3: Gauss's Law July 7, 2008

3.1 Electric Flux

In order to understand electric flux, it is helpful to take field lines very seriously. Think of them almost as real things that "stream out" from positive charges to be "absorbed" by negative ones. Think of the arrows on the field lines we have been drawing as denoting some sort of motion. There isn't actually anything moving, but the mental picture helps.

The idea of a flux is fairly general in physics, but it normally refers to something more concrete than the electric field. For example, we can talk about the flux of water flow, which is just the amount of water passing through some specified area, or the flux of light on a solar panel, which is just the amount of light which strikes the panel. Since the concept is exactly the same, lets think about flux in these cases first, then use what we learn to talk about electric flux.

3.1.1 Flux: Water Flow

First lets consider water flow. I said before that this refers to the amount of water passing through a specified area. What I mean by this is that if I were to take, say, a wire loop, stick it in a river, and measure the water passing through (somehow), that would be the flux. Simple enough, right? Well, yes and no. What does the flux depend on in this situation?

1. How fast the river is flowing
2. How big the loop is (the area water can flow through)
3. The angle between the loop and the direction of flow of the river

The first two should be pretty obvious: the faster the river and the bigger the loop, the more water flows through. The angle is also simple but not quite as obvious. Remember, we aren't asking how much water *could* flow through the loop, we are asking how much *does* flow. So imagine you held the loop (which we are assuming is flat) such that it was edge-on to the flow of the river. Well, aside from maybe some minor eddies in the current, no water is going to flow through in this case: it just goes right past. At the other extreme, if the loop is directly face-on to the flow of the river, the maximum possible amount of water passes through. This flow is given by $\Phi_{water} = v_{river} \times A_{loop}$. Φ is the flux, v the velocity, and A the area.

In order to describe both of these extremes and everything in between, define an *area vector* $\vec{A}$ whose magnitude is the area of the loop and whose direction is perpendicular to the plane of the loop. So if the loop is lying on a table, the area vector will point up. At this point, down would be just as good an answer. Eventually we will be talking about surfaces with insides and outsides that let us define the direction completely. For now, just use the definition that gives a positive flow and don't worry about it.

With this definition,

$\Phi_{water} = \vec{v}_{river} \cdot \vec{A}_{loop} = v_{river} A_{loop} \cos(\theta)$ $\quad [\Phi_{water}] = \frac{m}{s} m^2 = \frac{m^3}{s} = \frac{volume}{s}$

where θ is the angle between the river flow direction and the loop area vector. Note that this is consistent with our previous discussion: $\cos(\theta)$ is at a maximum for $\theta = 0$ and a minimum for $\theta = \pi/2$.

3.1.2 Flux: Sunlight

Sunlight is conceptually one step closer to electric flux. Instead of a physical material flowing, we have a non-material something flowing. We no longer have a velocity, because light all travels at the same speed, but instead we have an intensity or brightness. It doesn't matter what we use as the area: our loop continues to work fine, or we can think of a light-collecting solar panel.

The entire previous argument still holds, except that conceptually we have a variable "density" of light (the brightness) instead of a variable speed. The math is identical within optical constants we ignore:

$\Phi_{light} \propto \vec{I}_{light} \cdot \vec{A}_{panel}$

This expression isn't actually what you will use when we do optics, because in practical applications things like the energy of the light is more important and getting the details of that is entirely beside the point I am trying to make. That point is this: even tho light and water are completely different, and light is much less tangible than water, they both have a "flux" which can be described in almost exactly the same language. The flux of an electric field is just one more step in the direction of an abstract "material" which "flows" through an area to generate a flux.

3.1.3 Actual Electric Field Flux, Finally

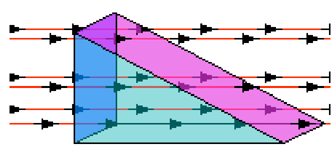

Figure 3.1: Electric Flux through Surfaces

For field flux, the field lines are what is passing through our area. Rather than velocity or intensity, we have field strength, as represented by the density of field lines. The flow direction is defined by the direction of the electric field, just as the flow direction of water is defined by the direction of the river's flow. Everything else is the same, so:

$\Phi_E = \vec{E} \cdot \vec{A}$ $[\Phi_E] = \frac{N}{C}m^2$

Done! The units aren't as intuitive as they were for water, but that's just because we don't have any sort of intuition for electric fields, but we do for water.

Unfortunately, when we want to use the flux concept to do anything, complications arise. The problem is that we are going to want to know the flux through surfaces that aren't simple planes like a loop of wire in the river. We are going to the the equivalent of throwing a straw hat into a churning rapids and ask what the flux through the hat is. The hat isn't a plane, and the water flow isn't uniform. With both of these differences, the simple product above stops being useful. You can't use a single field strength or area anymore.

The solution is to break up our complex surface into smaller surfaces $\Delta\vec{A}_i$ such that $\vec{A} = \sum_i \Delta\vec{A}$. If we make the pieces small enough, each one is effectively flat, and the field won't vary across it. Thus, we can treat each individual small surface with the simple formula above and add them together. Sometimes this is as simple as it sounds: if you have a box, for instance, you can fully describe it using a few square flat surfaces, do the calculation for each one, and add it all together.

$\Phi_E = \sum_{i=1}^{n} \vec{E}_i \cdot \Delta\vec{A}_i$

Unfortunately, we deal more often with surfaces with curves than not. In this case, we use infinitesimal area elements and integrate them together.

$\Phi_E = \int \vec{E} \cdot d\vec{A}$

In the most general possible case, this can be a bit of a nightmare and ends up getting done on computers. However, there are in fact lots of situations where we'll be able to make use of reasonably simple surfaces for which the integrals simplify significantly. In particular, Gauss's law will deal with *closed* surfaces. Closed surfaces are just surfaces that have an inside, an outside, and no holes. A balloon is a great example, but any bag, case, sealed jar, etc. is also a closed surface. When we are dealing with closed surfaces, we give ourselves a notational hint:

$\Phi_E = \oint \vec{E} \cdot d\vec{A}$

or sometimes

$\Phi_E = \oint_S \vec{E} \cdot d\vec{A}.$

The circle on the integral reminds us that our surface must be closed. The S names a particular surface. Now that we have specified this, we can resolve that ambiguity I mentioned before about the direction of $\vec{A}$. For a closed surface, we define $d\vec{A}$ to point *outward*. This is arbitrary, but you have to be consistent both with your own work and the results you may reference, so it is essential that you follow this convention at all times. The result of this definition is that flux *leaving* an enclosed surface is *positive*, while flux *entering* the surface is *negative*.

We can use that simple fact to instantly make a huge simplification.

Observe that any field line which passes *through* a surface cannot contribute *anything* to the total flux. A single line will have an equal negative (entering) and positive (exiting) contribution to the flux, canceling itself out. If we remember that we are talking about electric field lines, we know that they have some special properties. One of those properties is that field lines only begin and end at charges. Putting this together, we see that a charge outside of a surface won't contribute to the flux: its lines will go in and back out again. Likewise, if a charge is inside a surface, all of its field lines must pass out of (or into if the charge is negative) the surface. Together with the principal of superposition, these two observations been that the total flux of an enclosed surface is determined entirely by the charges *inside* of it. We can ignore everything else!

3.2 Definition of Gauss's Law

Gauss's Law is the formulation of the observation at the end of the last section:

$$\Phi_E = \oint_S \vec{E} \cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0}$$

where we again find the permittivity of free space, ϵ_0. Q_{enc} is the sum of all the charge enclosed within the surface S.

There are two main steps to do a calculation using Gauss's law. The first is to identify a *Gaussian Surface* that will make the problem as easy as possible to do, and properly set up the integral using that surface. The second step is to actually do the integral. Either step can be simple or complex, but it is generally the case that simpler surfaces give simpler integrals.

The advice I gave on integrals back in Sec [2.2.4] is still relevant here. You want to choose your surface such that the integral is easy to set up. Generally, this means picking a surface that can be simply described in some coordinate system, and has some significance to the system itself. For instance, if you have a single point charge with field lines extending from it, what should your surface look like? In this case the important fact to notice is that the electric field is simplest when we describe it in *spherical* coordinates. The strength depends only on r, and the direction is always $\hat{r}$ (or $-\hat{r}$ depending on the sign of the charge). That means that if we define a surface which is all at the same r, the electric field will have a constant magnitude everywhere! The surface with this property is a sphere. A sphere also has the handy feature that at every point, the direction of $d\vec{A}$ (outward) is the same in spherical coordinates: $\hat{r}$. We call this *using the symmetry of the problem*. With this choice of Gaussian surface, the integral becomes trivial. If we try the exact same problem with, say, a cube, it becomes fairly complex and annoying. So if you find yourself confronted with what seems like an unreasonably difficult integral, take a step back and look for a better Gaussian surface before you dig in and try and muscle through the problem or give up.

Lets use the surface I just described to prove that Coulomb's law and Gauss's law agree with each other. In fact, we are going to derive Coulomb's law from Gauss's law! Coulomb's law tells us about the force between two charges, or equivalently the field due to a single charge. Therefore, our system is a single charge alone in space. We assume that a) the field is directed radially and b) it depends only on r, not θ or ϕ. Gauss's law gives

$$\begin{aligned}
\oint_{Sphere} \vec{E} \cdot d\vec{A} &= \frac{q}{\epsilon_0} \\
\oint_{Sphere} E\, dA &= \\
E \oint_{Sphere} dA &= \\
E\, A &= \\
E\, 4\pi r^2 &= \frac{q}{\epsilon_0} \\
E &= \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}
\end{aligned}$$

Voila! Remember that $k = \frac{1}{4\pi\epsilon_0}$, so this is Coulomb's law! We have used the fact that the integral of the area element over a sphere just gives the area

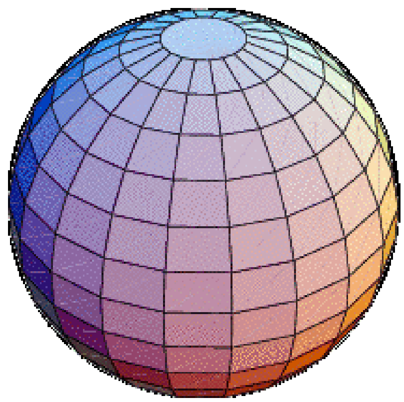

Figure 3.2: Sphere broken into rectangular elements

of a sphere, which is $4\pi r^2$. If you want to see the calculus for this explicitly, see below. It isn't necessary, however, and you can skip over it if you like.

The first step is to figure out what your area element dA is in terms of your coordinate system, (r, θ, ϕ). The are element is an infinitesimal rectangle on the surface of a sphere. Its area, unfortunately, depends on where on the sphere it is (this is unavoidable and is the fact that it is impossible to make accurate world maps which are flat). To see how, look at Fig [3.2].

If we pick one of the "rectangular" segments and ask what the length and width are, we quickly see that a segment will be wider near the equator and narrower near the poles, but the length will not vary. Trigonometry can give you that the length is always going to be $rd\phi$, where ϕ is the angle corresponding to latitude. By the same token, the width is given by $r\cos(\phi)\, d\theta$. You can see at least that this is plausible because it gives a maximum value at the equator, $\phi = 0$.

$$
\begin{aligned}
\oint_{Sphere} dA &= \int_{-\pi/2}^{\pi/2} r d\phi \int_0^{2\pi} r\cos(\phi)\, d\theta \\
&= r^2 \int_{-\pi/2}^{\pi/2} \cos(\phi)\, d\phi \int_0^{2\pi} d\theta \\
&= r^2 \left(\sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right)\right)(2\pi) \\
&= 4\pi r^2
\end{aligned}
$$

In this case going into this degree of detail is unnecessary because we know that the integral of the area element has to give you the area, otherwise its not the are element! However, this process of converting one differential element into differential elements of the coordinates comes up all the time in physics, especially E&M, so it would be wise to familiarize yourself with how this worked.

3.3 Applications

Gauss's Law is extremely powerful so long as there is some symmetry we can take advantage of. Generally, there are 3 types of symmetry we will encounter for which Gauss's law is well suited: Spherical (as with the point charge above), Cylindrical (for a long wire, for example), and planar (for large flat surfaces). Sometimes problems can have a combination of these symmetries. The best way to get a feel for how these problems work is to go ahead and do a bunch, so lets do that.

3.3.1 Examples

Solid Charged Sphere

We have a non-conducting sphere with a uniform charge distribution ρ throughout. The radius of the sphere is R_0. Since this is a sphere, it should be pretty clear that the Gaussian surface we want is a sphere. We can find the field through all of space due to this charge distribution with Gauss's law. Finding the field at even a single point using Coulomb's law would require a fairly annoying three dimensional integral.

Set the origin at the center of the sphere. Because of the spherical symmetry of the system, the field will depend only on r. To find that dependence, set the radius of your Gaussian surface to R.

$$\oint_{S_R} \vec{E} \cdot d\vec{A} = E\left(4\pi R^2\right) = \frac{Q_{enc}}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R^2}$$

But what is Q_{enc}? This is another integral. We need to integrate the charge density over the volume in order to find the total. If $R_0 > R$,

$$Q_{enc} = \int_0^R dr \oint_{S_r} \rho dA = \rho \int_0^R dr\left(4\pi r^2\right) = 4\pi\rho \int_0^R r^2 dr = \rho\frac{4}{3}\pi R^3$$

If $R_0 < R$,

$$Q_{enc} = \rho\frac{4}{3}\pi R_0^3$$

This is just because outside of R_0, $\rho = 0$. If we call the total charge of the sphere Q then $\rho = \frac{3}{4\pi}\frac{Q}{R_0^3}$. Putting this together,

$$E\left(R < R_0\right) = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}\rho\frac{1}{R^2}R^3 = \frac{\rho}{3\epsilon_0}R = \frac{1}{4\pi\epsilon_0}Q\frac{R}{R_0^3}$$

$$E\left(R > R_0\right) = \frac{1}{4\pi\epsilon_0}\frac{4\pi}{3}\rho\frac{1}{R^2}R_0^3 = \frac{\rho R_0^3}{3\epsilon_0}\frac{1}{R^2} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$$

Charged Conducting Sphere

If we charge a conductor instead of an insulator, all of the charge will be on the surface instead of spread throughout the sphere.

The first part of the problem is the same as for the insulator,

$$\oint_{S_R} \vec{E} \cdot d\vec{A} = E\left(4\pi R^2\right) = \frac{Q_{enc}}{\epsilon_0}$$

$$E = \frac{1}{4\pi\epsilon_0}\frac{Q_{enc}}{R^2}$$

However, the Q_{enc} is different for the 2 ranges. Now, for $R < R_0$, $Q_{enc} = 0$ and for $R > R_0$, $Q_{enc} = Q$. So,

$$E\left(R < R_0\right) = 0$$

$$E\left(R > R_0\right) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$$

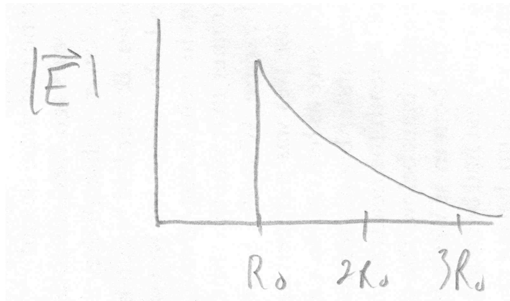

Figure 3.3: Electric Field outside a Charged Conducting Sphere

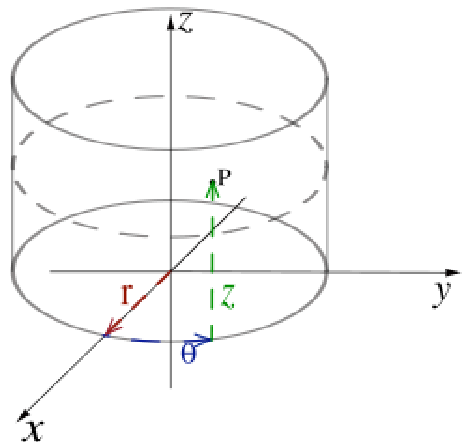

Figure 3.4: Cylindrical Coordinates

Long Uniform Line of Charge

A uniform line charge has charge λ per meter, such that a line of length L has charge $L\lambda$.

For this example we need a cylindrical symmetry instead of spherical. That means that we need to use cylindrical coordinates (r, ϕ, z). Cylindrical coordinates are just 2D polar coordinates with a z axis. We orient the coordinate system such that the line of charge is on that z axis. This way, $r = 0$ for all the charge and ϕ is arbitrary.

We define a Gaussian surface which is a cylinder of radius R and length L, centered around the origin and z axis.

If we draw the field lines from the charge line according to the rules we discussed earlier, we can see that they all extend straight out. This is because the line is uniformly charged, and as such the field lines coming from it will be evening spaced. Since the line extends forever, there is nowhere for the field lines to spread out on the z axis (tho they will still spread radially). This means that the field lines are all in the $\hat{r}$ direction, which happens to be tangent to the curved surface of the cylinder and parallel to the end caps. As such, there is no flux through the end caps, and the dot product between the field and the curved surface is trivial.

$$\oint_{C_R} \vec{E} \cdot d\vec{A} = \oint_{C_R} E\hat{r} \cdot d\vec{A} = E \oint_{C_R} dA$$

Get dA using the same logic as we used for the sphere before: think of it as a small square and find the lengths of the sides in terms of the actual coordinates.

$$dA = dz\, r d\theta$$

$$E \oint_{C_R} r dz\, d\theta = ER \int_0^{2\pi} d\theta \int_{-L/2}^{L/2} dz = ER\,(2\pi)\,(L) = 2\pi RLE$$

$$2\pi RLE = \frac{Q_{enc}}{\epsilon_0}$$

$$E = \frac{\lambda L}{2\pi\epsilon_0}\frac{1}{RL} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{R}$$

$$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{R}\hat{r}$$

Uniformly Charged Disk

This is another case with cylindrical symmetry, but a very different charge distribution. There is a surface charge $\sigma = \frac{Q}{\pi R_0^2}$ on a disk of radius R_0.

Dipole

4:

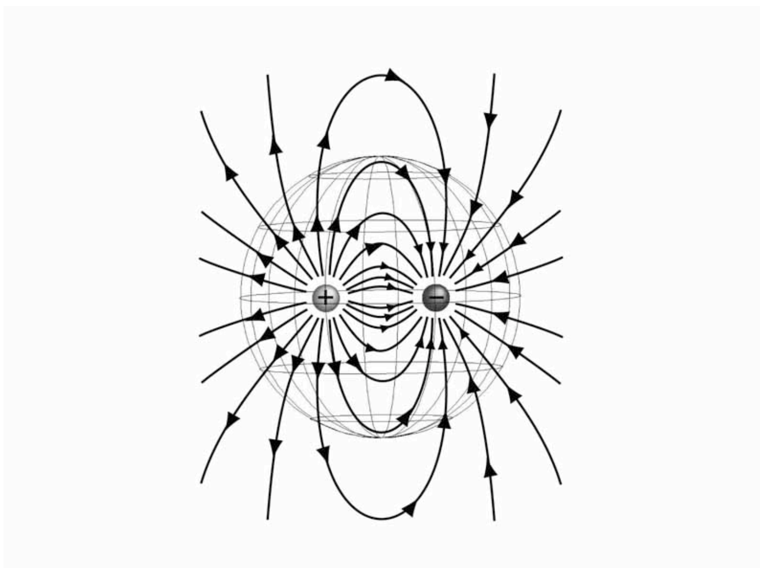

Figure 3.5: Electric Field of a Dipole