7-14-08

Exam Solutions 1

1. a) $\vec{F}_{12} = k \frac{q_1 q_2}{r^2} \hat{r}_{12}$

This is attractive if q_1 + q_2 have opposite signs, repulsive for same sign.

b) $\vec{F}_{12} = -G \frac{m_1 m_2}{r_2} \hat{r}_{12}$

Always attractive.

c) Both drop off as $\frac{1}{r^2}$. Both are proportional to both "charges", where for gravitation mass is the "charge".

Gravitation can only be attractive, while Coulomb's Law can give a repulsive force. Thus, neutral objects have canceling electrostatic forces.

2. a) $\vec{E}_q = k \frac{q_1}{r^2} \hat{r}$

b) I emphasize difference to remind that a reference point is needed.

If $V(\infty) = 0$, then

$$V(r) = -\int_\infty^r \vec{E} \cdot d\vec{l} = -\int_\infty^r k \frac{q}{r^2} dr \, \hat{r} \cdot \hat{r}$$

$$V(r) = -kq \left(-\frac{1}{r}\right)\Big|_\infty^r = -kq\left(-\frac{1}{r} - -\frac{1}{\infty}\right)$$

$$\boxed{V(r) = \frac{kq}{r}}$$

7-14-08

c) $W = -\nabla U = q_2 \nabla V = -q_2 \int_\infty^r \vec{E}\cdot d\vec{l}$

$$= -q_2 \int k \frac{q_1}{r^2}\, dr\, \hat{r}\cdot\hat{r}$$

$$= -kq_1 q_2 \int_\infty^r \frac{dr}{r^2}$$

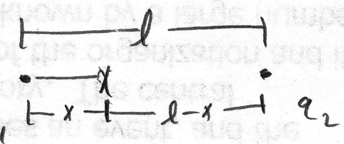
$$= -kq_1 q_2 \left[-\frac{1}{r} - -\frac{1}{\infty}\right]$$

$$\boxed{W = \frac{kq_1 q_2}{r}}$$

The signs here can be tricky. Remember it should be positive for same sign charges because pulling them together opposes their repulsion: we have to work to make it happen.

d) ~~c)~~ If they have the same sign, the field will cancel somewhere between the charges.

$$|\vec{E}_1| = |\vec{E}_2|$$



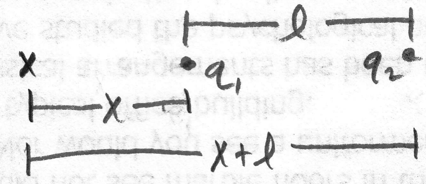
$$k\frac{q_1}{x^2} = k\frac{q_2}{(l-x)^2}$$

$$\left(\frac{l-x}{x}\right)^2 = \frac{q_2}{q_1} \longrightarrow \left(\frac{l}{x} - 1\right)^2 = \frac{q_2}{q_1}$$

$$\frac{l}{x} - 1 = \sqrt{\frac{q_2}{q_1}}$$

$$\frac{l}{x} = \sqrt{\frac{q_2}{q_1}} + 1 \longrightarrow \boxed{x = \frac{l}{1+\sqrt{\frac{q_1}{q_2}}}}$$

ii) For opposite signs, the point needs to be on one or the other side of both charges.



$q_2 > q_1$

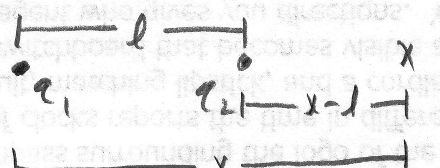
$$|\vec{E}_1| = |\vec{E}_2|$$

$$k\frac{q_1}{x^2} = k\frac{q_2}{(l+x)^2}$$

$$\left(\frac{l+x}{x}\right)^2 = \frac{q_2}{q_1}$$

$$\sqrt{\frac{q_2}{q_1}} - 1 = \frac{l}{x}$$

$$\boxed{x = \frac{l}{\sqrt{\frac{q_2}{q_1}} - 1}}$$



$q_1 > q_2$

$$|\vec{E}_1| = |\vec{E}_2|$$

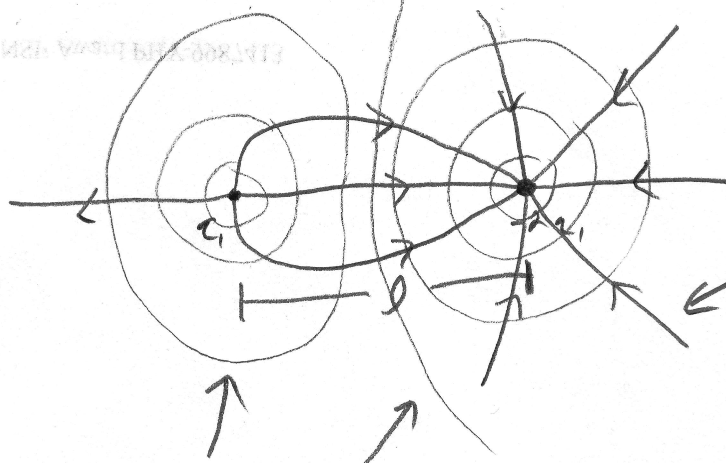
$$k\frac{q_1}{x^2} = k\frac{q_2}{(x-l)^2} \rightarrow \left(1-\frac{l}{x}\right)^2 = \frac{q_2}{q_1}$$

$$1-\sqrt{\frac{q_2}{q_1}} = \frac{l}{x}$$

$$\boxed{x = \frac{l}{1-\sqrt{\frac{q_2}{q_1}}}}$$

7-14-08

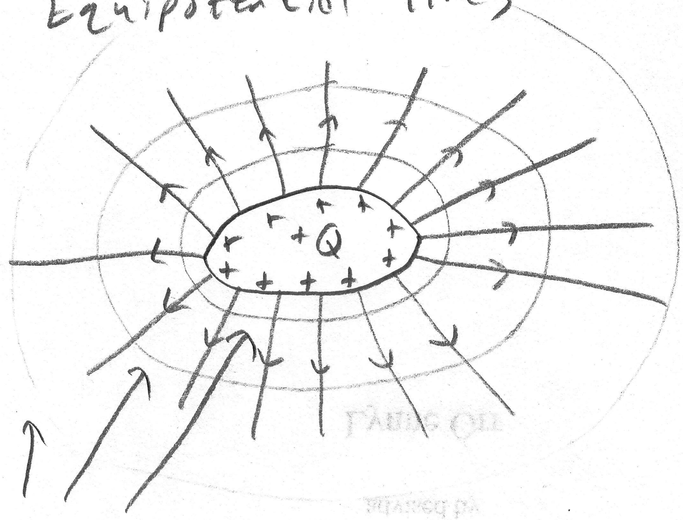
③ a)



$\vec{E}$ field lines

b) Equipotential lines

c)



d) Equipotential Lines

④ a) $\sigma_1 = \frac{-q}{A_1}$ $\boxed{\sigma_1 = \frac{-q}{4\pi R_1^2}}$ $A = 4\pi r^2$

$\sigma_2 = \frac{q+Q}{A_2}$ $\boxed{\sigma_2 = \frac{q+Q}{4\pi R_2^2}}$

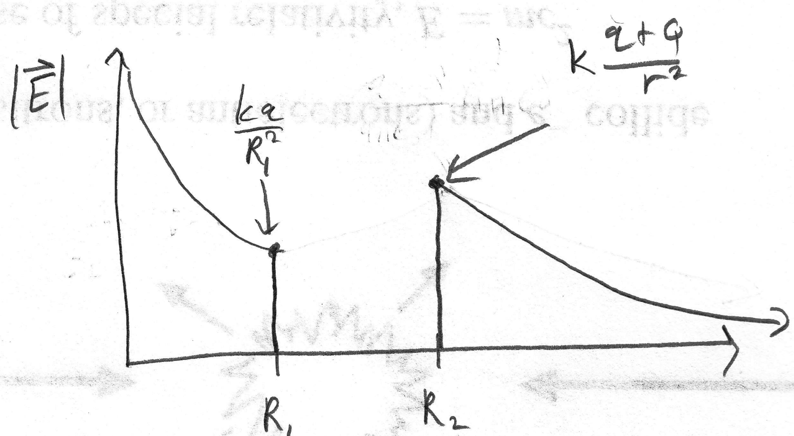
b) i) $\boxed{\vec{E} = k\frac{q}{r^2}\hat{r}}$

ii) $\boxed{\vec{E} = 0}$ inside a conductor

7-14-08

iii) $\boxed{\vec{E} = k\frac{q+Q}{r^2}\hat{r}}$

iv)



c) i) $V = -\int_{\infty}^{r} \vec{E}\cdot d\vec{l}$

$$= -k(q+Q)\int_{\infty}^{r}\frac{dr'}{r'^2}\hat{r}\cdot\hat{r}$$

$$= -k(q+Q)\left(-\frac{1}{r} - -\frac{1}{\infty}\right)$$

$$\boxed{V = k\frac{q+Q}{r}}$$

ii) $V(r) = V(R_2) - \int_{R_2}^{r}\vec{E}\cdot d\vec{l}$

$$= V(R_2) - 0$$

$$\boxed{V(r) = k\frac{q+Q}{R_2}}$$

iii) $V(r) = V(R_1) - \int_{R_1}^{r}\vec{E}\cdot d\vec{l}$

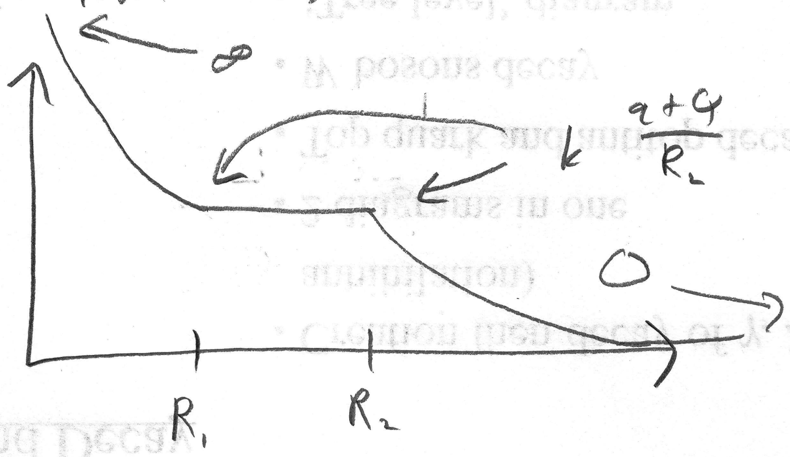
$$= k\frac{q+Q}{R_2} - kq\int_{R_1}^{r}\frac{dr'}{r'^2}\hat{r}\cdot\hat{r}$$

$$= k\frac{q+Q}{R_2} + kq\left(\frac{1}{r} - \frac{1}{R_1}\right)$$

$$\boxed{V(r) = k\left[\frac{q+Q}{R_2} + \frac{q}{r} - \frac{q}{R_1}\right]}$$

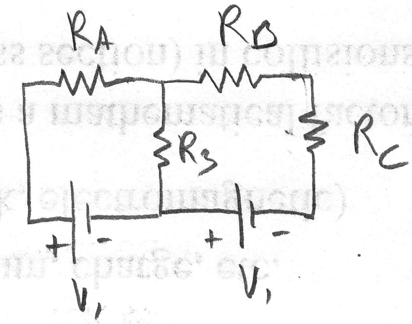
iv) Because each successive region builds on the last. This should remind you to work your way in from $V(\infty) = 0$.

v)

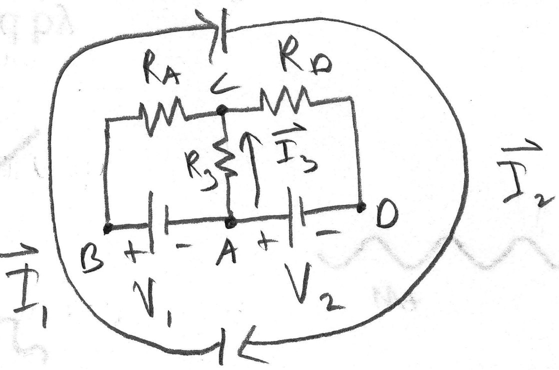


⑤

$$\left.\begin{aligned} R_1 + R_2 &= R_A \\ \frac{1}{R_4} + \frac{1}{R_5} &= \frac{1}{R_B} \\ \frac{1}{R_6} + \frac{1}{R_7} &= \frac{1}{R_C} \end{aligned}\right\} \longrightarrow$$



$$R_B + R_C = R_D \longrightarrow$$



$R_A = 3\Omega$

$R_B = \frac{4}{3}\Omega$

$R_C = \frac{4}{3}\Omega$

$R_D = \frac{8}{3}\Omega$

Unknowns: I_1, I_2, I_3

7-14-08

Junction Rule: A

$$\left.\begin{array}{lr} BA & -I_1 \\ +CA & -I_3 \\ +DA & I_2 \\ \hline & 0 \end{array}\right\} \quad \overset{\text{eq. 1}}{I_2 = I_1 + I_3}$$

Loop Rule:

ABCA:

$$\left.\begin{array}{ll} AB: & V_1 \\ -BC: & R_A I_1 \\ -CA: & -R_3 I_3 \\ \hline & 0 \end{array}\right\} \quad \overset{\text{eq. 2}}{V_1 + R_3 I_3 = R_A I_1}$$

ACDA:

$$\left.\begin{array}{ll} -AC: & R_3 I_3 \\ -CD: & R_D I_2 \\ DA: & V_2 \\ \hline & 0 \end{array}\right\} \quad \overset{\text{eq. 3}}{V_2 = R_3 I_3 + R_D I_2}$$

eq. 2

$$I_1 = \frac{V_1 + R_3 I_3}{R_A} = \frac{V_1}{R_A} + \frac{R_3}{R_A} I_3 = 2\frac{V}{\Omega} + 2I_3$$

eq 2 → eq 1

$$I_2 = 2\frac{V}{\Omega} + 2I_3 + I_3$$

$$I_2 = 2\frac{V}{\Omega} + 3I_3$$

$$I_2 = \frac{V_1}{R_A} + \left(\frac{R_3}{R_A} + 1\right) I_3$$

7-14-08

eq. 3

$$V_2 = R_3 I_3 + R_D I_2$$

$$12V = 6\Omega I_3 + \frac{8}{3}\Omega\left[2\frac{V}{\Omega} + 3I_3\right]$$

$$12V = (6\Omega + 8\Omega) I_3 + \frac{16}{3} V$$

$$\frac{36-16}{3} V = 14\Omega I_3$$

$$I_3 = \frac{20}{42}\frac{V}{\Omega}$$

$$\boxed{I_3 = \frac{10}{21}\frac{V}{\Omega}}$$

$$V_2 = R_3 I_3 + R_D\left[\frac{V_1}{R_A} + \left(\frac{R_3}{R_A} + 1\right) I_3\right]$$

$$= I_3\left[R_3 + R_D\left(1 + \frac{R_3}{R_A}\right)\right] + V_1 \frac{R_D}{R_A}$$

$$I_3 = \frac{V_2 - V_1 \frac{R_D}{R_A}}{R_3 + R_D\left(1 + \frac{R_3}{R_A}\right)}$$

eq. 2 again

$$I_2 = \frac{V_1}{R_A} + \left(\frac{R_3}{R_A} + 1\right) I_3$$

$$I_2 = 2\frac{V}{\Omega} + 3I_3$$

$$= \left(2 + 3\frac{10}{21}\right)\frac{V}{\Omega}$$

$$\boxed{I_2 = \frac{24}{7}\frac{V}{\Omega}}$$

eq. 1

$$I_1 = I_2 - I_3$$

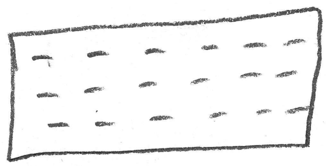
$$= \left(\frac{24}{7} - \frac{10}{21}\right)\frac{V}{\Omega}$$

$$= \frac{72-10}{21}\frac{V}{\Omega}$$

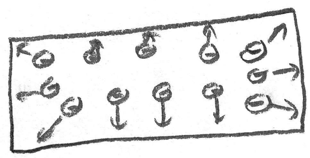
$$\boxed{I_1 = \frac{62}{21}\frac{V}{\Omega}}$$

b) $V = IR$ applied for each resistor

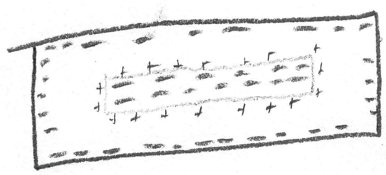
(6) a)



b)



c)



(7) a) i) $V = -\int_l^0 \vec{E}\cdot d\vec{l} = -\int_l^0 \frac{Q}{\epsilon_0 A}\hat{x}\cdot d\vec{l} = \boxed{\frac{Ql}{\epsilon_0 A}}$

$E = \frac{Q}{\epsilon_0 A}$ from sheet

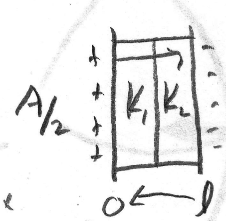
ii) $C_0 = \frac{Q}{V}$

$C_0 = \frac{lQ}{E}$ $\longrightarrow$ $\boxed{\epsilon_0 \frac{A}{l}}$

iii) $U_0 = \frac{Q^2}{2C_0}$

$$\boxed{U_0 = \frac{Q^2}{2}\frac{l}{\epsilon_0 A}}$$

b) i) $C = \frac{Q}{V}$



$$V = -\int_l^{l/2} \frac{Q}{\epsilon_2}\frac{\hat{x}}{A/2}\,dx - \int_{l/2}^0 \frac{Q}{\epsilon_1}\frac{\hat{x}}{A/2}\,dx$$

$$= \frac{\cancel{2}Q}{A\epsilon_2}\frac{l}{\cancel{2}} + \frac{\cancel{2}Q}{A\epsilon_1}\frac{l}{\cancel{2}}$$

$$= \frac{Ql}{A}\left(\frac{1}{\epsilon_1}+\frac{1}{\epsilon_2}\right) = \boxed{\frac{Ql}{A}\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}}$$

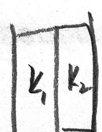
$$\boxed{C = \frac{l}{A}\frac{\epsilon_1\epsilon_2}{\epsilon_1+\epsilon_2}}$$

7-14-08

ii) $\boxed{C_{tot} = \epsilon_0 \frac{A}{2l} + \frac{1}{A}\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}}$

iii) $\boxed{U = \frac{1}{2}\frac{Q^2}{C_{tot}}}$

i) Alternate soln:



are capacitors in series, so

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \frac{1}{\epsilon_1 \frac{A/2}{l/2}} + \frac{1}{\epsilon_2 \frac{A/2}{l/2}}$$

$$= \frac{l}{\epsilon_1 A} + \frac{l}{\epsilon_2 A}$$

$$= \frac{l}{A}\left(\frac{\epsilon_2 + \epsilon_1}{\epsilon_1 \epsilon_2}\right)$$

$$\boxed{C = \frac{A}{l}\frac{\epsilon_1 \epsilon_2}{\epsilon_1 + \epsilon_2}}$$