5: Capacitors

July 8, 2008

5.1 Definition

A capacitor is a structure which has a certain *capacity* to hold an electric charge. It is essentially the simplest possible battery. The typical example of a capacitor, and the typical actual design, is two parallel charged plates. There are variations and clever extensions, but this is the basic idea, and we'll see why.

The capacity to store charge is defined in relation to the electrical potential necessary for a given amount of charge. Theoretically, any material object can store an almost unlimited number of electrons and it doesn't matter what the material is made of: you can pack electrons anywhere. However, when you build up larger and larger charges on things, you create fields and the charge starts looking for some way to escape. This is what happens when you build up a charge and get zapped when you reach for the doorknob. Your body has built up a charge, which gives rise to a potential difference between you and the door. When you stand far away, you have the capacitance to hold on to that charge. However, when you reach for the doorknob, you reduce the barrier for the charge to jump: you no longer have the capacity to hold on to all of that charge. In the language of capacitors, the *capacitance* of the system consisting of your body and the doorknob has dropped below the charge stored on your body, and the capacitor (yes, you and the doorknob form a capacitor) discharges.

But this is physics class, lets see some equations! The quantification of the above is simple. Capacitance is the amount of charge a system can hold be Volt of potential difference:

$$C = \frac{Q}{V} \qquad [C] = \frac{C}{V} = F = farad$$

(Don't you just love how all of these things start off with innocuous, simple definitions, and then turn out to have the potential for unlimited complexity and headaches? Perhaps you find this annoying, but its that

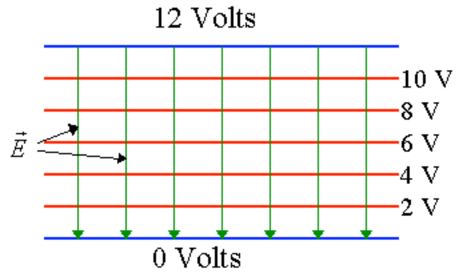

Figure 5.1: Parallel Plate Capacitor

tendency of simple physical systems to give rise to ridiculously complex behavior that makes the universe an interesting place with the dizzying variety of objects and life and phenomena. OK, end of pointless aside. Sorry.)

A $1F$ capacitor can hold an entire Coulomb of charge with only a single Volt of potential difference. Most of the capacitors in everyday electronics are measured in microfarad (μF).

That's all there is to the definition of a capacitor. They can take any shape, and in fact any shape *is* a capacitor, but certain shapes make much *better* capacitors than others, which is why the parallel plate capacitor is so common.

5.2 Examples

Any system with two objects between which you can find the potential difference can be analytically studied as a capacitor. There are many objects for which this calculation is impossible or impractical, in which case experiment can discern the capacitance, but we have plenty of examples we can consider.

5.2.1 Parallel Plate Capacitor

The parallel plate capacitor is that rare example in which case the one of the most easily solved problems actually looks a lot like the practical physical manifestation. Enjoy the novelty.

The definition of capacitance depends only on the charge held and the corresponding potential. For the parallel plate capacitor this is easily solved. Construct our system of 2 conducting plates. Each holds charge Q and is a plate of unspecified shape (within reason. Its not a cheese grater with holes nor a snowflake cutout with lots of corners) but with area A. This is all we need to calculate the potential difference. The positive charge is on the plate at point b.

$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{\ell} = -E \int_d^0 dx = Ed$$

Remember from solving for the field near a charged plane with Gauss's Law that

$$E = \frac{Q}{\epsilon_0} \frac{1}{A}$$

so

$$V_{ba} = \frac{Qd}{\epsilon_0 A}$$

$$C = \frac{Q}{V_{ba}} = \epsilon_0 \frac{A}{d}$$

5.2.2 Cylindrical Capacitor

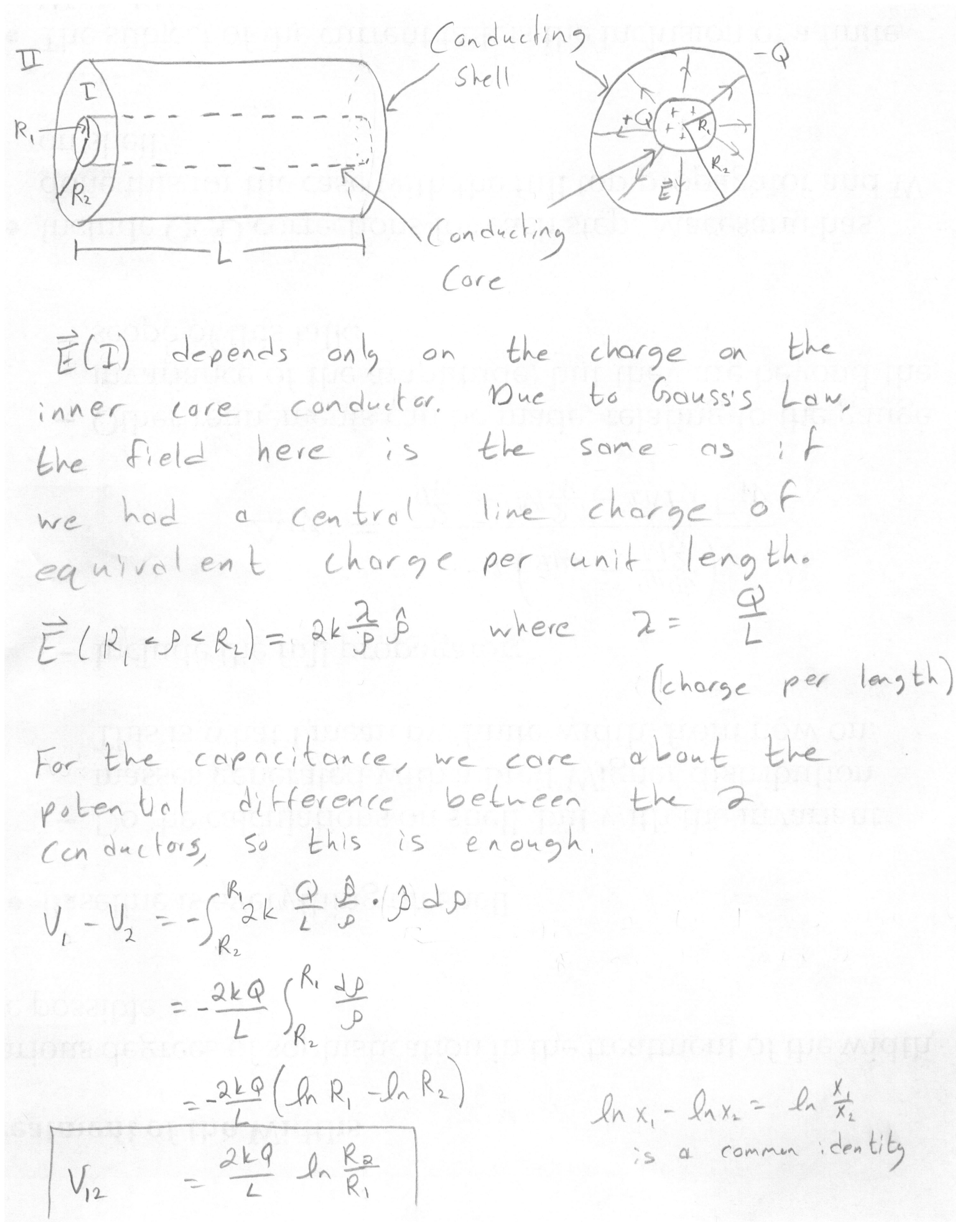

Figure 5.2:

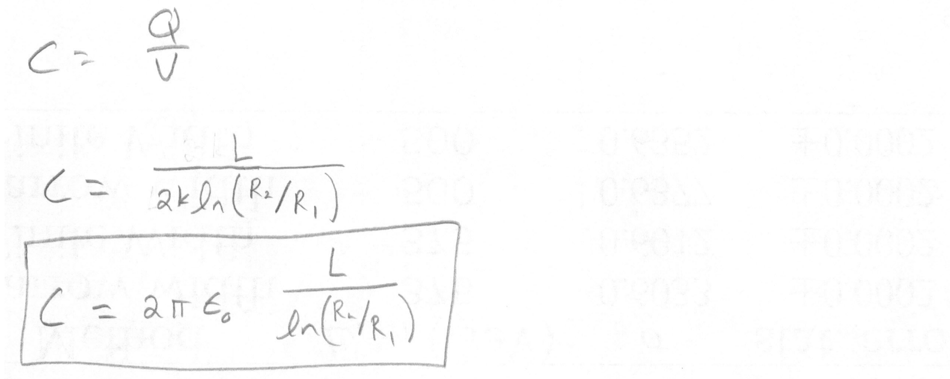

Figure 5.3:

$C = 2\pi\epsilon_0 \frac{L}{\ln(R_2/R_1)}$

Compare this to the case for the parallel plates. The length of the cylinder takes the place of the area (both give a measure of the size of the capacitor. The cylindrical capacitor is like the line charge to a parallel plate capacitor's surface charge) while the natural log of the ratio of the radii takes the place of the separation d (both give a measure of the separation of the charge carrying surfaces). Note that because the natural logarithm is a very slowly varying function, the cylindrical configuration doesn't gain capacitance nearly as well as the parallel plates by being brought close together.

5.2.3 Spherical Capacitor

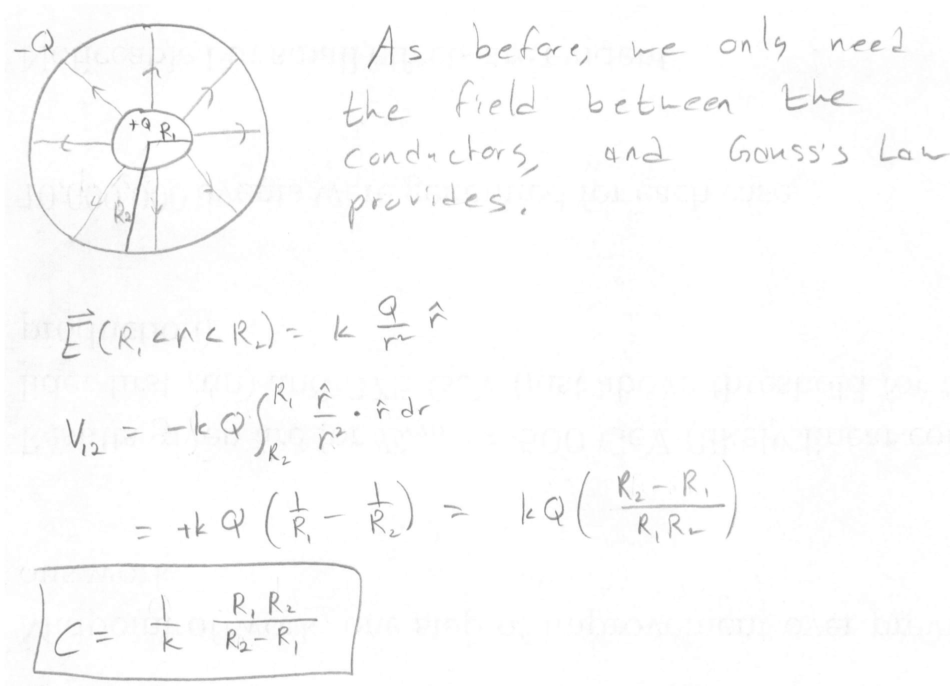

Figure 5.4:

$C = 4\pi\epsilon_0 \dfrac{R_1 R_2}{R_2 - R_1}$

Again we have an area like quantity over a separation of the charge carriers. Same idea, different realization for a different system.

5.2.4 Parallel Wires

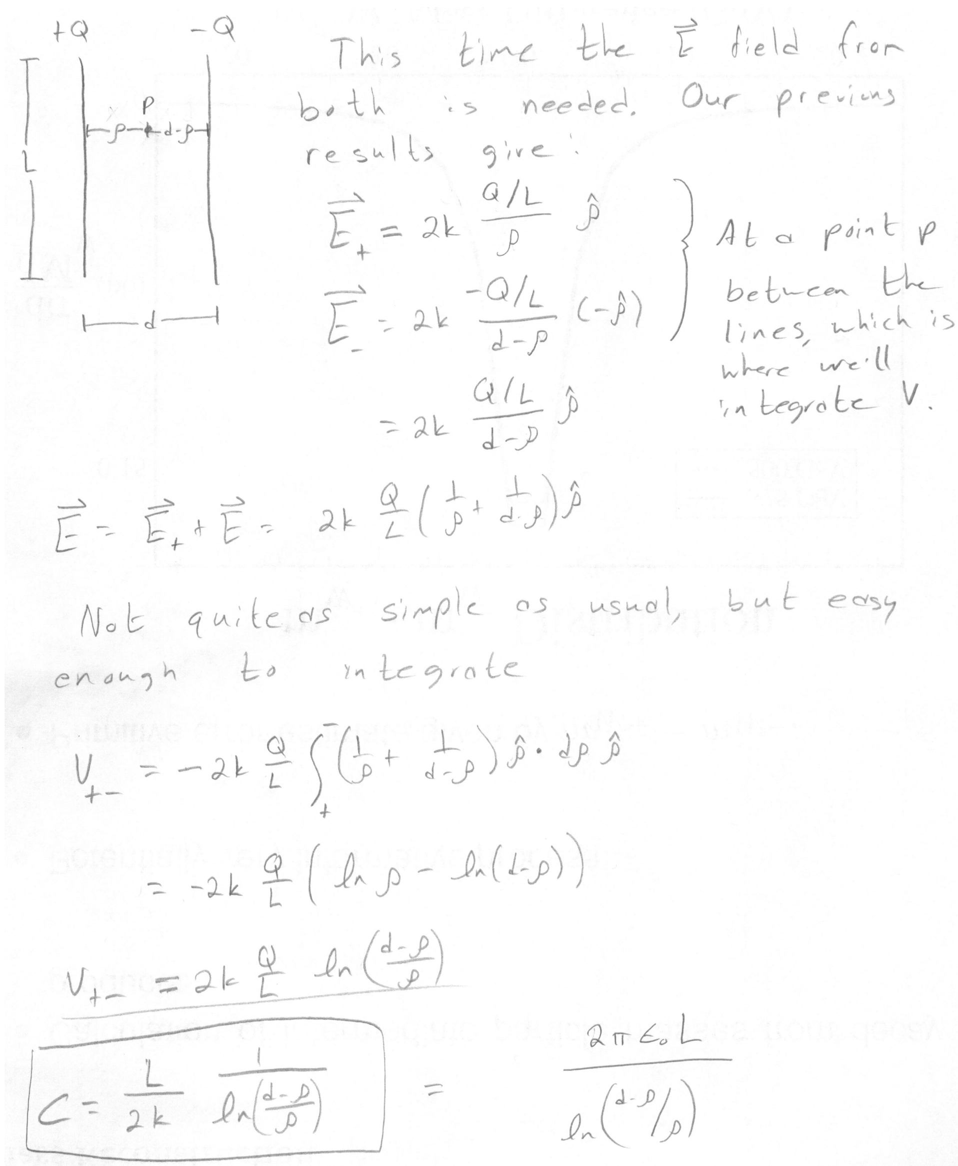

Figure 5.5:

$C = 2\pi\epsilon_0 \frac{L}{\ln(d-\rho/\rho)}$

This looks an awful lot like the cylindrical capacitor case, doesn't it?

They both have a cylindrical symmetry, so this shouldn't surprise you entirely.

5.3 Putting Capacitors Together: The Beginning of Circuits

Generally in the context of a purpose-built capacitor (rather than some random collection of objects which has a capacitance), the potential difference between the plates is maintained by wires connected to each and some sort of power source such as a battery or a generator. Since each capacitor is then connected to a wire on each end, it can be integrated into a circuit (which is just a loop which conducts and has a power source to drive it). We can wire capacitors together in 2 different ways: series and parallel. A series connection just means sticking one capacitor after the other in a line. More specifically, the positive lead from one capacitor leads into the negative lead of the next, and so on. In a parallel connection, the positive leads of a number of capacitors are all wired together, and the negative leads of all the same capacitors are also wired together.

These two different arrangements make for very different behaviors. Fortunately, we can calculate them!

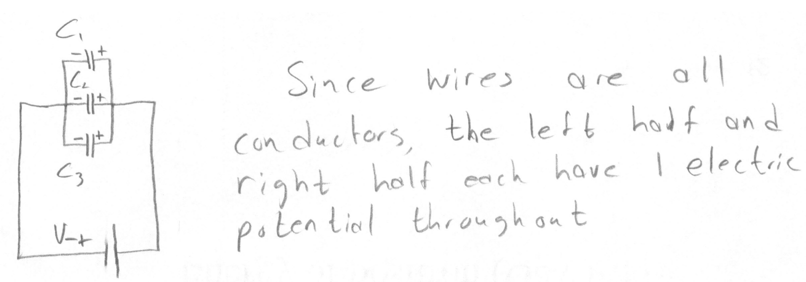

Figure 5.6: Capacitors in Parallel

Imagine first that I connect 3 capacitors C_1, C_2, C_3 in parallel. Because their positive and negative leads are connected, they all have the same potential difference V_{-+} across them. The question we want to answer, is what is the equivalent capacitance of this collection of capacitors? That is, what would the capacitance of a single capacitor need to be to function like this collection of 3. Remember, any system has a capacitance, so there has to be *some* capacitance that is equivalent to the 3 in parallel.

We know the voltage (potential difference) for each is the same, so all that remains is to find the total charge:

$Q = Q_1 + Q_2 + Q_3 = C_1 V_{-+} + C_2 V_{-+} + C_3 V_{-+} = V_{-+}(C_1 + C_2 + C_3)$

With the definition of capacitance, this gives:

$C_{equiv} = C_1 + C_2 + C_3$

Note that despite its similarity to expressions of the principal of superposition, this is result does not in fact follow (at least in any reasonably clear way) from that principal. We can make sense of this result by thinking in terms of parallel plate capacitors, however. In that case, the capacitance is proportional to the are of the capacitors. If we just stick a couple of capacitors in parallel, we are essentially adding their areas, so the capacitance would also add. This is just a plausibility argument, however, since it neglects the plate separation and other configurations of capacitor.

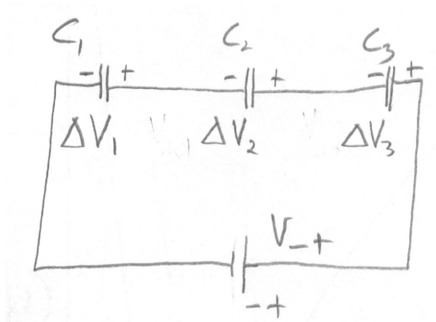

Figure 5.7: Capacitors in Series

That was easy enough. What about capacitors in a series? Again lets work with 3 capacitors. Since they are in series, they no longer have the same potential difference across them (necessarily). However, we know that their potential differences must add up to the total maintained potential difference, V_{-+}. Why? Because the circuit is a complete loop and the electric potential *can't have discontinuities.* So, the power supply provides a potential jump of V_{-+}. The rest of the circuit (in this case the 3 capacitors) must bring us all the way back down (but no further) so that the potential matches up.

$V_{-+} = \Delta V_1 + \Delta V_2 + \Delta V_3$

The notation ΔV_i refers to the potential difference between the two plates of capacitor C_i. Writing V_i would be ambiguous given our previous use of that symbol.

Looking at Fig[5.7], we can see that the charge on each capacitor must be the same. Why is this? The applied voltage from the wire connected to the negative lead of C_1 will induce some charge $-Q$ on C_1. This charge must induce an equal and opposite charge on the other plate, giving Q. Because there is a wire connecting C_1 and C_2, the $+$ side of C_1 must have the same electric potential as the $-$ side of C_2. Equal potential means even charge distribution in a conductor, so C_2 must *also* have charge Q. Likewise, we see that C_3 has charge Q.

We can now use the definition of capacitance to give

$C = \frac{Q}{V} \longrightarrow \frac{1}{C} = \frac{V}{Q} \longrightarrow \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{\Delta V_1}{Q} + \frac{\Delta V_2}{Q} + \frac{\Delta V_3}{Q}$

$\frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} = \frac{1}{Q}\left(\Delta V_1 + \Delta V_2 + \Delta V_3\right) = \frac{V_{-+}}{Q} = C_{equiv}$

$\frac{1}{C_{equiv}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$

Quite different from the addition rule for a parallel arrangement! In fact, the equivalent capacitance in series is smaller than the smallest capacitor in the series. This is not what you do if you are looking to build up a big capacitance in your circuit.

5.4 Energy Stored in Capacitors

We learned earlier that we can calculate the work required to construct a system of charges using the scalar potential. We found that

$U = \int dU = \int_0^Q V\,dq$

Rather than use the explicit expression in terms of charge and separation for the scalar potential, we can write it in terms of the capacitance,

$V = \frac{Q}{C} \Rightarrow \frac{q}{C}$

where q in the integral will be how much charge we've already added up,

$U = \int_0^Q \frac{q}{C} dq = \frac{1}{C}\int_0^Q q dq = \frac{1}{C}\frac{Q^2}{2}$

$U = \frac{1}{2}\frac{Q^2}{C}$

We can write this another way by simply substituting in for Q:

$U = \frac{1}{2}CV^2 = \frac{1}{2}QV$

5.4.1 Electric Field

I made a bit of a point when I introduced the electric field that it has a physical existence and isn't just a mathematical trick. This is the first place we see real evidence for that, and the evidence is that the electric field itself is the entity which contains the energy stored in a capacitor! This invisible, intangible "object" is storing energy in the gap between two conducting plates.

Combining the expression for the potential energy content of a capacitor, $U = \frac{1}{2}CV^2$ and the parallel plate expression for electric potential $V = Ed$, we find

$U = \frac{1}{2}\frac{\epsilon_0 A}{d}E^2d^2 = \frac{1}{2}\epsilon_0 AdE^2$

If we note that Ad is the volume between the capacitor's plates, we can define an *energy density* between the plates,

$u = \frac{U}{Ad} = \frac{1}{2}\epsilon_0 E^2$

This turns out to be a general expression for the energy density of an electric field (where it is constant), tho we derived it from this special case.

5.4.2 Examples

Pull capacitor plates apart

Pull plates apart while plugged in

5.5 Dielectrics (and Insulators)

Experimentally, we find that we can dramatically increase the capacitance of a system by inserting insulating materials between the charged surfaces. Your book has a section which explains how this works at a microscopic level, but the basic idea is just that even insulators have a kind of induced charge which reduces the field in a capacitor. This reduced field, with the same charge buildup, means that the charges feel less motivation to leap from one plate to the other, discharging out capacitor. And because dielectrics (materials which have this type of induced charge) are also (often) insulators, they also keep the electrons on their own side even if the field *is* cranked up extra high. Since this whole process depends upon complex molecular level physics of an aggregate material, we don't even pretend to try and calculate it (well, *we* don't. Professionals with huge computers and such do). Instead, we just look up measured values from experiment.

This effect can be described with a single constant (for each material) factor:

$C = KC_0$

where C_0 is the capacitance we have been calculating so far and corresponds to a vacuum existing between the plates. Air's dielectric constant K is almost exactly the same as that of the vacuum (1.0006 rather than 1), so everything we've done works for most purposes in atmosphere as well.

For reasons we won't get into, the dielectric constant is often rolled into another parameter

$\epsilon = K\epsilon_0$

ϵ_0 is the permittivity of free space (vacuum), while ϵ is the permittivity of whatever material we are worried about at the moment. The upshot of this is that if you've done a calculation for a capacitor in vacuum and want to turn it into one with a dielectric, you can just replace ϵ_0 with ϵ.

Functionally, this constant increases the capacitance of a system.Since K is always 1 or larger, it increases the charge required to produce a given voltage.

$Q = KQ_0$

and it increases the energy contained in a conductor,

$u = Ku_0 = \frac{1}{2}\epsilon E^2$.

However, the electric field is decreased

$E = \frac{E_0}{K}$

as is the potential difference

$V = \frac{V_0}{K}$

These make sense because the point of the dielectric is to reduce the forces pulling the electrons from one surface to the other.