle v_x^2 \right\rangle = \left\langle v_y^2 \right\rangle = \left\langle v_z^2 \right\rangle = \frac{v_{\text{rms}}^2}{3}$$

$$\frac{1}{2} m \left\langle v_x^2 \right\rangle = \frac{K_\beta T}{2}$$

Thus,

$$v_{rms} = \sqrt{\frac{3K_\beta T}{m}} = \sqrt{\frac{3(1.38\times10^{-23})(300)}{(0.91\times10^{-30})}}$$

$$v_{\text{rms}} = 1.17 \times 10^5\,\text{m/s}$$

When a constant electric field is applied, a drift velocity (v_{d}) superimposes in the opposite direction of the field.

1.2) Compute the magnitude of the drift velocity and conductivity of electrons in a copper wire of radius 0.815 mm carrying a current of 1 A and with mean free path of 0.38 nm. (Assume one free electron per atom, $Z = 1$, and a mass density of 8.92 g/cm^3.)

Solution

Using $eq.$1.1 $$n = (0.6022 \times 10^{24})\frac{z\rho_m}{A}$$

$$\rho_m = 8.92\ \text{g/cm}^3 \qquad A = 63.5\,\text{g/mol} \qquad z = 1$$

$$n = 8.47 \times 10^{22}\ \text{atoms/cm}^3$$

$$= 8.47 \times 10^{28}\ \text{atoms/m}^3$$

$$I = J(Area) = -nev_d(\pi r^2)$$

$$v_d = \frac{1\text{c/s}}{\pi(0.000815\,\text{m})^2(8.47\times10^{28}\text{m}^{-3}(1.60\times10^{-19}\text{c}))}$$

$$v_d \cong 3.54 \times 10^{-5}\,\text{m/s}$$

1.3) Compute the electron conductivity assuming a mean free path of 0.38 nm.

Solution

The conductivity of copper at room temperature is given by

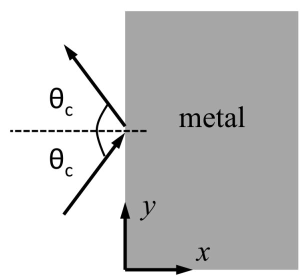
$$\delta = \frac{ne^2\tau}{m} \text{ where } \tau = \frac{l}{v_{rms}} = \frac{0.38\times10^{-9}\text{m}}{1.08\times10^5 m/s} = 3.52\times10^{-15}\,\text{s}$$

$$\delta = 8.33\times10^6(\Omega m)^{-1}$$

This value is seven time smaller than the measured value.

2 X-Ray Metal Mirror *(20 points)*

X-rays which strike a metal surface at an angle of incidence to the normal greater than a critical angle θ_c are totally reflected. Consider a metal (see figure) that occupies the region $x > 0$ and X-rays that propagate in the $x - y$ plane (the plane of the picture) with their polarization vector pointing the $+z$-direction (coming out of the picture). Assume that the metal contains n free electrons per unit volume and apply the Drude model.



2.1) Calculate θ_c as a function of the frequency ω of the X-rays.

Solution

The reflectivity of a metal drops significantly at the plasma frequency, and above ω_p the material becomes transparent (so-called transparent region). The Drude model of metals shows this behaviour in the dielectric functions [eq.(1.37)].

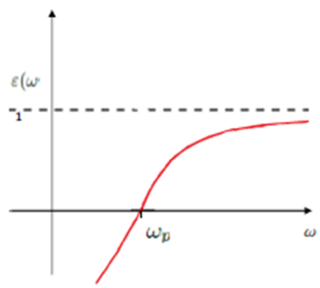
$$\frac{1}{\tau} \ll \omega_p < \omega \qquad \varepsilon(\omega) = 1 - \frac{\omega_p^2}{\omega_2}$$

$$\omega_p = \frac{4\pi ml^2}{m}, \qquad \text{for } m \simeq 10^{22}, \qquad \omega_p \simeq 5.7\times10^{17}\,\text{s}^{-1}$$

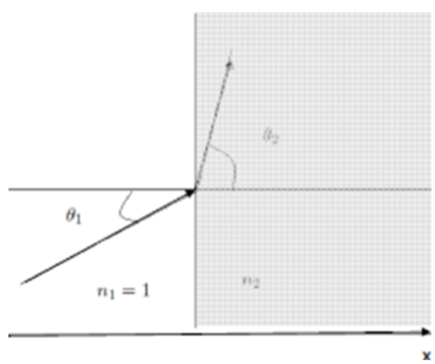
Generally such a Drude term describes only part of the behaviour and more refined theories are needed. Nonetheless, experimentally most of the metals reflect light in the visible and are transparent to light in the UV. for x-rays holds the condition.

$$\frac{1}{\tau} \ll \omega_p < \omega$$

and



Assuming $\mu = 1$ in the metal and $\mu_1 = 1$ for $x < 1$, we have



$$n_2 = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} < 1$$

The critical angle θ_c is

$$\sin\theta_c = n_2 \sin\pi/2; \qquad \theta_c = sin^{-1}\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

2.2)Experiments on gold show that X-rays with an energy of 1 keV have a critical reflection angle of 3.72 degrees. Calculate the plasma frequency ω_{p} using the result in 2.1) and comment on the striking discrepancy between this result and the fact that ω_{p} for gold actually lies in the visible range.

Solution

The energy associated with $\hbar\omega = 10^3 eV$ is

$$\omega \simeq 1.5 \times 10^{18} Hz$$

From the energy dependence of the critical angle obtained in the Drude model we have.

$$\begin{aligned}(sin\theta_c)^2 &= \varepsilon(\omega) \\ &= 1-\frac{\omega_p^2}{\omega^2}\end{aligned} \Longrightarrow \omega_p = \omega\sqrt{1-sin^2\theta_c}$$

$$\omega_p \simeq 1.5 \times 10^{18}\ \mathrm{Hz}$$

Such a ω_{p} is much larger than the real , which lies in the visible. Due to relativistic effects that effect the orbitals around gold atoms we cannot describe correctly the plasma frequency with the Drude model and $\varepsilon(\omega)$ is not given by the above mention formula.

3 Wave Attenuation at Low Frequency *(24 points)*

A plane electromagnetic wave of frequency $\omega/2\pi$ and electric field amplitude E_0 is normally incident on the flat surface of a semi-infinite metal of conductivity σ. Use the Drude model and assume the magnetic permeability of the metal $\mu = 1$ and the frequency regime $\omega\tau \ll 1$, i.e., the displacement current inside the metal can be neglected.

3.1) Using Maxwell's equations, derive the skin depth, or characteristic penetration depth, of the field (δ) and evaluate it for copper at $\omega = 60\,\mathrm{Hz}$ and $\omega = 100\,\mathrm{MHz}$.

Solution

From Maxwell's equations we get (using harmonic time dependence, $e^{-i\omega}t$)

$$\bigtriangledown^2 E + \frac{\omega^2}{c^2}\varepsilon(\omega)E = 0$$

$$\Longrightarrow \varepsilon(\omega) = 1 + i\frac{4\pi\sigma(\omega)}{\omega}$$

The complex wave vector is

$$k = \frac{\omega}{c}m = \frac{\omega}{c}\sqrt{\varepsilon(\omega)} \qquad \mu = 1$$

At low frequency ($\omega\tau \ll 1$)

$$\left.\begin{array}{r} n^2 = 1 + i\frac{k4\pi\sigma(\omega)}{\omega} \\ \sigma(\omega) = \frac{\sigma_0}{1-i\omega\tau} \simeq \sigma_0 \end{array}\right| \Rightarrow \quad \mu^2 \ll i\frac{4\pi\sigma_0}{\omega}$$

Neglecting the displacement current means writing.

$$1 + i\frac{4\pi\sigma(\omega)}{\omega} \ll i\frac{4\pi\sigma(\omega)}{\omega}$$

Thus we have

$$\mu = \sqrt{i}\sqrt{\frac{4\pi\sigma_0}{\omega}} = \sqrt{\frac{2\pi\sigma_0}{\omega}}(1+i) \qquad \sqrt{i} = \frac{1+i}{\sqrt{2}}$$

and the complex wave vector is

$$k = \sqrt{\frac{2\pi\sigma_0\omega}{c}}(1+i)$$

Assuming the E-field in the incident wave is polarized in the +x-direction and the H-field in the +y-direction, we can obtain the fields inside the metal($\overrightarrow{H_t}, \overrightarrow{E_t}$)

$$\overrightarrow{H_t} = H_t\widehat{y}e^{i(kz-\omega t)} = H_t\widehat{y}e^{-\frac{z}{\sigma}}e^{i(\frac{z}{\sigma}-\omega t)}$$

where

$$\sigma \equiv \frac{c}{\sqrt{2\pi\sigma_0\omega}} \text{ is the characteristics penetration depth of the field.}$$

$$H_t^{in} = H_t^{out} = E_0 \text{ we have}$$

$$\overrightarrow{H_t} = E_0\widehat{y} = exp[-\frac{z}{\sigma}]exp[i(\frac{z}{\sigma} - \omega t)]$$

The electric field inside the metal $\overrightarrow{E_t}$:

$$\overrightarrow{E_t} = E_t\widehat{x} = \frac{c}{4\pi\sigma_0}(\nabla X H_t) = -\frac{c}{4}\pi\sigma_0\frac{\partial H_t}{\partial t}\widehat{x}$$

$$\overrightarrow{E_t} = \sqrt{\frac{\omega}{8\pi\sigma_0}}(1-i)E_0\widehat{x}exp[-\frac{z}{\sigma}]exp[i(\frac{z}{\sigma} - \omega t)]$$

$$= \sqrt{\frac{\omega}{8\pi\sigma_0}}E_0\widehat{x}exp[-\frac{z}{\sigma}]exp[i(\frac{z}{\sigma} - \omega t - \frac{\pi}{4})]$$

where the phase shift $-\frac{\pi}{4}$ comes from the factor $(1+i)/\sqrt{2}$. Therefore, $\overrightarrow{H}$ and $\overrightarrow{E}$ in the conductor are $\frac{\pi}{4}$ out of phase. The fields that propagate into the metal are damped, phase shifted, and transverse.

3.2 ■) Derive expressions for the transversal components of the electric and magnetic fields inside the metal. What is the ratio of the magnetic field amplitude to the electric field amplitude inside the metal?

Solution

The ratio of the amplitudes inside the metal is in this approximation.

$$\frac{|H_t|}{|E_t|} = \sqrt{\frac{4\pi\sigma_0}{\omega}}$$

Therefore the field is mostly the magnetic field.

3.3) For sea water ($\sigma = 4.5 \times 10^{10}\,\mathrm{s}^{-1}$ in cgs units) and using radio waves of long wavelength ($\omega = 0.5\,\mathrm{MHz}$) calculate the characteristic penetration depth (δ) and the intensity attenuation at 10 m. Which frequency range would you use to communicate with a deep submerged object like a submarine?

Solution

$$\sigma = \frac{c}{\sqrt{2\pi\sigma_{0\omega}}} = \frac{3\times10^{10}}{\sqrt{2\pi(5\times10^5)(4.5\times10^{10})}} cm \simeq 80\,\mathrm{cm}$$

At a depth of 10 cm below the surface, the intensity attenuation at this frequency will be

$$\frac{p}{p_0} \cong exp[-\frac{2.10}{0.8}] \simeq 10^{-11},$$

which implies that the transmission of signals to submerged objects will require such lower frequency is, $f < 10 - 10^2\,\mathrm{Hz}$

4 Fermi Energy of Gold *(12 points)*

Electrons in a piece of gold metal can be assumed to behave like an ideal Fermi gas and follow the Sommerfeld theory of metals. Gold metal in the solid state has a mass density of 19.30 $\mathrm{g/cm^3}$. Assume that each gold atom donates one electron to the Fermi gas. Assume the system is in the ground-state ($T = 0\,\mathrm{K}$).

4.1) Compute the Fermi speed, Fermi energy (in eV), the Fermi temperature.

Solution

The number of electrons per cubic centimetre, $n = \frac{N}{V}$ is

$$n = 0.6022 \times 10^{24} \frac{z\rho_m}{A}$$

Where ρ_m is the mass density, z the number of electrons contributing to the Fermi gas, A is the atomic mass of the element (in this case A_u), and 0.6022×1010^{24} is the number of atoms per mole (Avocados' number) i.e. the inverse of the atomic mass unit (1.66×10^{-27} kg). (formula 1.1 in A.M. book)

For gold

$$n \cong 5.90 \times 10^{22} \text{ cm}^{-3}$$

Using equ. (2.21); (2.24); (2.25); and (2.33)

$$k_f = (3\pi^2 m)^{\frac{1}{3}}; \qquad v_f = \frac{\hbar k_f}{m}; \qquad \varepsilon_f = \frac{\hbar^2 k_{f^2}}{2m}; \qquad T_f = \frac{\varepsilon_f}{k_\beta}$$

We compute

$$v_f = 1.40 \times 10^8 \text{ cm/s}$$

$$\varepsilon_f = 5.53\,\text{eV}$$

$$T_f = 6.42 \times 10^4\,\text{K}$$

5 Two-Dimensional and One-Dimensional Ideal Fermi Gas *(24 points)*

Nowadays it is experimentally possible to confine electrons in thin layers forming two-dimensional (2D) systems or in thin wires forming or one-dimensional (1D) systems. In this problem we want to study few properties of non-interacting electrons (i.e. ideal gas of spin 1/2 fermions) in 2D and 1D. Assume the electrons to be confined in 2D within a square of area $A = L^2$ and in 1D within a line of length L.

5.1) Express the Fermi wave vector (k_{F}), the Fermi energy (ε_{F}), and the total energy per unit of area of the system as a function of the electron density ($n_{\mathrm{2D}} = N/A$ or $n_{\mathrm{1D}} = N/L$).

Solution

In 3D we have the equation (2.21)

$$m_{3D} = \frac{2}{(2\pi)^3} \int_{K<K_f} \, d^3K = \frac{2}{(2\pi)^3} \int_f^k 4\pi k^2 \, d^2K = \frac{1}{3\pi^2} k_f^3$$

Similarly, in 2D

$$m_{2D} \cong \frac{N}{A} = \frac{2}{(2\pi)^2} \int_{K<K_f} \, d^2K = \frac{2}{(2\pi)^2} \int^{k_f} 2\pi k \, dK = \frac{1}{2\pi} k_f^2$$

$$m_{2D} = \frac{1}{2\pi} k_f^2;$$

and

$$\varepsilon_f = \frac{\hbar^2 k^2 f}{2m} \Rightarrow \varepsilon_f = \frac{\pi \hbar^2 k_f^2}{m} m_{2D}$$

The total energy per unit of area (density of energy)is

$$\frac{E}{A} = \frac{2}{(2\pi)^2} \int_{k<k_f} d^2k \frac{\hbar^2 k^2}{2m} = \frac{2}{(2\pi)^2} \int_0^{k_f} 2\pi k \, dk \, \frac{\hbar^2 k^2}{2m} = \frac{m}{2\pi\hbar^2} \varepsilon_f^2,$$

which can be written as,

$$\frac{E}{A} = \frac{1}{2} m_{\mathrm{2D}} \varepsilon_f$$

5.2) Calculate the density of states of the system $g_{2\mathrm{D}}(\varepsilon)$ and $g_{1\mathrm{D}}(\varepsilon)$.

Solution

$$m_{2D} = \frac{2}{(2\pi)^2}\int_k d^2k = \frac{2}{(2\pi)^2}\int_0^\infty 2\pi\ k\,dk = \int_{-\infty}^{+\infty}\ \rho_{2D}(\varepsilon)d\varepsilon$$

Given the energy relation $\varepsilon = \frac{\hbar^2 k^2}{2m}$ for free electrons,

we have

$$2kdk = \frac{2m}{\hbar^2}d\varepsilon$$

$$m_{2D} = \int_0^\infty\ \frac{1}{2\pi}\,2kdk = \int_{-\infty}^{+\infty}\ \frac{m}{\pi\hbar^2}\,dk.$$

Thus

$$\begin{aligned} \rho_{2D}(\varepsilon) &= \frac{m}{\pi\hbar^2} \quad \varepsilon > 0 \\ &= 0 \quad \varepsilon < 0 \end{aligned}$$

The 2D free electron density of levels is a constant.