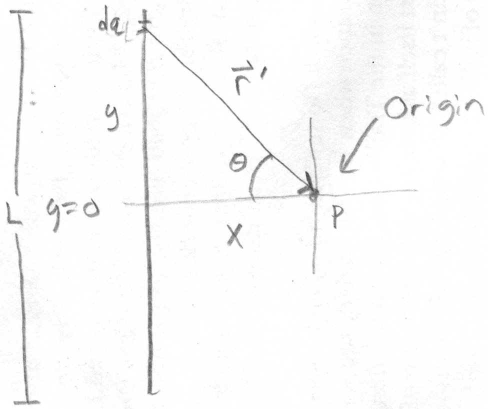
Single, Uniform Line Charge



$$dq = \lambda\, dy$$

$$\vec{r}\,' = y(-\hat{y}) + x\hat{x}$$
$$= r(-\hat{r})$$

$$\vec{E} = \int d\vec{E} = \int k \frac{dq}{(r')^2}\hat{r}\,' = -k\int \frac{\lambda dy}{r^2}\hat{r}$$
$$= -k\lambda \int \frac{dy}{r^2}\hat{r}$$

We want to change dy and r to simplify the integral. We want to express dy in terms of 1 thing that changes over the integral, and stuff that doesn't. $r\sin\theta$ doesn't work because r varies with θ.

$$y = x\tan\theta$$

$$\frac{dy}{d\theta} = x\sec^2\theta \longrightarrow \underline{dy = x\sec^2\theta\, d\theta}$$

$$x = -r\cos\theta \rightarrow r = \frac{-x}{\cos\theta}$$

← The sign is because we're on the opposite side of the origin.

$$\vec{E} = -k\lambda \int \frac{x\,\cancel{\cos^2\theta}\, d\theta}{x^2\,\cancel{\cos^2\theta}}\hat{r}$$

$$= \frac{-k\lambda}{x}\int d\theta\, \hat{r}$$

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}$$

$$\vec{E} = \frac{-k\lambda}{x}\int (\cos\theta\,\hat{x} + \sin\theta\,\hat{y})\,d\theta$$

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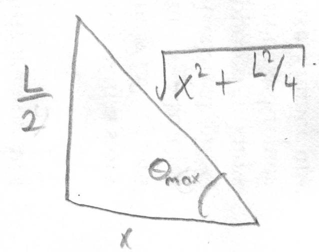
We can kill of the $\hat{y}$ part due to the symmetry of the problem.

$$\vec{E} = \frac{k\lambda}{x} \int \cos\theta \, d\theta \, \hat{x}$$

$$= \frac{k\lambda}{x} \sin\theta \Big|_{\theta_{min}}^{\theta_{max}} \hat{x}$$

Find θ_{min}, θ_{max} by looking at the diagram:



$$\sin\theta_{max} = \frac{L/2}{\sqrt{x^2 + L^2/4}}$$

$$\sin\theta_{min} = -\frac{L/2}{\sqrt{x^2 + L^2/4}}$$

$$\vec{E} = \frac{k\lambda}{x} \frac{L}{\sqrt{x^2 + L^2/4}} \hat{x}$$

If we look at the large L limit,

$$\boxed{\vec{E} = \frac{k\lambda}{x} \hat{x}}$$

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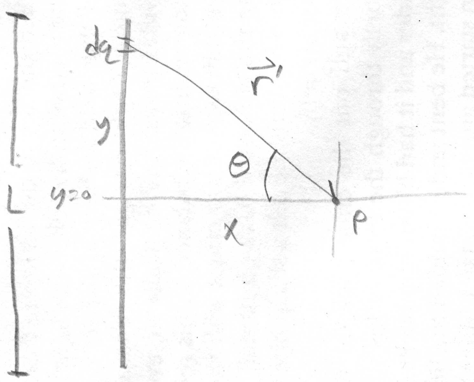
Single, Non-uniform Line Charge



$$dq = \underline{\lambda(y)}\, dy$$

This is the difference.

lets say $\lambda(y) = \lambda_0 \sin^2\theta$

$$dq = \lambda_0 \sin^2\theta\, dy$$

$$\vec{E} = -k\lambda_0 \int \frac{\sin^2\theta\, dy}{r^2}\,\hat{r}$$

$$= -k\lambda_0 \int \frac{x\,\cancel{\cos^2\theta}\,\sin^2\theta}{x\,\cancel{\cos\theta}}\, d\theta\,\hat{r}$$

$$= \frac{k\lambda_0}{x} \int \sin^2\theta \cos\theta\, d\theta\, \hat{x}$$

$$\frac{d\sin\theta}{d\theta} = \cos\theta$$

$$d\theta = \frac{d(\sin\theta)}{\cos\theta}$$

$$= \frac{k\lambda_0}{x} \int \sin^2\theta\, d(\sin\theta)\,\hat{x}$$

$$= \frac{k\lambda_0}{x}\, \frac{\sin^3\theta}{3}\Bigg|_{\theta_{min}}^{\theta_{max}}$$

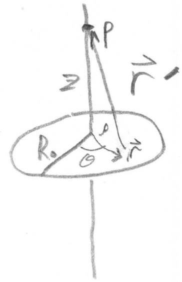
$$= \frac{k\lambda_0}{3x}\, 2\left(\frac{L/2}{\sqrt{x^2+L^2/4}}\right)^3$$

$$\underline{\vec{E}(y=0) = \frac{k\lambda_0}{12x}\, \frac{L^3}{(x^2+L^2/4)^{3/2}}}$$

$L \to \infty$

$$\underline{\vec{E}(y=0) = \frac{k\lambda_0}{3x}}$$

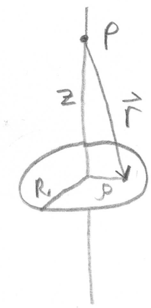
Ex. 2 Disc Charge (Uniform)



$\vec{r}' = z\hat{z} - \rho\hat{\rho}$

Radial coord. in cylindrical.

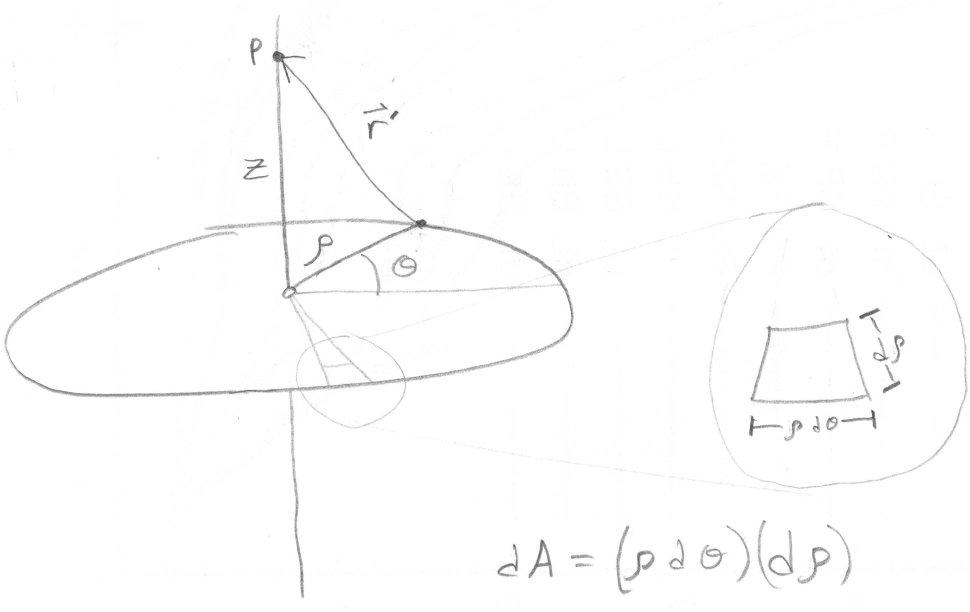
Origin on disk



$\vec{r} = -\vec{r}' = -z\hat{z} + \rho\hat{\rho}$

Origin at test point.

This problem is doable with either setup. I am going to use the first (origin on disk) mostly just in contrast to the previous example.



$dA = (\rho\, d\theta)(d\rho)$

$dq = \sigma \rho\, d\rho\, d\theta$

As usual, $\vec{E} = \int d\vec{E} = \int k \frac{dq}{(r')^2}\hat{r}'$

Note that I am using $\vec{r}'$ for the vector from the charge element to the test point.

We restrict ourselves to the test point being on the z-axis.

$$\vec{r}' = z\hat{z} - \rho\hat{\rho}$$

$$r' = \sqrt{z^2+\rho^2}$$

$$\vec{E} = k \int \frac{\sigma\, \rho\, d\rho\, d\theta}{z^2+\rho^2} \frac{z\hat{z} - \rho\hat{\rho}}{\sqrt{z^2+\rho^2}}$$

The $\hat{\rho}$ contributions will cancel by symmetry.

$$\vec{E} = \sigma k \int \frac{\rho z\, d\rho\, d\theta}{(z^2+\rho^2)^{3/2}} \hat{z}$$

Nothing depends on θ, so just integrate it.

$$\vec{E} = 2\pi\sigma k\, z \int \frac{\rho\, d\rho}{(z^2+\rho^2)^{3/2}} \hat{z}$$

$$= 2\pi\sigma k \frac{\cancel{z}}{\cancel{z}} \int_0^{R_0} \frac{\rho/z \; d\rho/z}{(1+(\rho/z)^2)^{3/2}} \hat{z}$$

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$\rho' = \rho/z \quad d\rho' = d\rho/z$ ← change variables

$$\vec{E} = 2\pi \sigma k \int_0^{R_0/z} \frac{\rho'\, d\rho'}{\left(1+(\rho')^2\right)^{3/2}} \hat{z}$$

$\tan\varphi = \rho' \quad d(\tan\varphi) = d\rho'$ ← change variables

$$\rightarrow \frac{\rho'\, d\rho'}{1+(\rho')^2} = \frac{\tan\varphi\, d(\tan\varphi)}{(1+\tan^2\varphi)^{3/2}} = \frac{\tan\varphi\, d(\tan\varphi)}{\sec^3\varphi}$$

$$= \frac{\sin\varphi}{\cancel{\cos\varphi}} \cos^{\cancel{3}2}\varphi\, d(\tan\varphi)$$

$$\frac{d(\tan\varphi)}{d\varphi} = \sec^2\varphi = \frac{1}{\cos^2\varphi}$$

$$\rightarrow \sin\varphi\cos^2\varphi\, d(\tan\varphi) = \frac{\sin\varphi\, \cancel{\cos^2\varphi}}{\cancel{\cos^2\varphi}} d\varphi$$

$$= \sin\varphi\, d\varphi$$

$$\vec{E} = 2\pi\sigma k \int_{\rho'=0}^{\rho'=R_0/z} \sin\varphi\, d\varphi\, \hat{z}$$

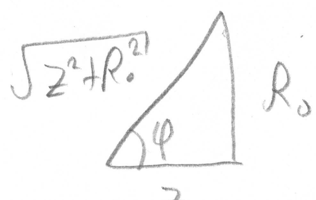
$\rho' = R_0/z \rightarrow \varphi = \tan^{-1}(R_0/z)$

$\rho' = 0 \rightarrow \varphi = \tan^{-1}(0)$

$$= 2\pi\sigma k \left(-\cos\varphi \Big|_0^{\tan^{-1}(R_0/z)}\right) \hat{z}$$

$$= -2\pi k\sigma \left[\cos\left(\tan^{-1} R_0/z\right) - 1\right] \hat{z}$$

How do we find $\cos(\tan^{-1} R_0/z)$?
Use the definitions of trig funcs, and make a triangle. tan is opposite over adjacent. $\tan^{-1} R_0/z \rightarrow \tan(\varphi) = R_0/z$



$$\rightarrow \cos\varphi = \frac{z}{\sqrt{z^2+R_0^2}}$$

$$\vec{E} = 2\pi k \sigma \left[1 - \frac{z}{\sqrt{z^2+R_0^2}} \right] \hat{z}$$

If we take the limit of either a very large disk $(R_0 \rightarrow \infty)$ or a point very near the disk $(z \rightarrow 0)$, this reduces to:

$$\boxed{\vec{E} = 2\pi k \sigma \hat{z}}$$

or

$$\vec{E} = \frac{2\cancel{\pi}}{4\cancel{\pi}\epsilon_0} \sigma \hat{z}$$

$$= \frac{\sigma}{2\epsilon_0} \hat{z}$$