12: Maxwell's Equations

July 22, 2008

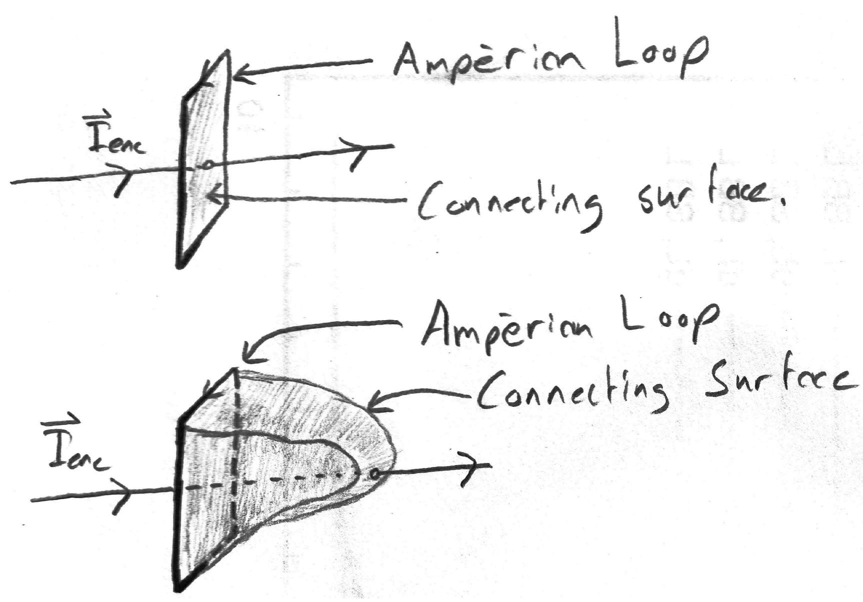
This is when we finally get to bring all of electricity and magnetism together. So far we've been learning about specific rules for this that and the other. Slowly, we've developed what seems like a hodgepodge of laws and rules that relate electric and magnetic phenomenon in a way that has some obvious symmetries (ways in which electric phenomena are similar to magnetic ones), but there are also obvious differences. Historically, the situation was even more unclear. The concepts of fields were not commonly used at the time when the state of the art understanding of the subject was roughly on par with what we have learned so far. It was only with the work of James Clerk Maxwell that it became clear that fields were the proper way to think about electricity and magnetism. Before this time, the symmetries that we've noticed along the way were far from obvious precisely because of the lack of the field formalism. The physics was the same, but because it was described differently it was harder to see what was really going on. Maxwell took the whole mess of behaviors we learned about, added some modifications, and packaged it together with a single clean, simple description which we have come to call "Maxwell's Equations". We've actually already seen almost all of Maxwell's equations because that is simply how the physics is expressed in the modern day, but we haven't seen how they fit together.

12.1 Changing Electric Fields

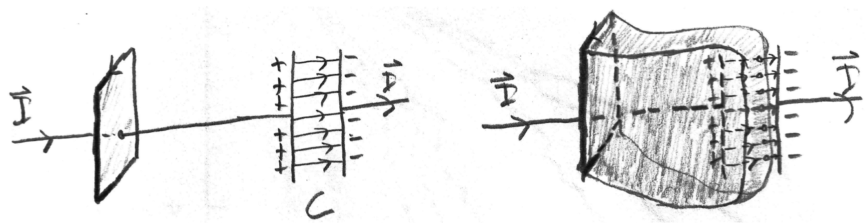
12.1.1 Setup

Faraday's Law tells us that a changing magnetic field induces an $\mathscr{E}$. Since an $\mathscr{E}$ is a potential difference, we realized that it must be associated with some electric field, meaning that a changing magnetic field really produces an electric field. One of Maxwell's new predictions was that, in a symmetry with this interpretation of Faraday's Law, a changing electric field will also induce a magnetic field. We can make a plausibility argument to support this idea by using a capacitor. I call this a plausibility argument rather than a proof, but we are going to accept the result as true. Experiment confirms the result, and at the end of the day that's all that really matters. It helps that the result can in fact be derived more generally, but its the experimental confirmation that should put our minds at ease.

Our argument starts with a single wire carrying current. We place an ampèrian loop around this wire to calculate the magnetic field, as usual. When we use ampère's law, the enclosed current we are talking about is actually the current which crosses a surface bounded by the closed ampèrian loop. When we first introduced Ampère's Law, I told you that this surface doesn't matter because they all give the same result, and so we've always just used nice flat surfaces. But, if we take that comment about the surface not mattering seriously, then presumably I can use a much-deformed surface like the one pictured.



In the case of a current carrying wire, this doesn't change anything. As expected, the wire still intersects the surface and I_{enc} is the same I as in the wire. But if we now introduce a parallel plate capacitor, we can get a stranger result. Lets draw 2 different possible surfaces for the same situation. First, the standard flat surface that will just cut through the wire. Second, a deformed surface which passes between the plates of the capacitor and comes back, without touching the wire. The first surface intersects total current I, while the second surface intersects no current.



What is going on here? This would seem to indicate that the magnetic field around our ampèrian loop depends on which surface we use to connect it, which would render the entire method useless! Luckily, we cansave Ampère's Law by modifying it a bit. Maxwell proposed that a term be added to the $\mu_0 I_{enc}$ side of Ampère's law to allow it to work in this (and, it turns out, many other) case. To figure out what that term is, consider under what circumstances the currant form fails for this configuration. The only time a discrepancy arises is if there is a current in the wire. Since the current can't pass through the capacitor, it corresponds to either a charging or discharging of the capacitor. Charging and discharging a capacitor implies that the charge on the capacitor must be changing. If the charge is changing, the electric field between the plates must likewise be changing. So, the surface of our loop intersects either a current, or a changing electric field. A surface intersecting a field defines a flux, and a surface intersecting a changing field defines a changing flux. We know that when a magnetic flux changes there is an induced electric field (the $\mathscr{E}$ of Faraday's Law implies an electric field, remember). Now we find that if a changing electric flux induces a magnetic field, Ampère's Law might not be broken after all. This recognition increases the symmetry between electrical and magnetic fields and brings us closer to Maxwell's Equations, but we first need to find precisely what the magnetic field induced by a changing electric field must be to satisfy our capacitor example.

12.1.2 Derivation

Our discussion above leads us to expect that we are going to somehow use the changing electric flux between the capacitor plates to take the place of the current which would otherwise intersect our surface. We can accomplish this by first starting with the charge on a capacitor,

$Q = CV = \left(\epsilon_0 \frac{A}{\ell}\right)(E\ell) = \epsilon_0 AE = \epsilon_0 \Phi_E.$

We use this form because it has an equivalence between charge and electric field. Our goal here is to relate current $\left(\frac{dq}{dt}\right)$ and changing electric field $\left(\frac{dE}{dt}\right)$ or flux $\left(\frac{d\Phi_E}{dt}\right)$. So, take the derivative on both sides to get

$\frac{dq}{dt} = \epsilon_0 \frac{d\Phi_E}{dt}.$

This can be written more helpfully in our context as

$I = \epsilon_0 \frac{d\Phi_E}{dt}$

What we have directly calculated here is that the current flowing into/out of a capacitor plate is in fact proportional to the derivative of the electrical flux. However, in the context of our earlier discussion we are going to interpret this $\epsilon_0 \frac{d\Phi_E}{dt}$ as being "like" a current, in that we stick it into Ampère's Law as we would a current:

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 \left(I_{enc} + \epsilon_0 \frac{d\Phi_E}{dt}\right).$ *Ampère's Law with Maxwell Correction*

We've just added this term in as tho it were a contribution to the enclosed current. We calculated above that for a parallel plate capacitor, it is in fact equal to the current along the wire, such that if we use the above in place of our old Ampère's Law, we get consistent results. It can in fact be shown, and experimentally verified, that this result is general.

Notice as a curiosity that the Maxwell correction term,

$\mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$

has the product $\mu_0 \epsilon_0$ in it. This is the first time we have seen both together, explicitly bringing magnetic and electric phenomena together. We will find later that this combination has a very surprising and important meaning, so watch out for it.

Apmère through Capacitor: See Worked Examples

Because of a since-discarded theoretical interpretation of Maxwell's correction to Ampère's Law, the new term is sometimes referred to as a *displacement current*. This name is useful in that it reminds us that the changing flux is acting the part of a current in some ways, but it is misleading because the changing flux is by no means an actual current. Treating it as such in all ways would get us into trouble, so it is safest to avoid referring to it this way. I only mention the name because it is sometimes used in the text and I don't want to cause confusion through omission. Try not to think of the new term as a current in general, but use it as one *only when inducing magnetic fields.*

12.2 Gauss's Law and Maxwell's Equations

I've mentioned a few times that Gauss's Law applied to magnetic fields doesn't tell us much interesting. The fact that there are no magnetic monopoles means that there can be no net flux into or out of a closed surface, and so we always find

$\oint \vec{B} \cdot d\vec{A} = 0.$ *Gauss's Law for Magnetism*

We say that this doesn't tell us anything interesting because we already *knew* that there weren't any magnetic dipoles. However, by including this equation, Maxwell expressed this fact precisely and as a symmetric part of the definition of electric and magnetic fields. Its now "in the math" rather than being an extra assumption we make explcitly all the time. On a theoretical level, having this kind of symmetry enables us to do a whole host of things that would otherwise be unclear.

Now that we've officially defined Gauss's Law, we have all the pieces that make up Maxwell's equations (in a vacuum):

$$
\begin{aligned}
\oint \vec{E} \cdot d\vec{A} &= \frac{Q}{\epsilon_0} \\
\oint \vec{B} \cdot d\vec{A} &= 0 \\
\oint \vec{E} \cdot d\vec{\ell} &= -\frac{d\Phi_B}{dt} \\
\oint \vec{B} \cdot d\vec{\ell} &= \mu_0 I_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}
\end{aligned}
$$

That's it! All of electricity and magnetism in the absence of materials is contained in these 4 equations. Everything we've learned so far is right there.

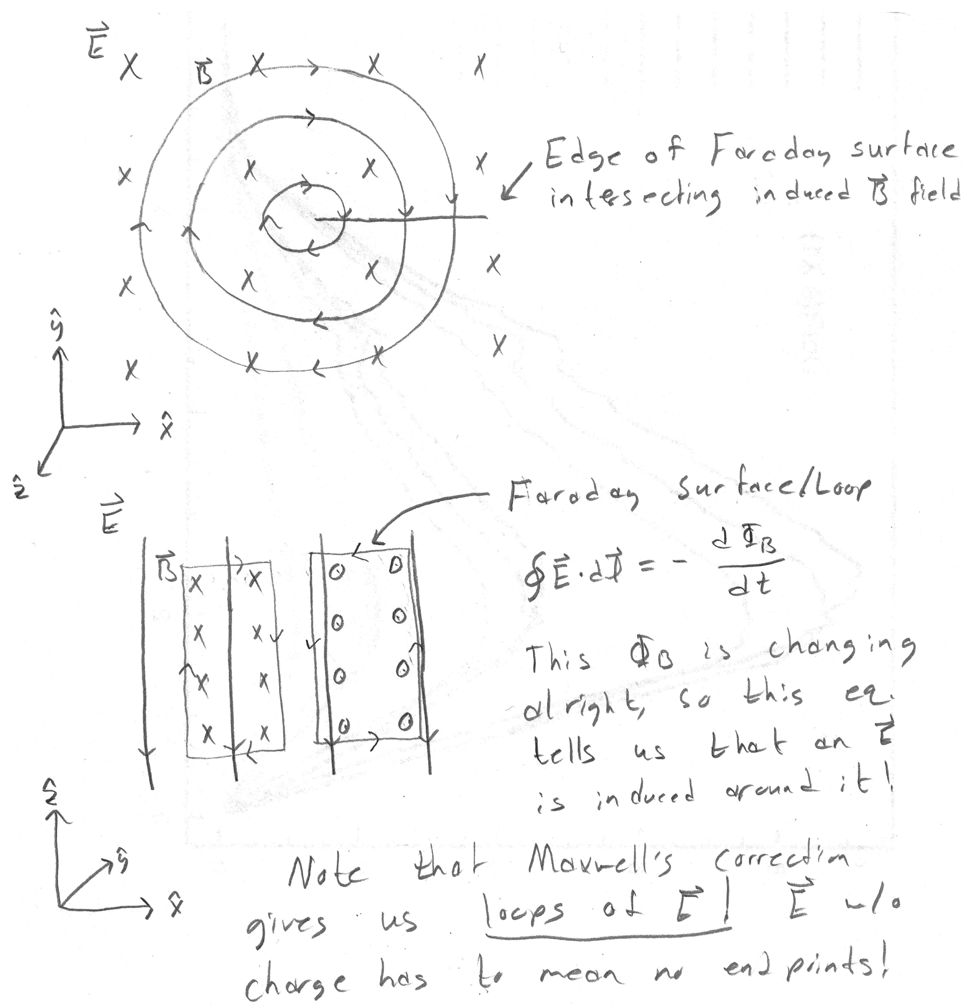
Notice the similarities and differences. Electric fields have a non-zero surface integral, but magnetic fields do not. Likewise, in the absence of changing fields the magnetic field has a non-zero loop integral, while the electric field does not. Both fields can be produced by a change in the other, differing only in the constants of proportionality that appear (and that is purely a matter of unit system choice and convention).

Maxwell's equations are some of the most fundamental equations in physics. This is a very significant piece of physics and getting to this point in your understanding is no mean feat. This essentially marks the beginning of the modern era of physics. The seeds of special relativity are within these equations, as we will see later. In addition, the study of the consequences of these equations led directly to the first contradictions that gave birth to quantum mechanics. Of course, it took about half a century for Einstein to produce Special Relativity and the first forays into quantum theory were at about the same time. These major shifts were not trivial, but Maxwell set the stage that forced the questions ultimately leading to what we now term "Modern Physics". Another way of saying this is that Maxwell's equations are the last piece of "Classical" physics.

The upshot of all of this is that I can now refer to "electromagnetism" instead of "electricity and magnetism" because we've now officially made it one big collection of interwoven stuff.

12.3 Production of Electromagnetic Waves

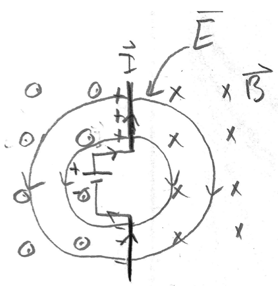
As usual, introducing a new effect into our theory of electromagnetism has unanticipated consequences. Keeping in mind Maxwell's correction to Ampère's Law, lets consider the case of a changing electric field more careful. Imagine that we have a large capacitor which generates a very strong magnetic field. Imagine also that this field is changing rapidly so that the derivative of the field is large.



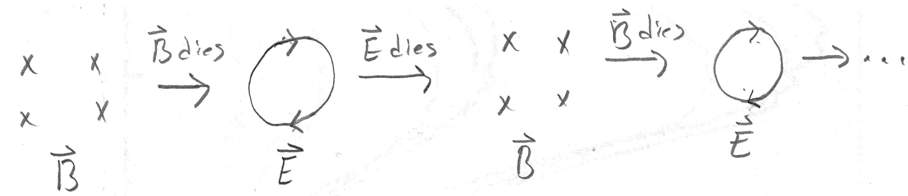
So now we have a changing $\vec{E}$, inducing a changing $\vec{B}$, which in turn induces another $\vec{E}$, which is naturally changing, so we end up with another $\vec{B}$ field... This never ends. It gets weaker w/ each step, but it goes on and on.

The ultimate effect of this infinite nesting of induced fields is the creation of *electromagnetic waves.* Electromagnetic waves, we will see, are self-propagating electromagnetic fields traveling through space. The electrical component induces a magnetic component, which induces an electric component, and so on and so forth. Through this process, the fields travel.

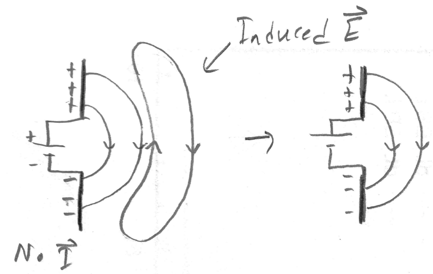
To see how this works, lets look at a very basic antenna. An antenna is just a collection of conductors designed to maximize the creation and/or reception of electromagnetic waves. The simple example shown is actually very similar to a capacitor: the $\mathscr{E}$ produces a charge difference between the two legs of the antenna much like the two plates of a parallel plate capacitor.



The charge distribution creates an $\vec{E}$ field. The current creates the $\vec{B}$ field.
In this case the $\vec{B}$ will be temporary because current can only flow briefly. <u>But</u> a dying $\vec{B}$ field means a changing flux



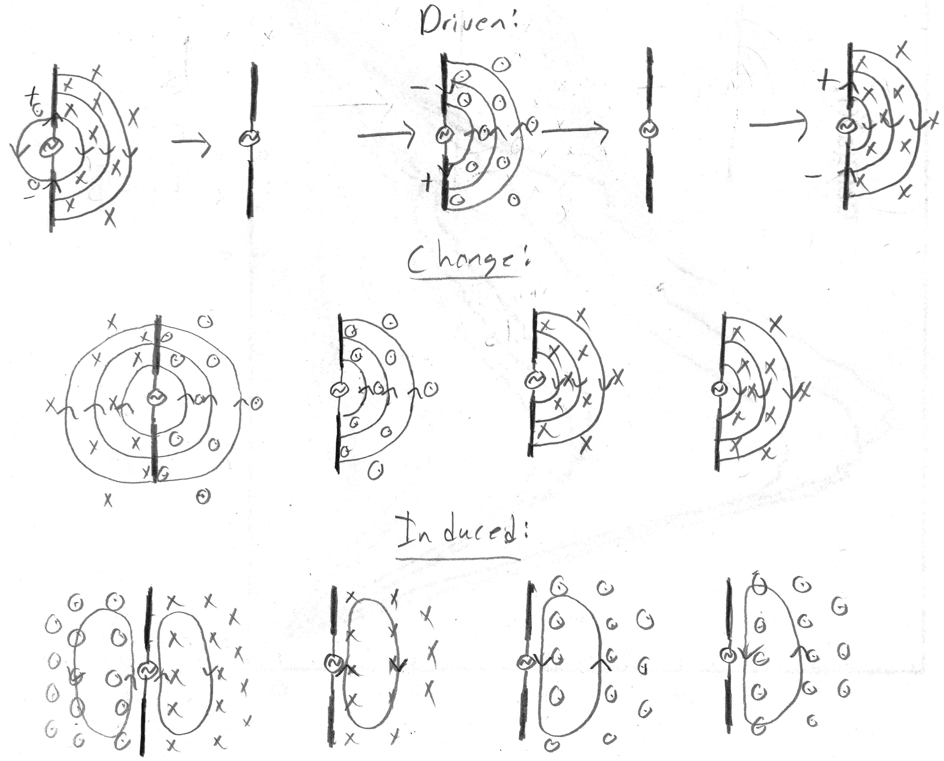
Three points to make here. First, how do we find the direction for each step in this chain? Use Lenz's Law. This is tricky, so remember that each step should produce fields like its precursor. So the 1st $\vec{E}$ loop should produce a $\vec{B}$ field into the page, as it does.



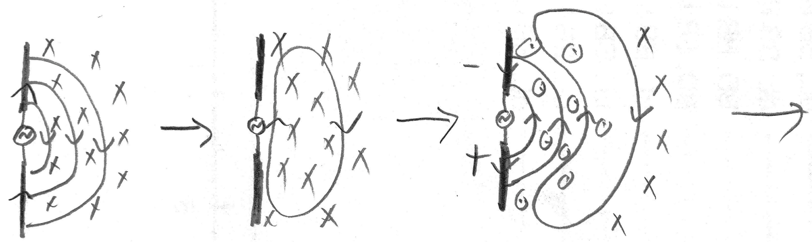
The induced $\vec{E}$ conflicts w/ the charge-based field the antenna is creating, and so dies off right away.

With a direct current source, not much happens. When the initial $\vec{B}$ field dies and induces an $\vec{E}$ field, that induced field is counter to the $\vec{E}$ field from the charges in the antenna and it gets killed off (mostly, good enough for now). However, if we attach an AC voltage source, then the charges on the antenna will switch back and forth. If the timing is correct, the electric field will have reversed direction just as the induced $\vec{E}$ reaches max strength. The two will then reinforce, rather than canceling. As the voltage oscillates, the induced fields will oscillate in time, and in fact spread out in space.

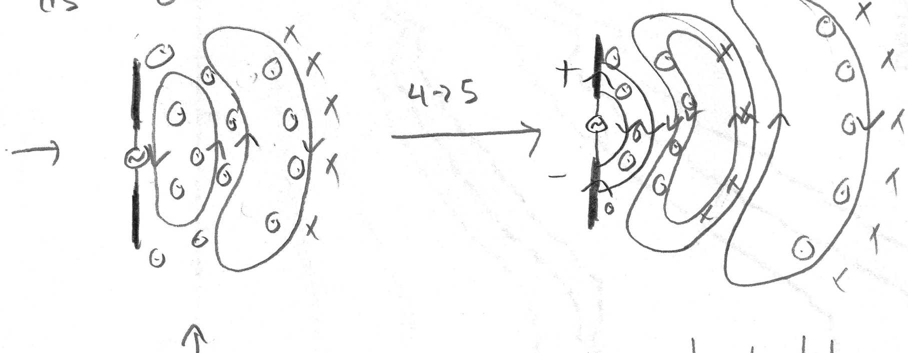
For the more complex AC case, consider w/o induced fields first.



Total = Driven + Induced



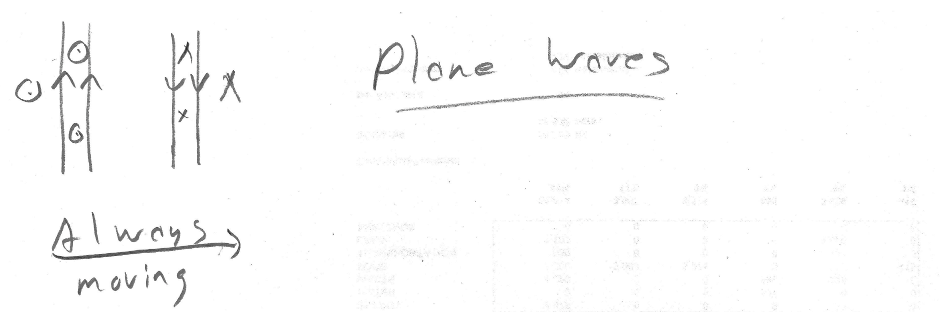
Now between the 3rd + forth step we need to add to this 3rd step both the change in the driven fields, as well as the induced fields for this step.



Remember that field lines don't like to be close, they repel, spread out! This pushes the electric "bubbles" from previous steps out from the antenna.

So an AC current in a simple antenna creates a set of electric and magnetic fields which are constantly traveling outward. These fields, we notice, are no longer connected to or supported by the antenna: they are self-supporting and self-propagating. In addition, the fields will continually oscillate around one another, each type of field inducing the other back and forth. This oscillation is a property of waves, and we call our new fields electromagnetic waves at last. Electromagnetic waves are very special because of their property of self-propagation. Unlike all of the waves known at the time of Maxwell, EM (electromagnetic) waves do not require a "medium" to "wave" or pass through. Sound waves must pass through a material because they are "waving" the density fluctuations of that medium. Water waves must obviously propagate through water because they are waving the water. EM waves, by contrast, are just fields which wave themselves. This strange fact led to enormous confusion in their early study.

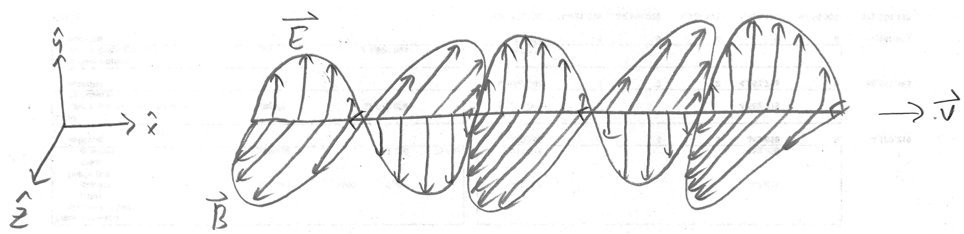
As you can see from our pictorial derivation of these waves, they can look quite complicated near the antenna. This is called "near field" radiation, and we won't do much with it. Far more interesting is the simpler "far field" radiation. At large distances, the EM waves settle down into a uniform and simple shape.



This far field radiation forms into very large flat sheets of fields, which we call *plane waves* because they are shaped like a plane at large distances. We won't calculate it here, but it can be shown that the fields in an EM wave fall off with distance as $\frac{1}{r}$. This should be compared with the more rapid dropoff of static fields from charges and currents of $\frac{1}{r^2}$. The coupled wave nature of the fields keeps them going further, essentially.

Inspecting our figures, we can pick up a few generally true facts about the fields in an EM wave. First: the $\vec{E}$ and $\vec{B}$ fields are always perpendicular to each other. Second: fields of each type will alternate directions. By this we mean that if we look at successive field lines, they will switch between, e.g., up and down with a regular period. You can have a region of up, but it will give way to a region of down, and back. This is the property that makes these fields *waves* rather than just a complicated jumble. A wave must return to its initial state. As the fields alternate from up to down and back again, they repeatedly return to their initial state.

Another true fact about these waves is that they travel with a velocity $\vec{v}$ which must be perpendicular to both $\vec{E}$ and $\vec{B}$. The right hand rule can give us this direction, as $\vec{v}$ is proportional to $\vec{E} \times \vec{B}$.



One last bit of information about these waves to note for now: we produced out wave by accelerating charges back and forth in an antenna. If we hadn't forced the charges back and forth, none of the field reinforcement would have taken place and a wave would not have formed. This hints at another general result about waves, which is that *electromagnetic waves are produced by accelerated electric charges*. Any time an electric charge is accelerated, there will be EM waves produced somewhere, they they may be very weak.