6: Currents and Resistance: Basic Electric Circuits

July 12, 2008

6.1 Circuits

6.1.1 Chemical Battery

Electric batteries work by using a chemical potential, an imbalance of chemical energy between two electrodes, to produce a roughly constant potential difference. The details of how this works can be interesting, but I'm going to leave them to the text. For our purposes at the moment, we only need to know that they produce a potential difference.

6.1.2 Current

Electric current is our first departure from the static case of the study of electric charges. An electric current is just a flow of charge. Usually we talk about a current of electrons through a wire or other conductor, but any flowing charge through any medium, including a vacuum, counts as a current. Specifically, current can be defined as an average over time,

$\overline{I} = \frac{\Delta Q}{\Delta t}$

where ΔQ is a quantity of charge which is moving, and Δt is the time duration over which we are averaging or sampling. This average can be as coarse- or fine-grained as we like, but we general talk about the fine grained limit of the instantaneous current,

$I = \frac{dq}{dt}$. $\qquad [I] = \frac{C}{s} = A\, or\, amp = ampere$

In many circumstances, the current will be a constant and these two definitions won't really differ. However, any time a power supply is turned on or a circuit is modified, the current may be modified, and then we will have to talk about a $I(t)$ rather than just a constant I.

Current exists because of a conductor's desire to reach a zero internal electric field. Remember we stated that in the static case, a conductor cannot have an internal electric field because the charges inside will keep moving until the cancel it. Usually this happens quite rapidly and we forget about it. However, if a constant potential difference is applied across a conductor (with, for instance, a battery), then the charges in the conductor will just *keep moving* in a futile attempt to cancel out the electric field they are now experiencing. Keeping the electrons moving drains energy from the battery, so eventually it will die and the electrons will win, the conductor will return to a static state with 0 internal field, and we can go home. But the presence of an applied potential difference (as opposed to an applied electric field) means that this static state will take an arbitrarily long time to come about.

6.1.3 Properties of a Circuit

A circuit refers to a collection of conductors and other elements which can potential form a continuous loop of charge flow. Current only flows when have a continuous route through which charge can flow, and some source of potential difference to drive the current. Normally we use the term circuit to refer to a *closed circuit*, which is an unbroken loop of, generally, conductors such that there is a clear path for charge flow. An *open circuit* doesn't have a path for charge to flow. By this definition, pretty much anything could be considered an open circuit (like the chalkboard), but it is generally used to refer to a temporary or modified state of what was or could easily be a closed circuit. We don't refer to the chalkboard as an open circuit, but the wire going from the wall socket to a switch in the "off" position to a light and back again *does* refer to an open circuit. The difference is of intent: the switch is obviously there to let the circuit alternate between open and closed, while the chalkboard isn't intended to have anything to do with currents. Also, a closed circuit which is damaged can become open.

Also, as the example of lightening strikes makes clear, electric current can flow anywhere if given a large enough potential difference, so any "open" circuit can be closed with a strong enough source. We call a circuit open if the built in or intended voltage source is too weak to cause the charge to "arc" across the break.

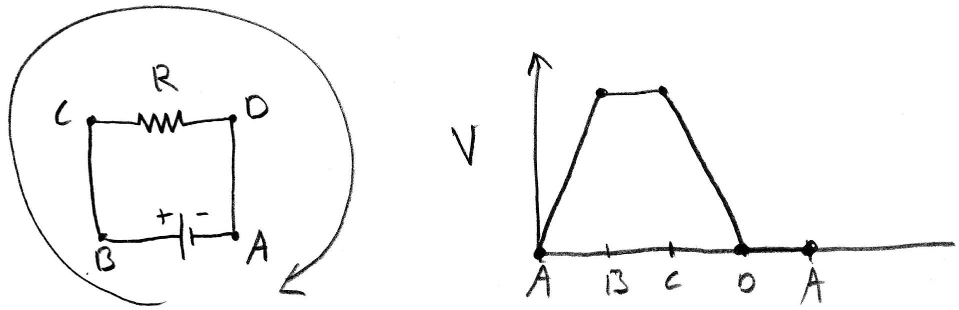

Figure 6.1: Current Convention (Battery Symbol)

Many circuits will also at some point have a connection to a large reservoir of charge referred to as the ground. In many applications, this is actually the earth. In other instances, it is simply a relatively large chunk of conductor which is nearby. The ground serves as a source for electrons or a place to dump them depending on what the current is doing. Think of it as a sort of charge buffer.

“Conventional” versus electron current

Because electric current was known and studied before the electron was discovered and the microscopic nature of typical current flows were understood, the definitions naturally assume the flow of positive charges. Because of this, current (unless specifically stated otherwise) always refers to the direction a positive charge would flow to have the net charge flow denoted. Since in conductors it is electrons which flow, the physical flow is in the opposite direction of the current. This usually doesn’t matter, but sometimes we need to keep it in mind.

6.2 Resistors

So far we have treated conductors as tho they allow for the completely free flow of charge. In fact, that’s is how we defined them when we wanted to make statements about their electric fields and induced charge. However, in the real world things are rarely so absolute. Every conductor (except the extra-special superconductors, but that's another story) offers some *resistance* to the flow of charge through it. In fact every material period offers some resistance, conductors are simply those substances which have a low resistance.

It is empirically true that if a potential difference is placed in a circuit, the current produced is usually proportional to that applied voltage:

$V = IR$ $\qquad [R] = \frac{V}{A} = \frac{V}{C/s} = \frac{Vs}{C} = \Omega = \ Ohm$

where R, the constant of proportionality, is the resistance. This relation is called Ohm's law and materials which obey it are called ohmic. There are many materials which are not ohmic (nonohmic), but most metals and thus most elements in a typical circuit are at least roughly ohmic, so we can use Ohm's law for simple circuits.

Remember that while all elements in a circuit will have some resistance (even the wires connecting them), the resistance may be negligible and ignored for most applications.

Conservation of Charge

Because charge is conserved, current must flow through a circuit without disappearing. Charge can build up or be drawn from a ground, but otherwise on a circuit the current must be the same at every point. This includes on either side of a resistor. Resistance to electrical flow simply means that a greater potential difference is required to keep the charges moving and more energy is expended. It does not mean that charge or current is sucked up somehow.

6.3 Potential around a Circuit

Remember our analogy of electric potential to elevation under a gravitational field. We can use the same analogy when talking about a circuit. Since a close circuit is a loop which ends where it started, it is like a close path around a landscape. The elevation may go up and down as we move along it, but it is always the same when we return to our starting point. This is also true of the electric potential. It goes up and down, sometimes abruptly,

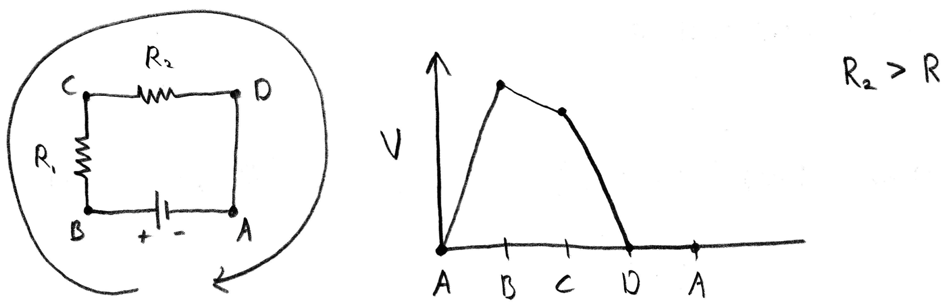

Figure 6.2: Potential Drop over Resistors

but always ends up where it started. This is equivalent to the statement that the electric force, like the gravitational force, is conservative.

Using this analogy, a source of potential is something which pushes us up a hill (increasing potential), while each element of resistance along our path corresponds to a downward slope: we must go downhill to overcome the resistance. If we think of the circuit as a pipe lying along the path and the current as a fluid flow through it, the voltage source is a water pump. Resistance can either be the thickness of the pipe, or the presence of obstacles within it which slow the flow of water. If there is an obstruction to the flow of water but the current must stay the same, we must have a compensating drop in the potential to pull the water through.

This is a key fact to remember: the *potential* will *decrease* across a *resistor* in the direction of the current. The amount of change is determined by Ohm's law (assuming the material is ohmic such as for a metal wire).

6.4 Conductivity and Resistivity

We've talked about materials as either conductors or resistors (or semiconductors) so far, but as mentioned earlier in this section some conductors are better than others. We can quantify this by specifying the *conductivity* of a material, σ. The resistance of a wire of length ℓ and cross sectional area A is given by

$R = \frac{1}{\sigma}\frac{\ell}{A}$.

The inverse of the conductivity is $\rho = \frac{1}{\sigma}$. These quantities have exactly the same information content and either can be used in any given circumstance. Unfortunately the conventional symbols are the same as those for charge distributions: such is life with limited alphabets.

The ratio here is also worth noting. The fact that the resistance of a wire is proportional to its length should not be a surprise: in fact, think how strange it would be if this were not the case. The inverse dependence on the area, however, is less obvious. This is analogous to a larger pipe having a greater capacity, and for roughly the same reason: there is more room for stuff to flow. If the wire is very wide, then there is more space to spread out and sail smoothly on through.

6.4.1 Temperature

We will see later that resistivity has to do with atoms in a material getting in the way of electron flow. When we heat up a conductor, its atoms will being moving more rapidly and they will become more disruptive. For this reason, conductors generally work better at lower temperatures,

$\rho = \rho_o \left(1 + \alpha \left(T - T_0\right)\right)$

where T_0 is some reference temperature at which the resistivity has been measured. Since α is typically quite small, this dependence doesn't matter for many applications. Many is by no means all, however, and you can even make a thermometer based on this variation.

6.5 Energy Transfer through Circuits: Power

Much of our modern existence is based upon our ability to transmit energy in the form of electric current across large distances through wires. Since everything on a circuit has a resistance and resistance is an opposition to the motion of electrons, overcoming that resistance must be using energy. To maintain a current requires a continuous expenditure of energy over time: power.

We already know the energy required to move charges through a potential difference,

$dU = V dq.$

Remember that resistors correspond to a voltage drop, so this tells us how much energy it takes to move a differential element of charge across a resistor. Power is energy expenditure per time, however, so divide both sides by dt

$\frac{dU}{dt} = V\frac{dq}{dt} \longrightarrow P = VI$ $\qquad [P] = \frac{J}{C}\frac{C}{s} = \frac{J}{s} = W = Watt$

where P is the power.

We can combine this with our existing formulas for a few convenient forms:

$P = I^2R$

$P = \frac{V^2}{R}$

This can of course be arranged to solve for whichever quantity you are interested in.

6.5.1 Practical Considerations

As we see above, any resistor in a circuit is using energy. This energy can be converted into pretty much any form, and the design of resistors which convert electrical power into useful functions (like a light bulb or refrigerator) is responsible for much of our modern way of life. However, there is inevitably some energy wasted in the generation of heat (tho if you are building a heater...). This is why computers need fans and poorly designed laptops can burn their users laps (also: exploding, but that's a different problem). It happens in weak resistors like the wires in your house too, however. If the current is large enough, the power dissipated by even the weakest resistor can be large, and for a simple wire all of that energy is going to heat. If the current were allowed to get too large, then, you could light everything nearby on fire. In order to prevent (or at least make less likely) this sort of thing, fuses and circuit breakers were invented. A fuse is a simple device you insert into a circuit which has the property that it will be harmlessly destroyed (rendered non-conductive) by a current above a certain limit. This is often accomplished by have a thin filament which is vaporized by the heat corresponding to too high a current (hence "to blow a fuse"). Fuses must be replaced once their limit has been exceeded a single time. Circuit breakers are more sophisticated devices which are "thrown" by a large current: they function as a switch which will automatically be forced into the off position by a large current. This switch can then be reset once the problem has been solved. This problem could be anything from too many devices plugged into the same circuit (AC in the summer, for example) to a "short" circuit. A short circuit is simply a circuit which has been functionally "shortened" because elements which were not intended to be in conductive contact now are (crossed wires). When this happens, the current will take the shorter, easier route rather than the full circuit. The reduced resistance of the whole circuit will increase the current, thus increasing the power dissipated by each resister still in use.

6.6 Alternating Current

So far we have been talking about circuits in which a *constant* potential difference is being applied. This gives rise to a constant current in one direction, which we call Direct Current (DC). For various reasons, it is often preferred to apply a potential difference which alternates, or oscillates, between a positive and negative value. This gives rise to a current which first flows one way, then the other, and back again. We call this Alternating Current (AC). Some devices or circuits can work with either alternating or direct current (AC/DC). The power delivered to your home is AC. The history behind the fact that the power grid is AC rather than DC is actually fairly interesting and the debate gave rise to the electric chair when Edison tried to give AC (and its proponent Westinghouse) a bad name by publicly electrocuting animals with an alternating current. And you thought modern politics could get out of hand!

A current can alternate in a huge variety of ways, and electrical engineers have uses for all of them, but the current coming into an American home has some standard characteristics and we'll stick to that basic picture. First and foremost, the applied voltage typically varies sinusoidally with time,

$$V(t) = V_0 \sin(\omega t) = V_0 \sin(2\pi f t) \qquad [f] = \frac{1}{s} = Hz = Hertz$$

where f is the number of oscillations per second, or frequency. Different frequencies are used in different places, but the US standard is $60Hz$. Since

Ohm's law works just as well for AC (meaning it still works for the same set of ohmic materials),

$I(t) = \frac{V(t)}{R} = \frac{V_0}{R}\sin(\omega t) = I_0 \sin(\omega t)$.

Power dissipation now also varies with time,

$P = I^2 R = I_0^2 R \sin^2(\omega t)$

but for many purposes we only care about the average power dissipated. We can find the average of the $\sin^2(\omega t)$ function by integrating it. The result is

$\overline{P} = \frac{1}{2} I_0^2 R$,

so the average is just half the peak value.

The average current or voltage is 0. That isn't a very informative quantity, so we instead use the root-mean-square, which is just the square root of the average of the square,

$I_{rms} = \sqrt{\overline{I^2}} = \frac{I_0}{\sqrt{2}}$

$V_{rms} = \frac{V_0}{\sqrt{2}}$.

This can frequently be used where the DC value would be appropriate,

$\overline{P} = I_{rms} V_{rms} = I_{rms}^2 R = \frac{V_{rms}^2}{R}$

6.7 Where it comes from: Microscopic Physics of Current

We already know that electric current is the flow of many microscopic charged particles. In a conductor, these will be electrons. In a gas or liquid or plasma, they could be any charged particle (ion). Living cells often generate currents based on various ions such as potassium, and these can potentially be positive or negative.

The current I refers to the bulk current through a circuit. On a microscopic level, things aren't necessarily so simple and uniform. And even if they are, once we begin treating a wire as a 3 dimensional object we will need to discuss a current *density* in the same way that we have volume charge densities for cylinders rather than just the total charge. For a uniform charge density in a wire, we define

$j = \frac{I}{A}$. $\qquad [j] = \frac{A}{m^2}$

However, neither the density nor its direction needs to be uniform, so we define in general

$I = \int \vec{j} \cdot d\vec{A}$.

This is another instance of a *flux*. This time it is the flux of electric charge through the cross sectional area of a wire.

Thinking for the moment in terms of electron current in a conductor, it is very tempting to simply define the current density in terms of the motion of the electrons. A current is just the movement of charge, so:

$\vec{j}_\rho = \rho \vec{v}_\rho$ $\qquad\qquad \left[\vec{j}_\rho\right] = \frac{C}{m^3}\frac{m}{s} = \frac{C/s}{m^2} = \frac{A}{m^2}$

While this formula is in the strictest sense *correct*, it is further removed from the macroscopic behavior we are interested in than you might realize. The reason is simple: electrons, the charge density above, do not move coherently together in a uniform direction. They are moving very quickly inside of the conductor, but bouncing crazily around off of the conductor walls and the bulk atoms of the conductor. Their average speed is *much* greater than the magnitude of their average velocity because they change directions so frequently. Thus, the $\vec{j}$ as defined above will vary rapidly over both space and time and is useless for most questions.

Instead, we realize that while the individual electrons velocities may be extremely random, the applied voltage and associated electric field will introduce a bias in which direction they randomly bounce. This bias will give rise to an overall average velocity, which we call the *drift velocity*, $\vec{v}_d$. We can use this velocity to define a more useful current density,

$\vec{j} = (-ne)\, \vec{v}_d$ $\qquad\qquad \left[\vec{j}\right] = \frac{1}{m^3} C \frac{m}{s} = \frac{A}{m^2}$

where n is the number of electrons per unit volume and $(-e)$ is the charge of each.

This definition is easy to generalize away from electrons, and we can even use it for collections of different kinds of ions:

$\vec{j} = \sum_i n_i q_i \vec{v}_{di}$

$I = A \sum_i n_i q_i v_{di}$

Note that drift velocities of electrons are $\mathcal{O}\left(5 \times 10^{-5} \frac{m}{s}\right)$ while average velocities are $\mathcal{O}\left(1 \times 10^{5} \frac{m}{s}\right)$. "Electricity", however, travels along transmission lines almost instantaneously because we don't need the specific electrons from the power plant in order to turn on the light, we just need an electric field to move the electrons which are already in the light. The electric field, it turns out, travels at about the speed of light, $c = 3 \times 10^8 \frac{m}{s}$

6.7.1 Electric Field in a Wire

We can think of a wire as an extremely thick parallel plate capacitor in the sense that it will have a constant electric field throughout because of the applied voltage on either end. Thus, we can use the result

$V = E\ell$

inside a wire. Combined with

$I = jA$, $V = IR$, $R = \frac{1}{\sigma}\frac{\ell}{A}$

we find that

$E\ell = \frac{I}{\sigma}\frac{\ell}{A} = \frac{jA}{\sigma}\frac{\ell}{A} = \frac{j\ell}{\sigma}$

$E = \frac{j}{\sigma} = \rho j$

such that we can find a microscopic version of Ohm's law,

$\vec{j} = \sigma\vec{E} = \frac{1}{\rho}\vec{E}$

7: DC Circuits

July 12, 2008

7.1 Batteries: "EMF" and Terminal Voltage

For historical reasons, the source of voltage difference (such as a battery) in a circuit is called the "electromotive force". I say historical reasons because at some point it got that name even tho everyone admits now that it is a terribly misleading name. The electromotive force, which we abbreviate emf and write $\mathscr{E}$, isn't a force at all. It is just a source of electrical potential difference. I believe the word force found its way in because the emf is what *drives* or *forces* current to pass through a circuit.

In any event, a battery is actually composed of 2 parts: an emf $\mathscr{E}$ and an internal resistance r. The external voltage, the voltage between the terminals or the "Terminal Voltage", is then $\Delta V = \mathscr{E} - Ir$. It is important to explicitly include the internal resistance rather than just using the terminal voltage because, for instance, when you charge a battery the effective terminal voltage is instead $\Delta V = \mathscr{E} + Ir$. For most problems, however, we can just use the terminal voltage.

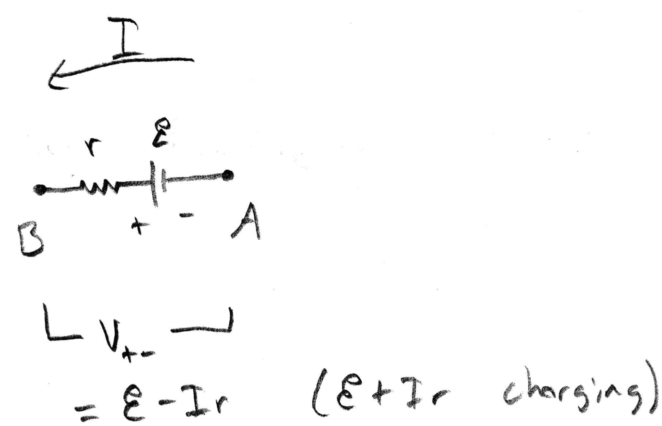

Figure 7.1: Internal Resistance