# 7: DC Circuits

July 12, 2008

## 7.1 Batteries: "EMF" and Terminal Voltage

For historical reasons, the source of voltage difference (such as a battery) in a circuit is called the "electromotive force". I say historical reasons because at some point it got that name even tho everyone admits now that it is a terribly misleading name. The electromotive force, which we abbreviate emf and write $\mathscr{E}$, isn't a force at all. It is just a source of electrical potential difference. I believe the word force found its way in because the emf is what *drives* or *forces* current to pass through a circuit.

In any event, a battery is actually composed of 2 parts: an emf $\mathscr{E}$ and an internal resistance $r$. The external voltage, the voltage between the terminals or the "Terminal Voltage", is then $\Delta V = \mathscr{E} - Ir$. It is important to explicitly include the internal resistance rather than just using the terminal voltage because, for instance, when you charge a battery the effective terminal voltage is instead $\Delta V = \mathscr{E} + Ir$. For most problems, however, we can just use the terminal voltage.

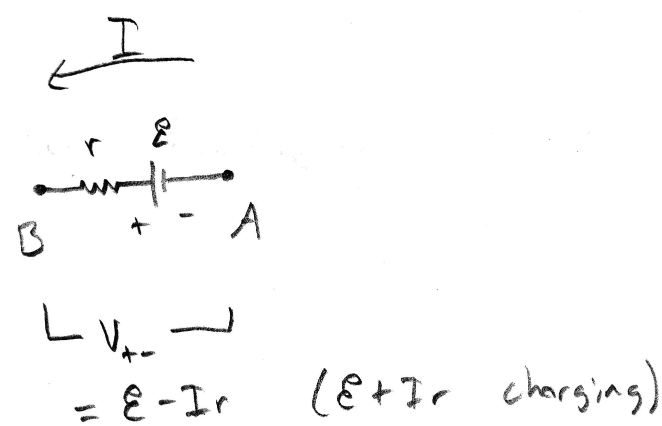


Figure 7.1: Internal Resistance

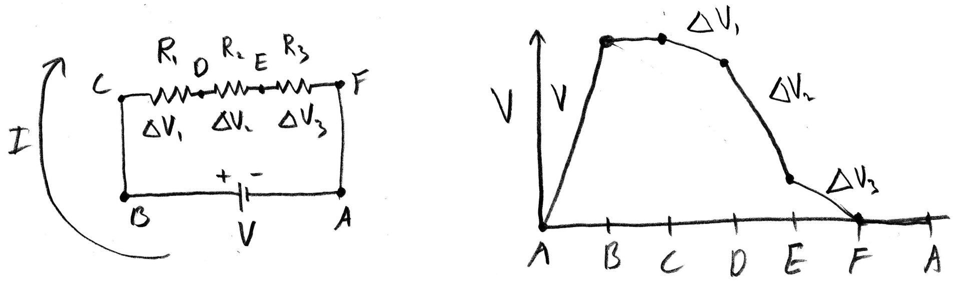


Figure 7.2: Potential drops over Resistors in Series

## 7.2 Resistors

Resistors, like capacitors, can be combined in circuits in two different ways: series and parallel. We cannot, however, assume that resistors add the same way as capacitors do. We have to work it out based on what we know, and will in fact discover that they do not at the same way.

### 7.2.1 Series

When connected in series, we know that the potential differences over the resistors must add up to the total potential difference of the source $V$, and using Ohm's Law we find:

$V = \Delta V_1 + \Delta V_2 + \Delta V_3 = IR_1 + IR_2 + IR_3$

where we have used conservation of charge to identify all three currents. Since charge cannot build up on a resistor and cannot be created or destroyed, the current that enters a resistor must come out, and will enter the next in line and so on and so forth.

$V = I\left(R_1 + R_2 + R_3\right) \longrightarrow V = IR_{equiv}$

$R_{equiv} = \sum_i R_i$

### 7.2.2 Parallel

Connecting resistors in parallel we once again find that they must share a single potential difference,

$\Delta V_1 = \Delta V_2 = \Delta V_3 = V.$

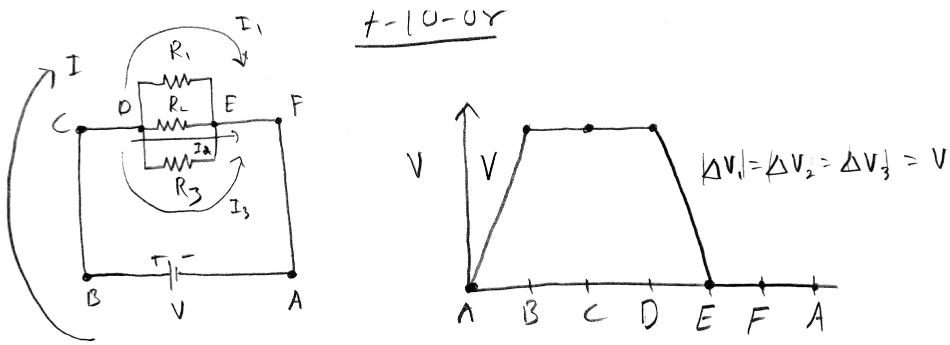


Figure 7.3: Potential over Resistors in Parallel

Keep in mind that we are dealing with magnitudes only.

We do not, however, know what the current through each resistor is. We still have conservation of charge to tell us that the *total* current is unchanged, however:

$I = I_1 + I_2 + I_3$

Lets put these together with Ohm's Law:

$I_i = \frac{V_i}{R_i}$

$I = \frac{V}{R_1} + \frac{V}{R_2} + \frac{V}{R_3} = V\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right) = \frac{V}{R_{equiv}}$

$$\frac{1}{R_{equiv}} = \sum_i \frac{1}{R_i}$$

### 7.2.3 Combinations

Many circuits can be fully understood by iteratively combining resistors in series and parallel combinations. Look at the diagram and search for subsets of resistors which are most obviously in one configuration or another. Find the equivalent resistance for each of these, then redraw the diagram with those subsets replaced by the equivalents you just found. Now, re-examine this diagram for obvious parallel and series combinations. Repeat the process until you've found an expression for the total resistance of the circuit or are convinced that doing so is impossible given the rules above.

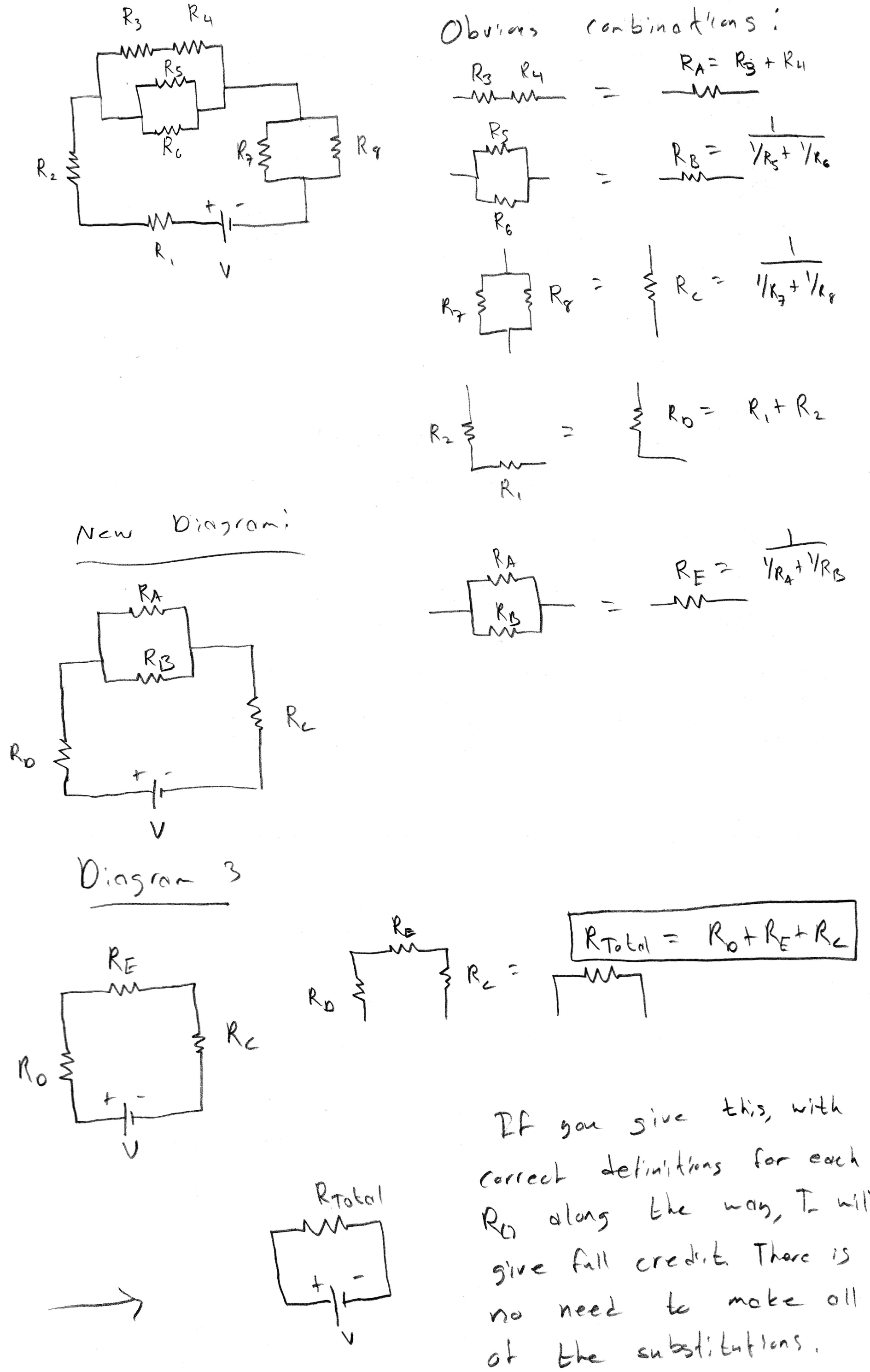




Figure 7.4: Breaking Down a Compound Circuit

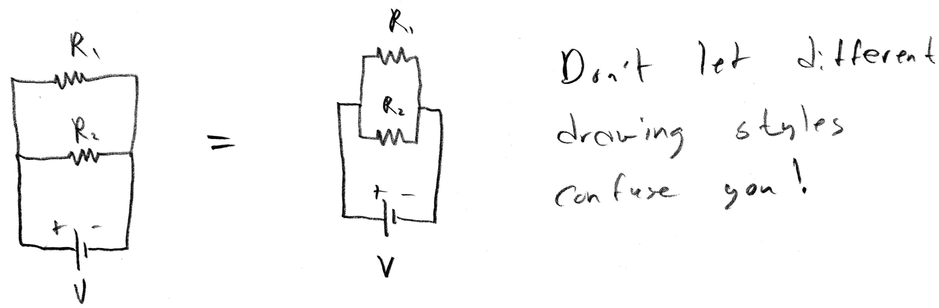


Figure 7.5: Other Drawing Styles

## 7.3 Kirchoff's Rules

Sometimes, particularly if there is more than one source of voltage, it is not only difficult but impossible to reduce an entire circuit by making combinations of resistors in series and parallel. In this case we use a more general application of the laws of conservation of charge and energy. For the series and parallel addition rules, we used these conservation laws for the specific configurations we were looking at, so that when we then recycle the addition rules, we carry along conservation of charge and energy for the ride. However, this restricted application prevents us from getting the most information possible out of the laws. We instead use Kirchoff's 2 Rules:

**Junction Rule:** Conservation of charge. The total current entering a junction must equal the total current leaving that junction.

**Loop Rule:** Conservation of energy. The total change in electric potential around a closed current loop, including potential sources and sinks, must be zero.

We in fact used both of these rules when deriving the addition laws for capacitors and resistors, tho not stated quite this way. The trick with kirchoff's rules is not the rules themselves, but how to apply them consistenty to circuits.

Follow these steps to use Kirchoff's rules:

0 (Optional): Do as much simplification as possible using the addition rules we learned earlier. These are simpler to use than kirchoff's laws, and can much reduce the number of junctions and loops you need to deal with using kirchoff.

1. Label currents. Pick one current per length of wire. You only need to label a new current when you reach a junction: resistors, batteries, and capacitors cannot change the current (no charge buildup).

2. Identify your unknowns. To be Rumsfeldian, Know your Unknowns. You will need as many *independant* equations as there are unknown quantities you are searching for.

3. Apply junction rule: Balance the currents at a junction or three. To start with, you'll guess wrong as to which junctions are most useful and independant, but do a bunch of problems and you'll start to see how to pick the most efficient choices earlier or even straight away. Note while using the junction rule that it simplifies the bookkeeping (keeping track of variables and their signs) if we add up the currents as either *all incoming* or *all outgoing* by applying appropriate signs, and set this quantity to 0.

4. Apply loop rule: There will be some choices of which loops to use that will be easier than others. Sometimes you need to use them all. Remember when using this rule to put in appropriate signs for resistors and voltage sources. The important thing is that resistors and batteries/voltage sources have opposite sign when going around the loop in the same direction, and to stay consistent within each loop. Also: you generally want to chose loop such that at least one of your loops passes through every circuit element! Otherwise, you aren't going to be able to use that information on your calculation and won't be able to find all of the answers.

5. Solve the equations you've written down in 3) and 4). You may find you either have more equations than you need, in which case the leftovers should reduce to something trivial (like $1 = 1$ or $0 = 0$ or $I_1 = I_1$ or...).

If you find one of your equations gives a result which is manifestly false ($1 = 0$, $I_1 = 2I_1$, etc.) then you have made a mistake somewhere. These extra equations can be a good consistency check to locate errors. You may also find you don't have enough equations if you accidentally picked 2 redundant equations in 3) and 4). Its easy to do; don't get discouraged, just try and identify another junction or loop that seems to involve just one of your remaining unknowns and which you haven't used yet.

### 7.3.1 Examples

See Worked Examples section of webpage.

## 7.4 RC Circuits

When we discussed capacitors before, we only specified how they act in a steady state, where they have been sitting in a circuit for an arbitrarily long time. The combination of resistance and capacitance allows us to look at the time dependent case. Let us start by using the one think we know already about all of our circuit elements: their potential change. In a circuit with an $\mathscr{E}$, 1 resistor $R$ and 1 capacitor $C$ in series,

$\mathscr{E} = IR + \frac{Q}{C}$.

Note that this follows from Kirchoff's Loop Rule, which is an expression of conservation of energy and thus holds regardless of whether the circuit elements are resistors or capacitors. Since we want to look at the behavior over time, make the substitution:

$\mathscr{E} = \frac{dq}{dt}R + \frac{q}{C}$.

Note that I have changed to lower case $q$ as usual when treating charge as a variable rather than a set quantity. The text keeps capital $Q$. The meaning is the same. Rearranging the above we find

$\mathscr{E} - \frac{q}{C} = \frac{dq}{dt}R$

$\frac{dt}{RC} = \frac{dq}{\mathscr{E}C - q}$

This is a simple *differential equation* and we can solve it by just integrating both sides:

$\int_0^t \frac{dt'}{RC} = \int_0^q \frac{dq'}{\mathscr{E}C - q'}$

$\frac{1}{RC}t = -\ln\left(C\mathscr{E} - q'\right)|_0^q = -\ln\left(C\mathscr{E} - q\right) + \ln\left(C\mathscr{E}\right) = \ln\left(\frac{C\mathscr{E}-q}{C\mathscr{E}}\right) = -\ln\left(1 - \frac{q}{C\mathscr{E}}\right)$

$\ln\left(1 - \frac{q}{C\mathscr{E}}\right) = -\frac{t}{RC}$

$e^{-t/RC} = 1 - \frac{q}{C\mathscr{E}}$

$q\left(t\right) = C\mathscr{E}\left(1 - e^{-t/RC}\right)$

$q\left(t\right) = C\mathscr{E}\left(1 - e^{-t/\tau}\right)$ where $\tau = RC$ is the time contant of the circuit. It gives the characteristic time scale over which things change.

We can use the definition of current to find its time dependence:

$I\left(t\right) = \frac{dq}{dt} = \mathscr{E}\frac{t}{R}e^{-\frac{t}{\tau}}$

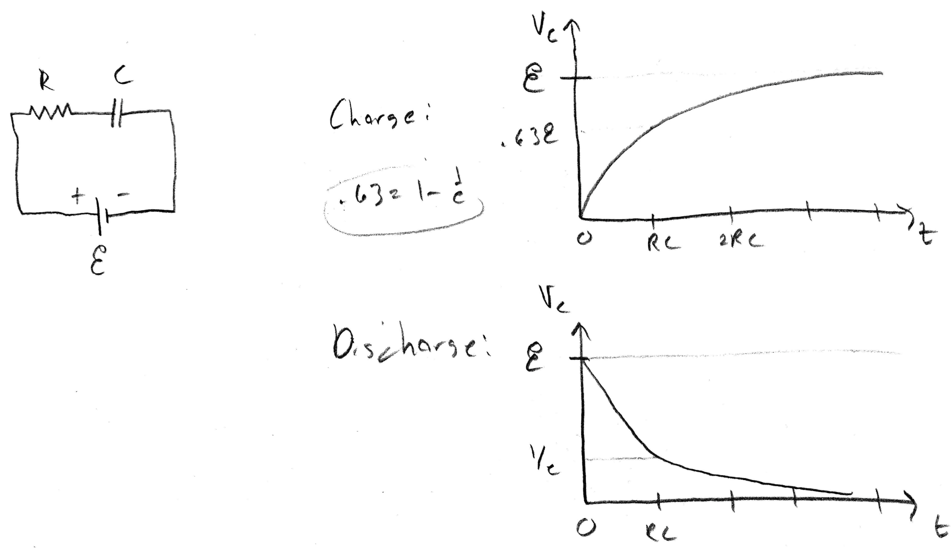


Figure 7.6: Capacitor Charging and Discharging

The math here is fairly straightforward. The physics underlying it is also relatively simple. We know that like charges repel and so it takes energy to build up a collection of them. We also know that the more charges in a given volume, the harder it is to add the next one. So, the more charge we add to the capacitor, the more slowly the $\mathscr{E}$ can force charge into it, since it applies a constant potential difference and the capacitor is slowly canceling that potential difference as it is charged up. This sort of rate problem leads time and time again to exponential behavior in physics, learn to expect it.

The discharge is similar, but the opposite logic applies. With nothing holding the charge on the capacitor, it will start pushing its charge out along the wire. However, the more charge it loses, the less the charges are repelling one another and the more slowly charge is expelled.