10: Faraday's Law and Induction July 17, 2008

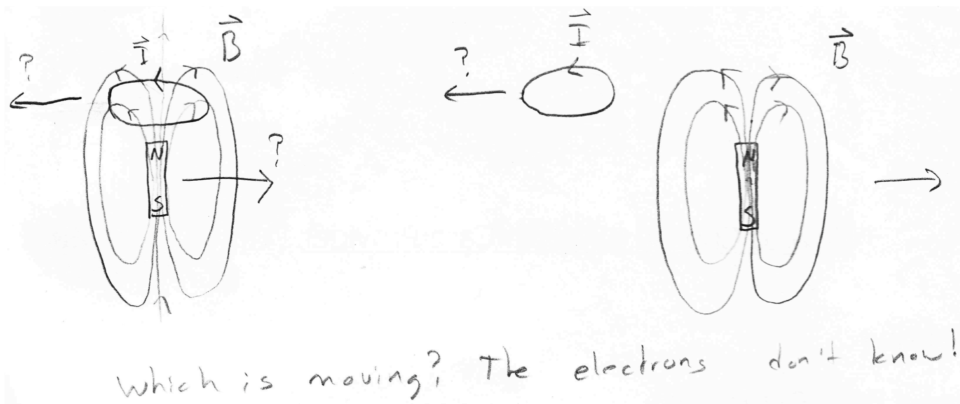
10.1 Induced $\mathscr{E}$

Once it was realized that a current produces a magnetic field, it was natural to ask if somehow the converse is possible: can a magnetic field generate a current? We know that a magnetic field exerts a force on a current because it is composed of moving charges, but this is inevitable perpendicular to the direction of current flow and so can neither increase nor decrease the current. Likewise, it is hard to see how the magnetic field, which influences only moving charges, could produce a new current.

Nevertheless, it somehow seemed to physicists at the time that there should be an equal footing of sorts: if a current can produce a $\vec{B}$ field, the field should be able to produce the current! The resolution to this dilemma turns out to be one of the first steps down the road the development of Einstein's theory of Relativity, strangely enough. That resolution can be most simply stated by the often misused summary of that later theory: everything is relative! What exactly does "moving charge" mean? Well, it turns out that it just means charge moving relative to the magnetic field. So if I leave the circuit on the table but instead wave the magnet around above it, ta-da! Current! The charges are moving *relative to the field*, and so feel a force. This is of course not quite how the phenomenon was understood at the time, because nobody had coined the phrase "everything is relative" and there was no formal expectation in physics that such a thing was true. We won't do special relativity for a while, but we don't need details, just to know that relative motion is important.

10.1.1 Description

After much trial and error involving constant magnetic fields, Faraday noticed by accident that when he turned his electromagnets on and off, it would produce the current he was hoping for. After fairly extensive investigation,



he was able to state a form of the law which now bears his name, "changing magnetic fields produce an $\mathscr{E}$". The $\mathscr{E}$ then creates a current. We call this an *induced* $\mathscr{E}$ because the field induces it, rather than a source $\mathscr{E}$ being plugged into the circuit.

I talked about relative motion before, and now we state the law in terms of changing field: what gives? The two are in fact equivalent. A charge can tell a moving field from a stationary one only because it sees a change in the field, and if it sees a field changing, that looks like a field moving. Both are mathematically equivalent, which means they are physically equivalent even if our mental picture of the situation is quite different.

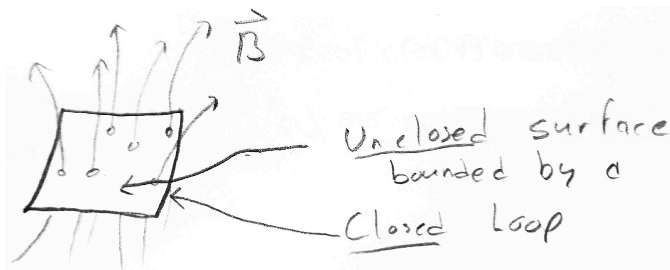
10.1.2 Faraday's Law

In order to define Faraday's Law more precisely, we need to define a *magnetic flux*. This is a real flux in direct analogy to the electric flux we defined for Gauss's Law, rather than the flux-like quantity I used to describe Ampère's Law.

$\Phi_B = \int \vec{B} \cdot d\vec{A}$

Note that we do *not* close the surface of integration. As we discussed last time, that would automatically be 0. Instead we integrate the flux passing through a circuit loop with an area built up of the $d\vec{A}$s of our integral. This is the same area vector we defined when discussing the magnetic dipole. In terms of integration, it is the same as the $d\vec{A}$ of Gauss's Law but this time we are building different whole surfaces (not closed) out of the pieces of area than before (closed).

So, once again, this flux is a measure of how much field, or how many field lines, pass through a loop. Remember that magnetic field lines are complete loops, so it is meaningful to talk about the field lines passing through a loop in the same way that it is meaningful to talk about electric field lines leaving a closed space, yet a closed loop would tell us nothing about electric field lines because they have ends and thus cannot be "enclosed" by a loop.



This definition allows the precise statement of Faraday's Law, which encapsulates both changing fields and moving fields:

$\mathscr{E} = -\frac{d\Phi_B}{dt}$. **Faraday's Law of Induction**

The $\mathscr{E}$ produced is *proportional to the change in magnetic flux through the circuit loop.* Anything which causes such a change will produce a current. So far I have mentioned changing the field (by turning on an electromagnet, for instance) and moving the field (waving a bar magnet around near a circuit). We will discuss a third option later, which is to change the shape of the circuit loop itself while the magnetic field remains entirely static. By changing shape or size, the magnetic flux will also be changed.

Note that in the many systems consisting of looped wires, we again just get a factor of N,

$\mathscr{E} = -N\frac{d\Phi_B}{dt}$

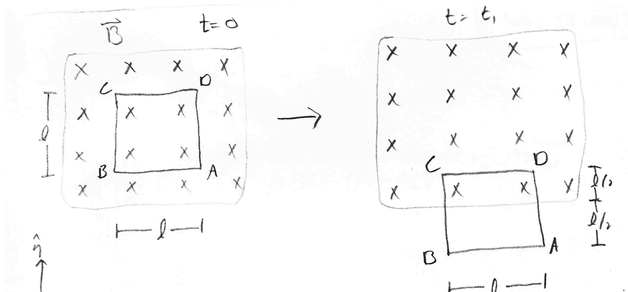
where Φ_B refers to the flux of a single loop.

We can use Faraday's Law to find averages without taking derivatives if we know the configuration of the system at an initial and final time. The *average* induced $\mathscr{E}$ is given by:

$\overline{\mathscr{E}} = -N\frac{\Delta\Phi_B}{\Delta t}$.

We can use this if either the precise form of the magnetic field equation isn't known, or is too difficult to take derivatives of.

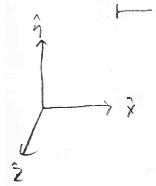
Examples



$$\vec{B} = B_0(-\hat{z})$$

$$\Phi_B(t=0) = B_0\, l^2$$

$$\Phi_B(t=t_1) = B_0\, l\, l/2 = B_0 \frac{l^2}{2}$$



$$|\bar{\mathcal{E}}| = \left| -\frac{\Delta\Phi_B}{\Delta t} \right| = \left| \frac{\Phi_B(0) - \Phi_B(t_1)}{0 - t_1} \right| = \left| -\frac{B_0 l^2 - B_0\, l^2/2}{-t_1} \right|$$

No defined direction yet.

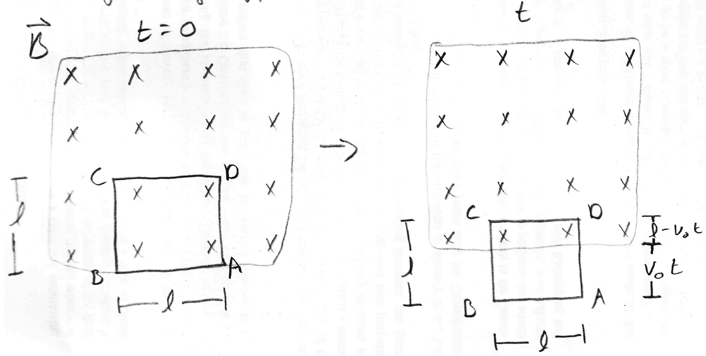
$$= \left| -\frac{l^2/2}{-t_1} \right|$$

$$\boxed{|\bar{\mathcal{E}}| = \frac{l^2}{2t_1} = \frac{A}{2t_1}}$$

$\overline{\mathscr{E}}$ in square loop

Same system, but now the loop is moving with a known $\vec{v}$.

$\vec{v} = v_0(-\hat{y})$



$\vec{B} = B_0(-\hat{z})$

$\Phi_B(0) = B_0 \ell^2$

$\Phi_B(t) = B_0 \ell(\ell - v_0 t)$

$$|\mathcal{E}(t)| = \left| -\frac{d\Phi_B(t)}{dt} \right| = \left| -B_0 \ell \frac{d}{dt}(\ell - v_0 t) \right|$$

$$= B_0 \ell \left| -(0 - v_0) \right|$$

$$\boxed{|\mathcal{E}(t)| = B_0 v_0 \ell}$$

True if AB is on the edge at $t=0$ and until $\ell = v_0 t$, $t = \frac{\ell}{v_0}$, which is when the loop is out and the flux stops changing

$\mathscr{E}(t)$ in a square loop

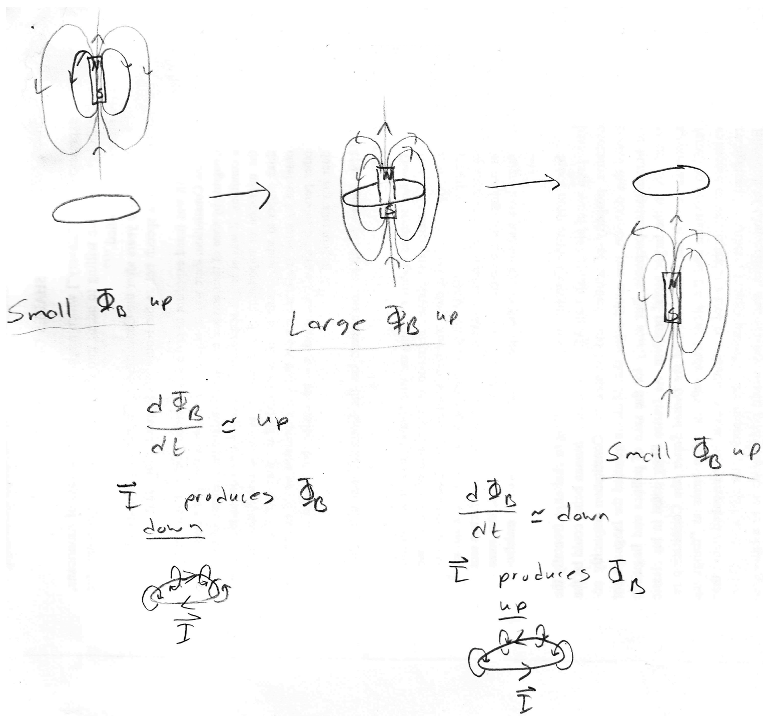
10.1.3 Lenz's Law

As always happens when we define area as a vector, we are left with some ambiguity as to which direction is positive, and we always need some rule to pick one or the other. As usual, that distinction is somewhat arbitrary but we must remain consistent. In the case of magnetic induction, the rule of thumb is known as Lenz's Law and isn't directly stated in terms of the area. Rather, Lenz's Law just goes ahead and tells us which direction the $\mathscr{E}$ will be in. We can assign whichever direction we want to the area to start with (as long as it is consistent without the problem!) and then at the end just use the magnitude of $\mathscr{E}$ given by Faraday's law and stick the direction on it that Lenz's Law dictates. If you want to think about the area directly, you can use the same definition for the direction of area that we did for the magnetic dipole, but that requires you to define an arbitrary direction for the current, which may end up being the opposite of what physically occurs. The only way to straighten all of this out is the following:

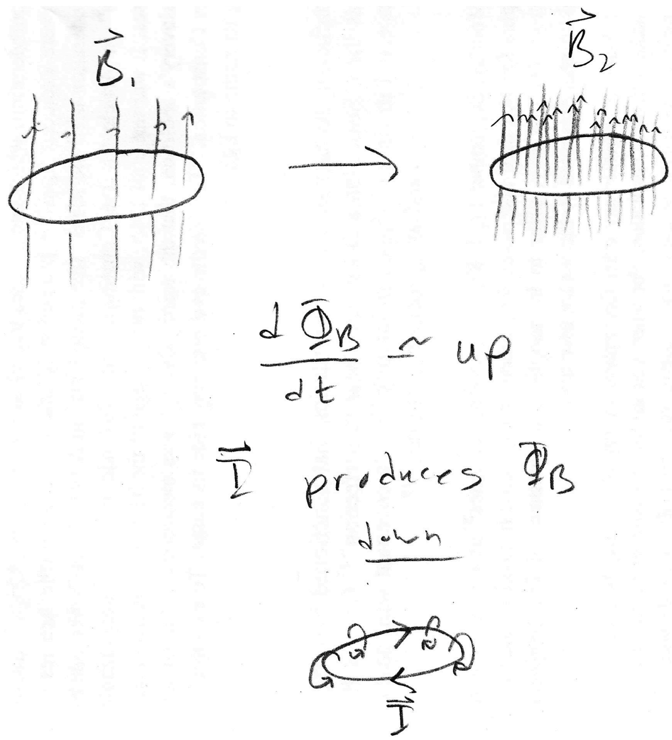
> An induced $\mathscr{E}$ will always *oppose* the change in the field which produced it. **Lenz's Law**

What does this mean? We are speaking here of the magnetic flux. Any change in flux will produce an $\mathscr{E}$. That change may be a reduction or an increase (which depends on our arbitrary choice of direction for the area vector). If the *change* was an increase, then the *induced* current will produce a field flux which tries to *reduce* the flux: the induced $\mathscr{E}$ produces a current, which produces a flux, which opposes the original change. Likewise, if there is a *decrease* in field flux, an $\mathscr{E}$ will be produced, which produces a current, which is in a direction so as to *increase* the field flux, in opposition to that original change. The principal is fairly simple, but applying Lenz's Law can be tricky because it takes a few steps to get from one end to the other in the logic. You can think of this effect as a sort of electrical momentum: the system wants to keep going in the whatever state it finds itself and resists changes to that state.

Examples



Bar Magnet Through Loop

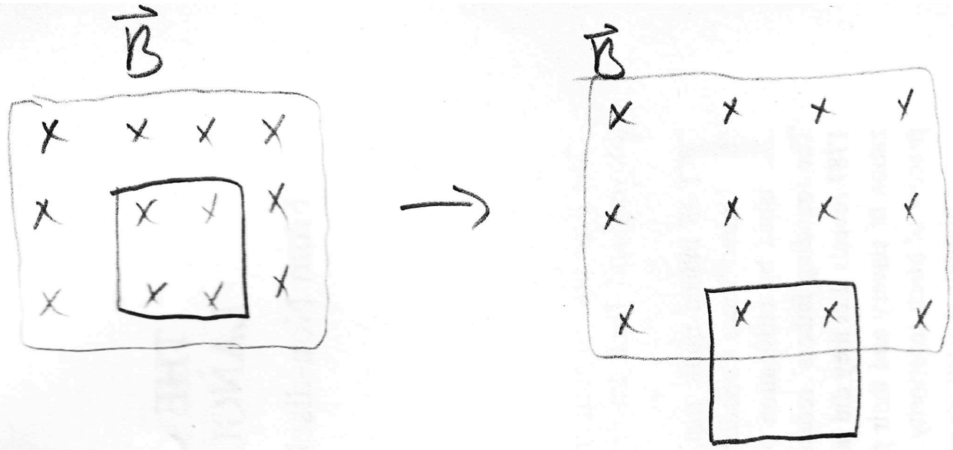


Ramped-Up Field

10.1.4 Change the Circuit Loop Instead

There are a couple of ways we can modify the circuit loop to change the magnetic flux and produce an $\mathscr{E}$. First, we can move the circuit through an existing field. If the field is uniform, then there will be no change in flux and no current will flow, but otherwise there will be some effect. Second, we can change the orientation of the circuit loop within a field. This will change all of the dot products $\vec{B} \cdot d\vec{A}$ without actually changing or relocating either field or circuit. Thirdly, we can physically modify the loop itself by either deforming it to change the shape, or constructing an adjustable circuit whose physical size is alterable.

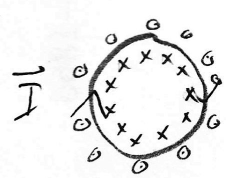
Examples



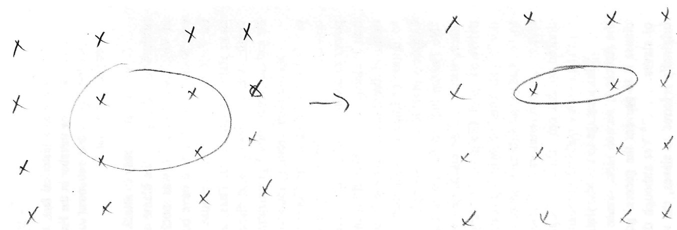
$$\frac{d\Phi_B}{dt} \simeq \odot$$

$\vec{I}$ produces Φ_B

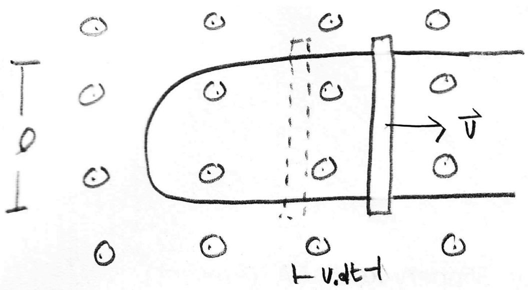
$\otimes$



Moving Circuit Through Field



Deformed Wire Loop



$$\vec{v} = v_0 \, dt$$

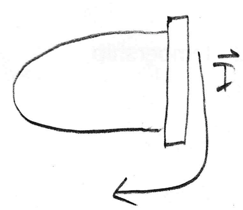
$$\frac{d\Phi_B}{dt} = B_0 \frac{dA}{dt}$$

$$dA = l \, v_0 \, dt$$

$$\frac{dA}{dt} = l v_0$$

$$\frac{d\Phi_B}{dt} = B_0 l v_0$$

$$\mathcal{E} = -B_0 l v_0$$



Rolling/Sliding Bar Circuit

10.2 Uses

There are tons of applications of applications of magnetic induction in modern technology, but we are just going to discuss two that are particularly direct manifestations of important aspects of magnetic induction. We are starting to see that despite its simple formulation, the fact that there are many different ways in which magnetic induction can arise means that there is a lot of variety in the final results.

10.2.1 Electric Generators

Most electric generators operate on the same basic principal: some source of energy is converted into mechanical rotation which is used to drive an axle. This can be as direct as a waterwheel under a waterfall or as indirect as nuclear energy heating a liquid which heats water which turns to steam which runs a turbine, and probably even more so. The axle is then attached to a collection of wire loops sitting in a magnetic field. As the axle rotates the loops, the flux through them changes due to their orientation and a current is produced. The energy needed for this process comes from the mechanical driver, not the magnet, so a permanent magnet can be used in a single generator for a long time. The simplest example produces an AC current, but it is possible to construct a DC generator with a similar design.

$\mathscr{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\int \vec{B}\cdot d\vec{A} = -\frac{d}{dt}\left(BA\cos\theta\right)$

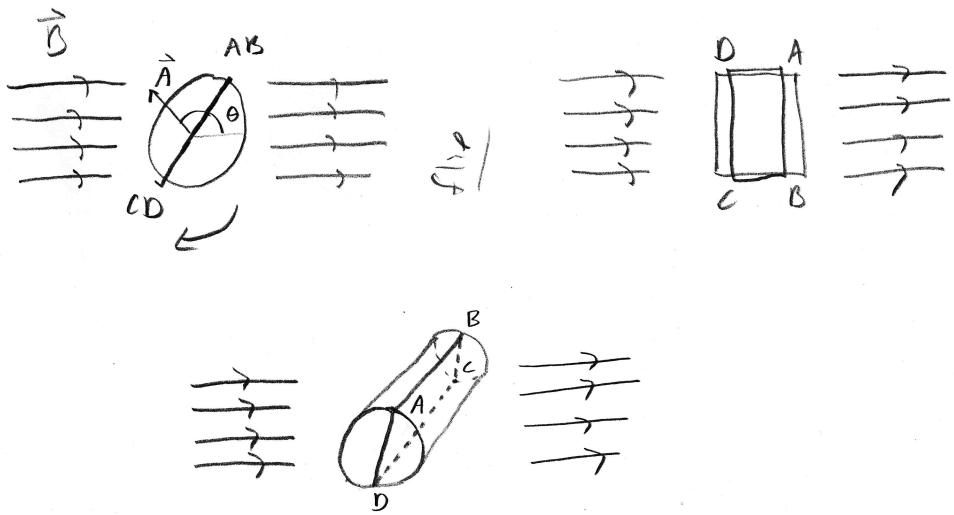
Where we have used the typical definitions. If we assume a constant field and loop area, the only thing that changes is the angle the loop makes with the field,

$\mathscr{E} = -BA\frac{d}{dt}\cos\omega t = BA\omega\sin\omega t$

where ω is the angular frequency.

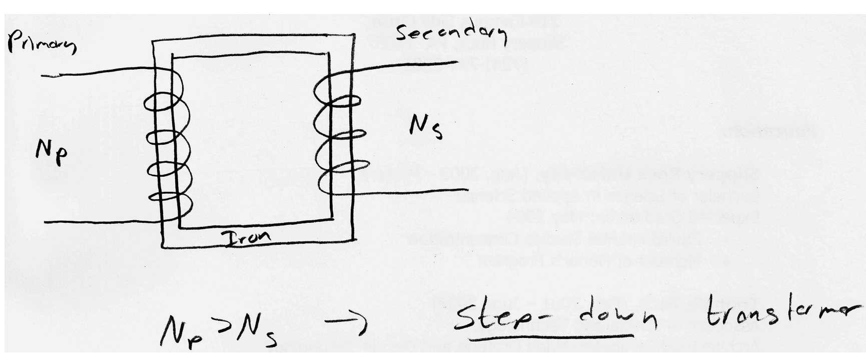
We can throw in another N and define a peak $\mathscr{E}$ for a generator with multiple loops:

$\mathscr{E} = BAN\omega\cos\omega t = \mathscr{E}_0\sin\omega t$



10.2.2 Electric Transformers

Earlier we discussed a system known as a solenoid. A solenoid is just a cylinder with a wire wrapped around it many times such that a very large and uniform magnetic field is produced on the inside. We won't discuss the details, but the strength of a solenoid can be great;y enhanced and extended if we fill it with a ferromagnetic material like iron. If we do this, and use the iron to connect two separate solenoids, we can actually transmit the current from one circuit to another without a conductive connection. With a little cleverness, we can even use this system to change the voltage in a circuit without losing (much) power. The resulting device is called a transformer.



First, recall the formula for induced $\mathscr{E}$ from a set of coils (solenoid):

$\mathscr{E}_{induced} = N\frac{d\Phi_B}{dt}$

Now, if I have 2 coils intermeshed, or around the same ferromagnetic core, we will have two separate solenoids with the same changing magnetic flux:

$V_P = N_P\frac{d\Phi_B}{dt} \longrightarrow \frac{V_P}{N_P} = \frac{d\Phi_B}{dt}$

$V_S = N_S\frac{d\Phi_B}{dt} \longrightarrow \frac{V_S}{N_S} = \frac{d\Phi_B}{dt}$

Putting these together,

$\frac{V_P}{N_P} = \frac{V_S}{N_S} \longrightarrow \frac{V_S}{V_P} = \frac{N_S}{N_P}$

We find that the voltage applied to the secondary coil by the primary through induction can be controlled just by adjusting the winding of the coils! So, if you want a higher voltage out than you put in (step-up transformer) you just make sure to wrap more coils of the secondary wire than the primary, in the proper ration. Likewise if you want to reduce the voltage (step-down), wrap fewer coils of the secondary wire.

Since energy must be conserved (we assume that the transformer is perfectly efficient and in reality they are usually very good), the current has to go down (up) if the voltage is stepped up (down).

$P = IV$

$\frac{I_S}{I_P} = \frac{N_P}{N_S}$

10.3 Faraday's Law with Magnetic Flux: Electric Fields Again!

We went through a series of logical steps and references to experimental results to arrive at the conclusion that changing magnetic fields create a current, and casually equated this with generating an $\mathscr{E}$. We then used that assumption in a variety of applications such as the electrical transformer. However, the jump from current to $\mathscr{E}$, while entirely correct and in fact logically necessary, hides a profound fact. An $\mathscr{E}$ is just a potential difference. A potential difference is an *electrical* potential difference, and necessarily *has* associated with it an *electric field* $\vec{E}$. Therefore, what Faraday's Law really says is that a changing magnetic flux induces an electric field! Remember from our study of potential difference that

$V = \mathscr{E} = \oint \vec{E} \cdot d\vec{\ell}$

Put this together with the form of Faraday's Law we earlier stated and we find that

$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi_B}{dt}$ **Faraday's Law, General Form**

Going one step further,

$\oint \vec{E} \cdot d\vec{\ell} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{A}.$

We use the first form for practical applications, but this second equation makes very explicit that this is a relationship between electric fields and magnetic fields. I've been saying and hinting there was a connection since day one, and here it is. Static electric and magnetic fields are unrelated, but as soon as things start changing and moving the two become closely entwined.