2: Electric Field July 6, 2008

2.1 Description

2.1.1 Physical significance of a field

The concept of a "field" is where electricity and magnetism start to depart from our intuitive sense of physics much more than does mechanics. Originally, the idea of electric (and magnetic) fields were developed as a mathematical tool and were not considered to have any physical meaning. Now we know, however, that fields are in fact real "objects". I put objects in quotes because I use the word in an abstract sense and they are nothing like what you think of as a physical object.

Coulomb's law as given above describes a force which acts instantaneously from afar. The buzzword for this is "action at a distance" (OK, buzz-phrase). Your intuition tells you that this is weird, somehow. Telekinesis is action-at-a-distance, as are things like levitation. We just don't expect objects to be able to influence each other without some sort of contact or communication. In the modern age we are somewhat more blasé about this because we are so used to radio signals and the like making things happen all over. But we still know that even tho we can't see them, there are real things (radio signals) traveling through the air to make stuff happen. Our intuition on this matter is in fact correct and remains a fundamental principal even in today's most advanced, bizarre theories. So, how do we reconcile this strangeness with Coulomb's Law?

Of course, the answer is the electric field. The electric field is physical and extends out from charges, transmitting the electric force between charges. Sounds strange? It is, a bit, but don't worry. You can use the electric field to solve problems without being entirely comfortable with the idea of it as a physical object.

2.1.2 Mathematical representation of a field

Lets take a step back again. There are two kinds of answer to the question "What is a field?" I've given you a description of the function the electric field serves as a physical object. The other answer is a description of the abstract math that describes that field and how it behaves. A field is an object which has a value at every point in a space. So, for instance, temperature is a field. If you check `weather.com`, you can find maps which have the temperature for every point on the map. Every point in the air in this room has a temperature. Every little bit of water in the ocean has a temperature. And so on. In this example, we are talking about a *scalar* field. That is, the value that exists at each point has no direction, just a size. This is why you can represent it with, say, a colorized map. An even more familiar example for this is elevation. The height above sea level of the ground at each point is also a scalar field, and it is represented on topological maps which either have contour lines or color gradients. You can thus look at these maps and immediately get a feel for the field it represents. We never talk about a topological map being a picture of an elevation field, but that's exactly what it is. When we write a field, we need to remember what it is: a different *value* at every *point* in space. So, if I want to talk about the temperature field, I need a notation that includes all of this, e.g. $T(lat, long) = t$, or for a specific point, $T(43^oN, 76^oW) = 83^oF$. Sometimes, depending on context, you will see an abbreviated notation where the field is just referred to as e.g. T. Note that even tho the fields in these two examples don't have directions, they do have units. The temperature field has units of, say, $[T] = {}^oF$, while the elevation(altitude) field has $[A] = m$.

A field can have any sort of value, with any sort of units (even no units). The electric field is a *vector* field, which means that it defines a direction at every point in space, as well as a magnitude (the strength of the field). A more familiar example of a vector field is air or water currents. `http://www.sat-ocean.com/`has a great animation of the vector field of the surface currents in the Gulf of Mexico. A single image is in Fig[2.1].

This particular map shows the magnitude of the vector (wind speed) as a color gradient, and the direction is shown with a grid of arrows. This is a

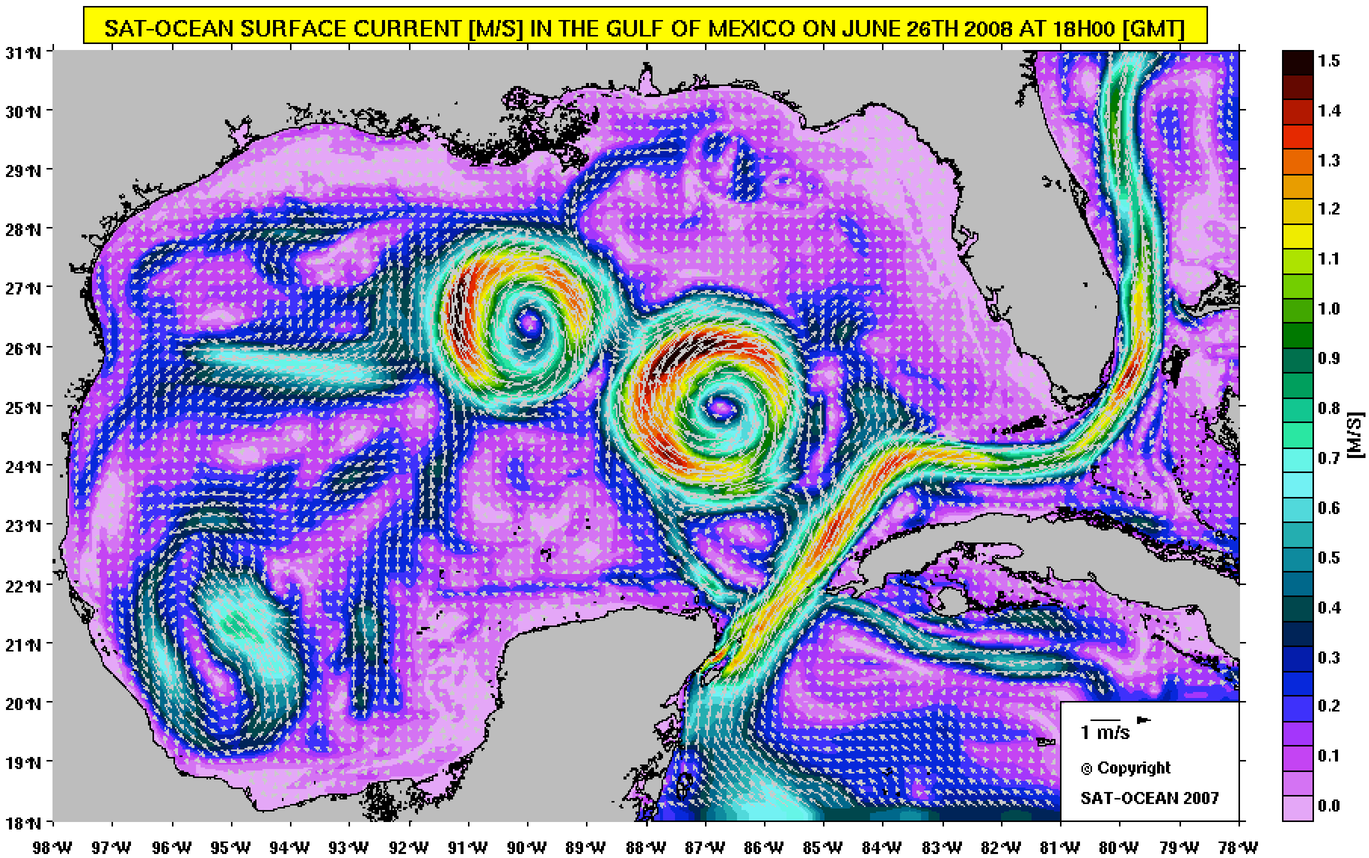


Figure 2.1: A vector field: surface currents in the Gulf of Mexico

fairly typical choice for this sort of application, tho often the arrows will be of different lengths to show the vector magnitude. ($[windspeed] = m/s$) You can do a Google search for wind or current maps to see a bunch of examples.

The electric field is generally represented differently than this. While the same information is contained (a direction and magnitude at every point), the electric field has a few special properties that allow us to use a more meaningful and suggestive depiction.

The electric field is defined as follows. Suppose we have a large charge, q_1. This charge exerts a force on the much smaller *test charge* q_2. A test charge is a charge small enough that it does not affect the charges which create the field it is "testing". Coulomb's law gives:

$$\vec{F}_{12} = -k\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12} \qquad \left[\vec{F}_{12}\right] = N.$$

The electric field is represented by $\vec{E}_1$ (the electric vector field due to charge q_1) and is defined has having the value

$$\vec{E}_1 \equiv \frac{\vec{F}_{12}}{q_2} = -k\frac{q_1}{r_{12}^2}\hat{r}_{12} \qquad \left[\vec{E}_1\right] = N/C.$$

This is the value of the field at the point where q_2 sits. Here I used the abbreviated notation mentioned above. But if I want a value for $\vec{E}_1$, I need to specify a point: $\vec{E}_1\left(\vec{r}_{12}\right) = -k\frac{q_1}{r_{12}^2}\hat{r}_{12}$. This makes it explicit *where* the field has this value: it is at the point $\hat{r}_{12}$ where we consider the charge q_1 to be the origin. (A note on vectors and notation: I could have been more explicit and written r_{12}^2 as $\vec{r}_{12}\cdot\vec{r}_{12}$. This is the *scalar product* or *dot product* of $\vec{r}_{12}$ with itself. Because dot products depend on the angle between two vectors, and $\vec{r}_{12}$ obviously points in the same direction as itself: $\vec{r}_{12}\cdot\vec{r}_{12} = |\vec{r}_{12}\times\vec{r}_{12}| = r_{12}^2$ where we use the *convention* that $r_{12} = |\vec{r}_{12}|$ is the magnitude of $\vec{r}_{12}$.

Don't let the fact that the field appears to differ from the force only by a factor of one of the charges: it is a powerful concept which makes most of our study of electricity and magnetism possible, and we'll see why soon.

2.1.3 Visualization of a field

OK, so lets think about what we've learned. A few things these equations tell us are that electric fields point *away* from positive charges and *towards* negative charges, while the *magnitude* of the field drops as $1/r_{12}^2$: it gets smaller pretty quickly as you move away. If we plot this on a grid as we did for the wind vector field, we get something like Fig[2.2].

However, this isn't the preferred visualization. Instead, we introduce the idea of a *field line*. These are also sometimes called *lines of force*, but we'll stick with field lines. A field line is somewhat like a joining of a series of the field arrows from Fig[2.2], but instead of drawing the arrows at set spots on a grid, you draw each arrow where the last one leaves off, tail to head. If you make your arrows short enough, you start to get nice smooth curves rather than a collection of jumble straight lines. Once you've made the arrows

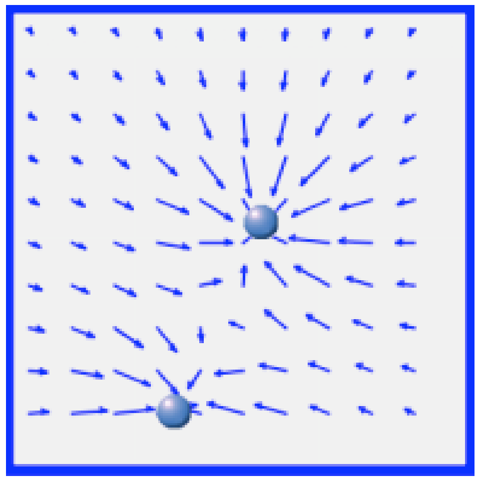


Figure 2.2: Electric field of two negative charges.

short enough that they form smooth lines, however, you can no-longer tell how long they are: how can you see how strong the field is at a given point? You keep track of the field direction by putting one arrow on each field line, since the field can't change direction along a single line, but that doesn't do you any good for magnitude.

This problem is one reason why field lines aren't used for generic vector fields. In the case of electrostatics, however, we can take advantage of the fact that the field must always point away from positive charges and towards negative ones. So we start at a charge, pick a number of field lines we want per unit charge (it doesn't matter as long as you are consistent within a problem), and draw those lines coming out of the charge. For a single point charge, the lines will extend straight out from the charge, always pointing out but getting further and further apart. This spreading of the field lines is what tells us how strong the field is at any given point: the closer together the lines, the stronger the electric field. You can see what this looks like for a few simple charge configurations in Fig[2.3].

This sounds like a pretty arbitrary choice for how to display the vector field, doesn't it? Take a look at Fig[2.4]. Here two charges (of opposite sign) are causing a collection of metal (conductive) shavings to stick together (via

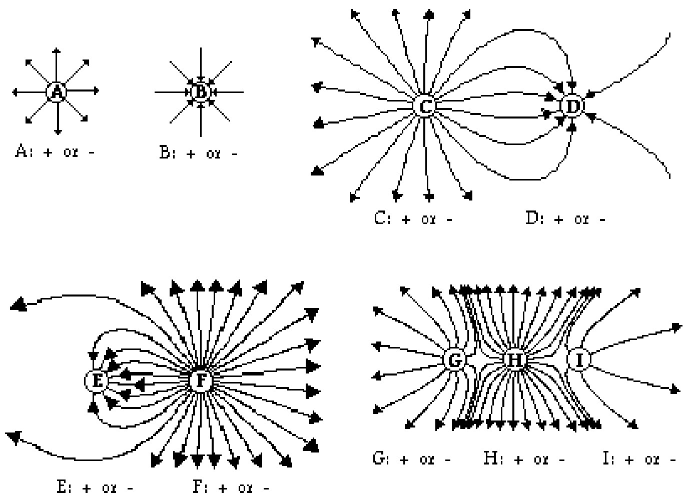


Figure 2.3: Field lines for simple charge configurations.

charge induction) and line up. In the process, they *actually draw field lines for us*! This sort of thing pops up all over the place, so the field line picture of the electric field starts to look a lot less arbitrary. It also turns out to be really handy for almost every application of electrostatics.

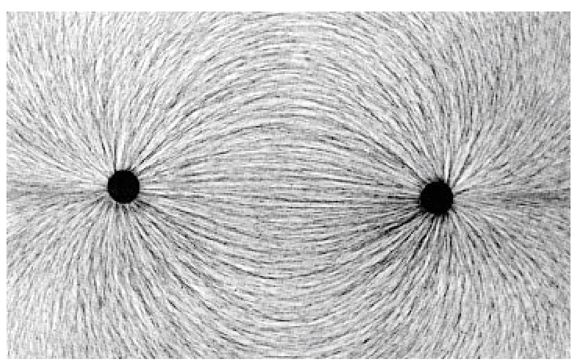


Figure 2.4: Electric field lines in metal shavings.

Demo

2.1.4 Interpretation and use of a field

So now you have a nice picture of a field. You have a series of field lines radiating out from positive charges, converging into negative charges, and labeled with arrows to make it clear which is which. What does this tell you? Well, looking at this picture, you instantly know where and how hard a test charge will get pushed/pulled if you drop it into your system. The field lines will give the direction with the arrows, naturally enough, and the relative strength of the force felt by the test charge is given by the density of lines around it. This is the same information you would get from Coulomb's Law, but most people find this visual form far easier to absorb in a qualitative fashion, which makes it many times easier to "sanity check" a result (decide if it makes sense or if there are clear errors). Also, the field picture allows you to more easily make qualitative predictions about the system.

When drawing a field diagram, there are a few "rules" to keep in mind:

- Field lines don't like each other: they will try and spread apart as much as possible. If you think of them as rubber bands (because they won't stretch arbitrarily) that repel each other, you'll have a good idea of how they will arrange themselves.

- Field lines must be anchored on at least one end by a charge. If you have net charge in your system, there will either be lines flowing in (net negative charge) or out (net positive charge) of your diagram. If your system has no net charge (equal positive and negative charges) then all lines will be anchored on both ends: leaving a positive charge and entering a negative charge.

- Field lines *cannot cross, ever*. Not only because they don't like each other, but because crossed lines is a fundamental contradiction. Remember that field lines are describing a vector field. That means that at each point, the field has a *single* vector value. If your field lines cross, then at that point where they cross, you are assigning 2 different directions to the same point.

- Field lines point in the direction of the force a *positive* charge would feel if placed in the field. A negative charge will feel a force in the opposite direction.
- Fun fact: when your hair becomes staticy (either because of a demo in physics class or a wool carpet or whatever), it will trace out the electric field lines leaving your body. If the charge is weak and your hair is long, the weight of your hair will overwhelm the effect, but if you look just at say your arm hair, even a small static buildup will cause it all to stick straight out. However, if you move near a metal object (conductor), the hair will try and point to it. This is because you are inducing a charge in the metal, which pulls the field lines towards it.

2.2 Application

2.2.1 Calculating things

The initial purpose of the electric field is that it makes it easier to calculate things. It serves other important functions (remember the bit at the beginning about how otherwise electrostatics would be action-at-a-distance), but practically, we want to calculate stuff. The first thing you can calculate from the electric field is the force felt by a charge q. Given an electric field $\vec{E}$, the force felt by q in $\vec{E}$ is given by

$\vec{F} = q\vec{E}$

This shouldn't be surprising, since we defined $\vec{E} \equiv \frac{\vec{F}}{q}$ earlier.

The electric field also makes it more convenient to use what is called the *principal of superposition.* The principal of superposition states that when you have multiple charges $q_{1,}q_2, q_3$, etc., each produces an electric field $\vec{E}_1, \vec{E}_2, \vec{E}_3$, etc. We don't have to deal with all of these different fields to solve a problem, however. We need only calculate the total field, $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \cdots$ once. Then, we can use that single field for any calculation we have left. The principal of superposition is incredibly powerful, and without it almost every non-trivial problem would be significantly to vastly more difficult. You need to remember and be aware of it for everything you do, because otherwise you will at best do extra work and at worst misunderstand a problem and get very incorrect results.

Examples

- Field near single charge.
- Field near 2 like charges.
- Field near 2 opposite charges (a *dipole*).
- Field near 3 like charges in a line.

2.2.2 Charges in a Field

The definition of a field was $\vec{E} = \frac{\vec{F}}{q}$. From this, it is easy to see that if you know the field in a region of space, and want to know what the force it feels is, just apply:

$\vec{F} = q\vec{E}$

This means that a + charge will be pushed in the direction of the field with a strength proportional to the charge's size, while a − charge will be pushed in the *opposite* direction from the field. This is clear enough when you are thinking about it, but because we often work with electrons without explicitly saying "negative charge" (this is particularly true later when we talk about currents and circuits), it is very easy to forget that electrons feel the force this way, and the arrows in our diagrams (for the field) often point in the opposite direction of the thing we want (force).

A charge will feel this force so long as it is within the field, and it will feel the force corresponding to the field *at the point where it currently lies.* This means that if a field isn't entirely uniform, or stops, the force felt will change whenever the charge moves to a region with a different electric field. This is one of the nice things about drawing field lines: you can picture how a charge will get pushed around in a system without having to think about each of the charges it interacts with.

Examples

- ▶ Uniform field: charge starts at rest.
- ▶ Uniform field: charge starts in perpendicular motion.
- ▶ 2 charges start at rest: one is much more massive.
- ▶ 2 charges start in relative motion: one is much more massive.

2.2.3 Electric Dipoles

Definition

A special system of charges is the combination of a positive and negative charge with equal magnitudes. We call this object a *dipole*, because it has two "poles". This object is the closest analogy in electrostatics to a magnet, which has two poles, North and South.

Dipoles are very important partially because of how often they show up in nature. For example, many molecules are "polar", which means that one side is slightly more positive and the opposite slightly more negative. This means that these molecules are (essentially) dipoles. The reason that polar fluids (like water) behave differently than non-polar ones (like oil) is due to this polarity.

Two things quantify a dipole: the magnitude of the charge, q, and the vector separating them, $\vec{\ell}$. Together, these give the *dipole moment,*

$$\vec{p} = q\vec{\ell} \qquad\qquad [\vec{p}] = Cm$$

$\vec{\ell}$ is the displacement vector from the negative to positive charge. Note that we can sometimes specify a dipole moment without bothering to give the q and $\vec{\ell}$ specifically. This is because once you get far enough away, it really doesn't matter in the slightest, and you can calculate everything from just the dipole moment itself.

Behavior in an External Field

Assume a uniform electric field $\vec{E} = E\hat{x}$ ($\hat{x}$ is the unit vector in the x direction), and a dipole with dipole moment $\vec{p}$ inside that field. The dipole consists of two charges $Q_1 = +q$, $Q_2 = -q$. Because the dipole has net 0 charge, there will be no net force. However, because the two charges are displaced from one another, they will experience a *torque*, τ. You should remember from mechanics that a torque is the product of the force applied and the length of the lever arm, $\tau = dF$. In our case there will be two forces $\vec{F}_+ = q\vec{E}$, $\vec{F}_- = -q\vec{E}$. $\vec{F}_+$ is of course the force felt by the positive charge, $\vec{F}_-$ by the negative. To find the lever arm, we need the distance from the *center of rotation* to the *charge*, measured *perpendicular to the force*. This is just from the definition of torque.

The lever arm in each case will come from the half of $\vec{\ell}$ extending from the center of the dipole to the charge in question. The orientation of the dipole is important, because the field doesn't necessarily act perpendicular to $\vec{\ell}$.

$d = \frac{\ell}{2}\sin(\theta)$ $\qquad [d] = m$

where θ is the angle between $\hat{x}$ and $\vec{\ell}$.

Since $\tau = dF$ ($[\tau] = mN$) and $F_+ = qE$, $F_- = -qE$,

$$\begin{aligned}
\tau &= qE\frac{\ell}{2}\sin(\theta) - qE\frac{\ell}{2}\left(-\sin(\theta+\pi)\right) \\
&= qE\frac{\ell}{2}\sin(\theta) + qE\frac{\ell}{2}\sin(\theta) \\
&= qE\ell\sin(\theta) = pE\sin(\theta)
\end{aligned}$$

This is a lot more straightforward if you just use the vector definition of torque:

$$\begin{aligned}
\vec{\tau} &= \vec{\tau}_+ + \vec{\tau}_- \\
&= \vec{p}_+ \times \vec{E} + \vec{p}_- \times \vec{E} \\
&= q\frac{\vec{\ell}}{2} \times \vec{E} + -q\frac{-\vec{\ell}}{2} \times \vec{E} \\
&= \vec{p} \times \vec{E}
\end{aligned}$$

$$[\vec{\tau}] = mC \times \frac{N}{C} = mN$$

Remember, the torque is an *axial* vector which points perpendicular to the plane of rotation (like an axle). Use the right-hand-rule to determine which perpendicular direction it should go:

1. Hold your thumb straight out from your hand.
2. Point your fingers, straight, in the direction of $\vec{p}$.
3. Curl your fingers in the direction of $\vec{E}$. Your fingers only curl one direction, you need to rotate your hand such that both step 2 and 3 are possible with the same hand orientation. This is the point!
4. After the rotation to make step 3 work, your thumb should now be pointing in the direction of $\vec{\tau}$.

These steps work for determining the direction of the vectors in any cross product (vector product). Your thumb will always be the result vector, straightened fingers always correspond to the first vector in the product, and you always curl your hand towards the second vector in the product.

If the dipole is free to spin, this torque will cause it to oscillate around the $\hat{x}$ direction. The initial position will determine how far from equilibrium the dipole will swing. If there is any sort of damping on the dipole rotation (as there inevitably is in a physical system), the dipole will settle down such that it points parallel to the external field.

Sometimes we will need to know how much *Work* it takes to rotate the dipole. We know that work is the force times the distance applied. Here we have a torque which will vary as the angle changes. You should remember from mechanics that the analogue of $force \times distance$ for a torque is $torque \times angle$ because the lever arm in the torque, combined with the angle of rotation, gives the distance traveled. Because of this variation, we need an integral to find the work:

$$W = \int_{\theta_i}^{\theta_f} \tau \left(-d\theta\right)$$

$$[W] = mN$$

θ_i is the initial angle, θ_f the final. Plugging in our expression for τ:

$$W = pE\int_{\theta_i}^{\theta_f} \sin(\theta)\,(-d\theta) = pE\cos(\theta)\Big|_{\theta_i}^{\theta_f} = pE\left(\cos(\theta_f) - \cos(\theta_i)\right)$$

Note the minus sign with $d\theta$. This is because the torque points in the direction of *decreasing* angle. The integral must match positive torque with the direction of integration if we are to integrate positive work done by the field. If $\theta_i > \theta_f$, this integral will give a positive result, which makes sense because the dipole is moving in the direction the force is pushing it. However, if $\theta_i < \theta_f$, then the movement is *against* the torque and the contribution will be negative. You can get the correct result by identifying the direction of integration that is going *with* the torque and building that into your differential element ($d\theta$) right off the bat. Then you can always integrate from the initial to the final position and the signs will work themselves out. It is, however, still *always a good idea* to sanity-check your result. Does it seem to give reasonable answers? Is it positive when you expect negative? Is there any way your expectation is wrong? Double-check if the result doesn't seem to make any sense.

Another interesting quantity of the dipole is its potential energy U. Dipole $\vec{p}$ has potential energy due to the torque the field applies to it, and is at a minimum when the dipole moment is parallel to the electric field (because the field has nowhere left to torque the dipole to). We don't have to set the minimum at $U = 0$, however. We are free to define the 0-point wherever we want, and the point that gives the simplest result is $\theta_i = \frac{\pi}{2}$. This is because U is related to W, and this choice of θ_i gives

$W = pE\cos(\theta_f) = \vec{p}\cdot\vec{E}$ $\qquad\qquad$ $[W] = [p]\,[E] = mC\frac{N}{C} = mN$

because the second term drops out since $\cos\left(\frac{\pi}{2}\right) = 0$.

Remember from mechanics that $\Delta U = -W$. Since θ_i is our starting point, that is where $\Delta U = 0$. Thus:

$\Delta U = -W = -\vec{p}\cdot\vec{E}$.

Field Produced by a Dipole

Of course, a dipole produces a field of its own. This field is just the superposition of the fields created by the two component charges, so we know how to calculate it.

$$\begin{aligned}\vec{E}_p &= \vec{E}_+ + \vec{E}_- \\ &= k\frac{q}{r_+^2}\hat{r}_+ + k\frac{-q}{r_-^2}\hat{r}_-\end{aligned}$$

Hold on! This isn't nearly as simple as it looks! Notice the subscripts I've put on the rs here. The vectors from the positive charge and negative charge aren't the same thing! Worrying about this complete solution isn't something we want to do at the moment, but we can get some interesting information out by taking advantage of the *symmetry* of the problem and restricting ourselves to only looking for part of the solution.

If we decide to only ask about points which lie on the plane equidistant between the two charges, life gets easier. Automatically, the magnitudes of $\vec{r}_+$ and $\vec{r}_-$ are the same. The unit vectors are still different, but we can deal with that, too. Notice that for any point on this plane, the field due to each charge will have radial components which *cancel exactly.* The positive charge creates an outward pointing field contribution, while the negative charge creates an equal contribution which is directed in. The z components, however, will reinforce one another (I have chosen my coordinates such that z is parallel to $\vec{p}$ and r points out radially along the plane between the two charges. The origin is at the center of the dipole.)

To find the $\hat{z}$ component of a vector, you take the dot product of the vector with $\hat{z}$.

$$
\begin{aligned}
\vec{E}_{+,z}(z=0) &= k\frac{q}{r_+^2}\hat{r}_+\cdot\hat{z} \\
&= k\frac{q}{r^2+\left(\frac{\ell}{2}\right)^2}\cos(\phi) \\
&= k\frac{q}{r^2+\left(\frac{\ell}{2}\right)^2}\left(\frac{\ell/2}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\right)
\end{aligned}
$$

Now we can use this to find the total field:

$$
\begin{aligned}
\vec{E}_p(z=0) &= k\frac{q}{r^2+\left(\frac{\ell}{2}\right)^2}\left(\frac{\ell/2}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\right)(-\hat{z}) - k\frac{q}{r^2+\left(\frac{\ell}{2}\right)^2}\left(\frac{\ell/2}{\sqrt{r^2+\left(\frac{\ell}{2}\right)^2}}\right)\hat{z} \\
&= -k\frac{q}{r^2+\frac{\ell^2}{4}}\left(\frac{\ell}{\sqrt{r^2+\frac{\ell^2}{4}}}\right)\hat{z} \\
&= -\frac{kq\ell}{\left(r^2+\ell^2/4\right)^{3/2}}\hat{z} \\
&= -k\frac{p}{r_\pm^3}\hat{z} \\
&\xrightarrow{r\gg\ell} -k\frac{p}{r^3}\hat{z}
\end{aligned}
$$

You may notice that I have a minus sign which the book doesn't. This isn't a disagreement: the text is only displaying the magnitude of the field.

We have found that the field due to a dipole drops as $\frac{1}{r^3}$ rather than $\frac{1}{r^2}$ as it does for a single charge. This makes sense: we knew that the field would have to drop off faster than for a single charge because the two should cancel each other out, but it isn't just 0 everywhere.

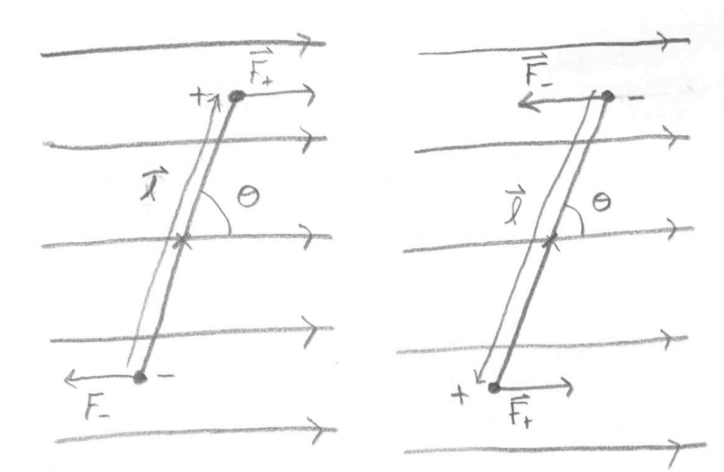
Clarification of the confusion in lecture.

1. I'm not sure what happened with the right-hand rule in class, but the discussion above is correct. I think I just switched $\vec{p}$ and$\vec{E}$ in the equation but didn't change the explanation (trying to avoid mistakes

by sticking to the notes!). Change one and not the other and you'll get disagreement.

2. $\int \sin(\theta)\, d\theta$ is of course $-\cos(\theta)$. That was just crazy talk.

3. The diagram I drew in class had the charges swapped relative to what I was working from in the notes. The textbook does an example with the diagram as I used in the lecture notes. The final result from class appeared to agree with the book because swapping the charges introduces a sign (see below) which canceled the incorrect sign on the integral. This is also why I was mumbling about how the work done seemed to be incorrect: it was! The result from class was off by a sign.



(a) (Notes/Book) (b) (Lecture)

Figure 2.5: Dipole in Field

Fig[2.5] shows the diagrams relevant for the Notes and Textbook (a), and for what was done in Lecture (b). The flipping of the signs in lecture flips the direction of the torque. Because of this, the integral for the work done,

$$W = \int_{\theta_i}^{\theta_f} \tau\,(-d\theta)$$

loses the minus sign on $d\theta$. The reason for the sign is that with the positive charge on top, as assumed by the derivation in the notes, the torque is in the direction of decreasing θ. To integrate up the work done, we need the differential element to be in the direction of positive work contributions. If the torque is in the $-\theta$ direction, that means $-d\theta$. If the torque is in the θ direction, that means $d\theta$. Since we used $d\theta$ in the lecture, along with the negative charge on top, that part was correct and self consistent. Had the sign on the silly cosine been right, the whole thing would have tied together and been fine.

2.2.4 Electric fields of Continuous Charge Distributions

This is one of those things that would be insane without the field formalism or the principal of superposition. The calculus can get quite tricky anyway, but its doable.

So far we have been talking about individual, distinct charges such as q, q_1, Q, Q_0, etc. This is fine for many situations, but eventually we have to deal with a system in which there is a distribution of charge, meaning a charge which isn't located at a single point. (We call the charges discussed before now *point charges* because they are concentrated into a point.) Point charges are accurate representations of fundamental charged particles like an electron, but in macroscopic systems, we often have so many electrons that (because of the principal of superposition) it looks like a spread out blob of charge (a continuous charge distribution), rather than a collection of lots and lots of point charges.

Since Coulomb's law gives the force between point charges and the electric field is derived from there, we need to relate continuous distributions to point distributions (point charges). We do this using the standard Calculus tactic of defining an *infinitesimal* charge, which generates an infinitesimal field element. You may hear these called the differential charge and field element. The infinitesimal field element is given by:

$$d\vec{E} = k\frac{dq}{r^2}\hat{r}$$

$d\vec{E}$ The infinitesimal field element. It is still a vector, and has a magnitude.

This magnitude approaches 0 (hence infinitesimal), but just like any dx in calculus, you can integrate an infinite number of infinitesimal pieces together to arrive at a finite (non-zero) result. $\left[d\vec{E}\right] = N/C$

dq The infinitesimal charge. As above, it can be integrated up to a finite charge, it just happens to be spread about. $[dq] = C$

In order to get a usable result, we need to integrate the infinitesimals.

$$\vec{E} = \int d\vec{E} = k \int \frac{dq}{r^2}\hat{r}$$

Note that dq could potentially vary with $\vec{r}$ if the charge distribution isn't uniform (continuous doesn't necessarily imply uniform). We won't be dealing with such situations much, but they do exist and are important. Also, as a simple matter of calculus that is of utmost importance to remember, *dq can depend on how you set up your coordinate system*. This might be the hardest thing to get right about setting up the integrals necessary to solve problems with charge distributions. The key is to think about which coordinate (x, y, z, θ, ϕ, r, *etc.*) most *simply* correlates to the charge distribution. For instance, say you have a ring of charge. If you were to decide to work in an x, y, z coordinate system, none of those coordinates would correlate well with the charge. What do I mean by this? What I mean is that if I want to describe a point moving along the ring, I have to continuously change both x and y (if the ring is in the $x - y$ plane). If I were to instead use an r, ϑ coordinate system, and with the origin at the center of the ring, I only have to change *one* coordinate, θ. Of course, you can't always set up the whole problem in such a simple fashion: some features might be simpler with a Cartesian (x, y, z) coordinate system, while others might be simpler with a polar (r, θ, ϕ) coordinate system. Sometimes you can write dq in one of these systems, and then *carefully* convert it to the other. This careful conversion is extremely important because it allows you to "scale" the charge element (dq) properly. I suggest paying particular attention to this issue when watching me do examples or reading them from the book.

This is an integral over all of space where there is a charge. Sometimes this is very simple, such as a single ring of uniform charge, yet there is no real limit to how complex the distribution can get. We will stick to fairly simple cases: this is a physics class, not a course on methods of integration. That said, the necessary calculus for electricity and magnetism is significantly more difficult than what you would have done in Mechanics, PHY113. If you have difficulties with this, you should work on it and come see me for help right away. There is a limit to how much you can learn on-the-fly, and I won't make an attempt to teach anyone calculus, but I am happy to help refresh or solidify what you have already learned.

Examples

- "Infinite" line of uniform charge.
- Disk of uniform charge.
- "Infinite", uniformly charged disk.

2.2.5 Conductors

Earlier we mentioned a few properties of conductors: they allow the free flow of electrons (charge), and as a result it is possible to induce charges in them. There is another direct consequence of the free flow of electrons which happens to be of a lot more practical use: *The electric field inside of a conductor is* 0 so long as nothing is moving (any nearby charges are static). It is actually pretty easy to see why this must be true. Imagine for a moment that you have a conductor, and there is a field inside, for instance pointing up. Since the property of conductivity is due to the presence of "free" electrons (electrons which are able to move about freely inside the conductor), these free electrons will of course feel a force due to the non-zero charge. When free electrons feel a force, they move (that's what it means to be free). A field which points up, as in our example, will exert a force down on the electrons. A net movement of negative charge downward will then cancel out the electric field (field lines point from $+$ to $-$ charges, so there must be $+$ charges on the downward side of the conductor. A flow of electrons towards them will reduce the charge imbalance, canceling out the field.). Electrons will continue to flow until the field is entirely canceled, so when everything settles down again, the field will be 0 inside the conductor.

Another direct result of the same line of reasoning is that any charge distributions in a conductor (say from induction) *must* be on the *surface*. We call these surface charges. Notation tip: surface charges are almost always represented by σ. Bulk charges (which can exist, just not statically in a conductor) are usually represented by ρ and both are called "charge densities".

Examples

- Charge in a ball.
- Charge in a box.
- Box in a field.