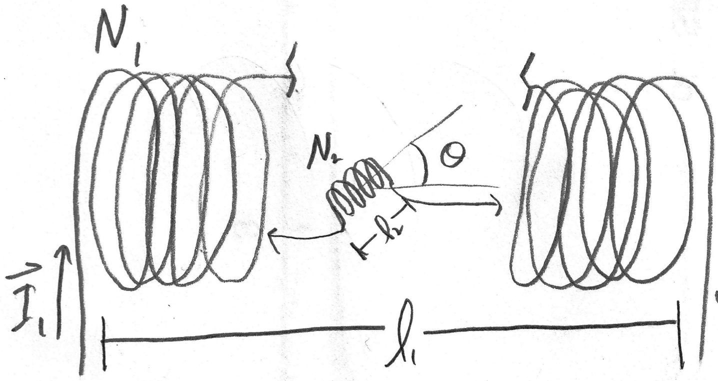
Inductance Examples



To find M, we work by analogy to the case where θ was 0. Lets just go through step-by-step and see what has changed. (Guess what it'll be now!)

First, as before lets define number densities to save a little writting:

$$n_1 = \frac{N_1}{l_1} \qquad n_2 = \frac{N_2}{l_2}$$

Next, we still know the field inside coil 1,

$$B_1 = \mu_0 n_1 I_1$$

The key step is finding the induced $\mathcal{E}$. This has to be different somehow,

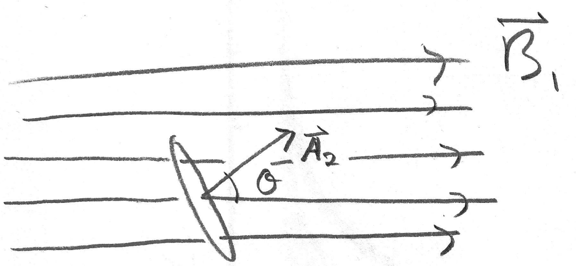
$$\mathcal{E}_2 = -M \frac{dI_1}{dt}.$$

But it isn't clear from this that anything is different until we remember that M depends on the geometry of the setup.

$$M = \frac{N_2}{I_1}\Phi_{21} = \frac{N_1}{I_2}\Phi_{12}$$

This is where things get interesting. The flux is going to be different. Φ_{21} is still easiest to calculate, so lets look at that.

Since coil 2 is small and inside of 1, lets treat the field from 1 as a uniform external field for the purposes of calculating the flux. Also, lets just look at one loop of 2 for now.



We can find the flux through this single loop with the dot product

$$\Phi_{21} = \vec{B}_1 \cdot \vec{A}_2$$
$$= B_1 A_2 \cos\theta$$
$$= \mu_0 n_1 I_1 A_2 \cos\theta$$

So if we want the whole M, just multiply by N_2 / I_1

$$M = \frac{N_2}{\cancel{I_1}} \mu_0 n_1 \cancel{I_1} A_2 \cos\theta$$

$$\boxed{M = \mu_0 \frac{N_1 N_2 A_2}{l_1} \cos\theta}$$

This $\cos\theta$ is the only difference when we rotate the inside coil, so:

$$\mathcal{E}_2 = -\mu_0 \frac{N_1 N_2 A_2}{l_1} \frac{dI_1}{dt} \cos\theta$$

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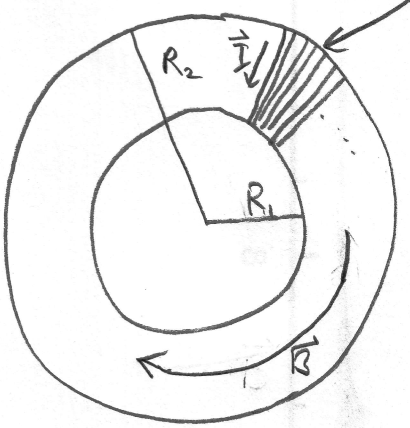
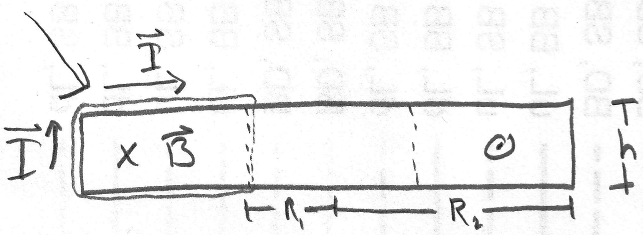
Top view

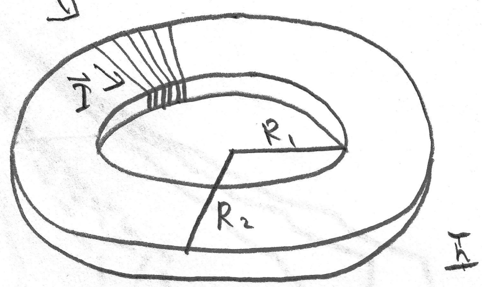


Side View

Wire Coils



Perspective View



N wire coils

Lets just find the self inductance L.

What is L? $\quad L = \dfrac{\Phi}{I}$.

We can find Φ by finding the flux through one coil and multiplying by N. The flux through a loop would be the flux passing through the plane of the paper as in the side view. We don't yet know the field inside, however.

$$\Phi = \int \vec{B} \cdot d\vec{A}$$

$\vec{A} = h(R_2 - R_1)\ \widehat{RHR}$

$d\vec{A} = h\,dr\ \widehat{RHR}$

(see the coaxial cable problem. Same logic here.)

$$\vec{B} = ?$$

7-30-08

Since this torus is a closed loop, none of the field inside escapes. In particular, the field created by each loop passes through every other loop.

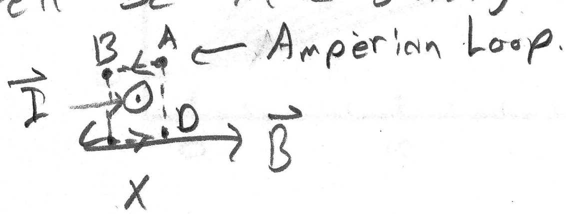
We calculate that field produced by each loop the same way we did a single loop of our first solonoid problem: Ampère's Law.

When we had a straight solonoid, we cut it in half to draw our Ampèrian Loop.

Ampère's Law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

We can do the same, but we need to deal with the fact that our toroid is curved. Luckily, we can cheat. Rather than do something curved, we just only look at 1 loop, which is so little of the ring that it might as well be in a straight solonoid!



$$\oint \vec{B} \cdot d\vec{l} = \int_A^B \vec{B} \cdot d\vec{l} + \int_B^C \vec{B} \cdot d\vec{l} + \int_C^D \vec{B} \cdot d\vec{l} + \int_D^A \vec{B} \cdot d\vec{l}$$

As in the case of a solonoid, we can take 1 of these to be 0 because we assume the field outside the coil to be small $\to 0$.

$$\int_A^B \vec{B}\cdot d\vec{l} = 0$$

Then we can toss the two sides because in them $d\vec{l}$ is perpendicular to $\vec{B}$, or $\vec{B}$ is 0.

$$\int_B^C \vec{B}\cdot d\vec{l} = \int_D^A \vec{B}\cdot d\vec{l} = 0$$

leaving

$$\int_C^D \vec{B}\cdot d\vec{l} = \mu_0 I$$

$$B\int_C^D dl = \mu_0 I$$

Ok, so how long is the segment CD of our loop? Well, it is on the top of our toroid and takes up $1/N$ th of the total circumference. However, that circumference depends on r. Circ. of a circle is $2\pi r$. At the outer edge this is $2\pi R_2$, inside edge gives $2\pi R_1$.

So the length of CD $= \dfrac{2\pi r}{N}$

$$B \int_C^0 dl = \mu_0 I$$

$$B \frac{2\pi r}{N} = \mu_0 I$$

$$\underline{B = \frac{\mu_0}{2\pi} \frac{NI}{r}} \qquad \underline{\vec{B} = \frac{\mu_0}{2\pi} \frac{NI}{r} \widehat{RHR}}$$

Now we can find Φ.

N loops!

$$\Phi = N \int \vec{B} \cdot d\vec{A}$$

$$= N \int_{R_1}^{R_2} \left(\frac{\mu_0}{2\pi} \frac{NI}{r} \widehat{RHR} \right) \cdot \left(h\, dr\, \widehat{RHR} \right)$$

$$= \frac{\mu_0}{2\pi} N^2 I h \int_{R_1}^{R_2} \frac{dr}{r} \cancel{\widehat{RHR} \cdot \widehat{RHR}} \to 1$$

Again, we have seen this integral before:

$$\underline{\Phi = \frac{\mu_0}{2\pi} N^2 I h \ln \frac{R_2}{R_1}}$$

$$\boxed{L = \frac{\mu_0}{2\pi} N^2 h \ln \frac{R_2}{R_1}}$$

We could also find the energy content of the toroid.

Recall that $u_B = \frac{1}{2\mu_0} B^2$.

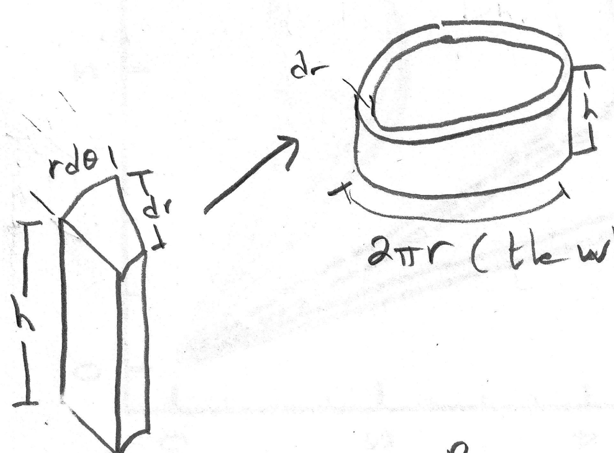
We know B as a function of r, so we can find the total U by integrating.

$$U = \int u_B \, dV_{ol}$$

since $u_B = \frac{1}{2\mu_0}\left(\frac{\mu_0}{2\pi}\frac{NI}{r}\right)^2$

$$= \frac{N^2I^2}{8\mu_0\pi^2}\frac{1}{r^2}$$

depends only on r, that should be the only integral that gives us trouble. The other two dimensions (h high and 2π around) can just be multiplied by.



$2\pi r$ (the whole way around)

$$dV_{ol} = 2\pi r h\, dr$$
$$= (2\pi h)(r\, dr)$$

$$U = \frac{N^2I^2}{8\mu_0\pi^2}\int_{R_1}^{R_2}\frac{1}{r^2}(2\pi h)(r\,dr) = \frac{hN^2I^2}{4\pi\mu_0}\int_{R_1}^{R_2}\frac{dr}{r}$$

$$\boxed{U = \frac{hN^2I^2}{4\pi\mu_0}\ln\frac{R_2}{R_1}}$$