Part II
Magnetostatics

8: Magnetic Phenomena

8.1 History

8.1.1 Relation to Electrostatics

Magnetic phenomena are a lot less common in nature than electrostatic effects. There are lots of ways to accidentally build up a static electric charge and it isn't hard to notice that something strange is happening. By contrast, it takes deliberate effort to produce magnetism where it didn't already exist, and there aren't a lot of naturally occuring magnets. Think about your own experience: when was the last time you happened to notice something behaving in a strange way that could have been a magnet, other than a man-made object? Probably never. On the other hand, electrostatic shocks can be a daily experience. Of course, these are also made more frequent by the proliferation of different fabrics in modern life, but with furs and such lying around for all of human history, someone was bound to notice the buildup of charge pretty quickly. Despite all of this, it turns out that electricty and magnetism were first studied seriously (at least as far as we know) around the same time in Greece. Electricity was named by the Greeks after amber because amber would readily build up electrostatic charge, while magnetism was named after the province (Magnesia) where magnetite (an inherrently magnetised iron ore) was found.

We study electricity and magnetism together because we now know that the two are intimitely connected phenomena and even understand (just wait!) what that connection is. But even in the ~600s BCE, it was clear that the two were similar and attempts were made to relate the two (unsuccessfully). It was only after the development of controlled electrical currents that scientists were able to understand the connection between the two.

8.1.2 Permanent Magnets

In any event, there *were* magnetised objects occurring naturally. Lodestones and magnetite are obvious examples, but it is also possible to produce a magnet accidentally in the course of working with metal. Rubbing two pieces of iron together in the same orientation for a very long time will slowly magnetise them both. Apparently you can use this to build a compass in an emergency, but I'd suggest just packing one for your trip instead.

It was realized pretty pretty quickly that magnets have the special property of reliably preferring to point (roughly) north. This is of course the entire basis for the functioning of a compass. This is also the origin for much of the nomenclature associated with magnetism.

8.2 Behavior

8.2.1 Permanent Magnets

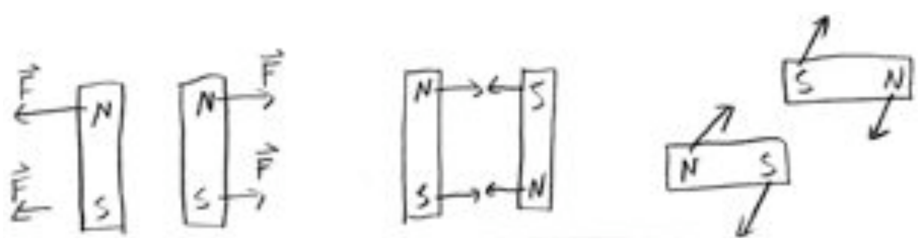


Figure 8.1: Opposites Attract

Every magnet has 2 different "ends", known as *poles* because of their tendency to orient towards the earth's poles. This tendency led to the obvious choice of names: the *north* pole of a magnet is the end that points toward the north pole of the earth, the *south* pole... points the other way. However, this choice leads to a confusing fact. If you take two magnets and identify the north pole of both (by testing where they point when allowed to rotate freely), you will find that when you bring the two together, the north poles repell, while the north poles attract the southern poles. Like repells like and opposites attract (as with electric charge). Following this logic, then, we find that what we have cleverled labeled the “north” pole of our magnet must be attracted to a “south” pole: the North Pole is magnetically south! If you thought it was a little odd that electrons ended up with a negative charge, screwing up our definition of current, this should put it into perspective. Of course, nobody refers to the north pole of the earth as the south magnetic pole. We use words like “geomagnetic north”, which means “we know its the north pole but the magnet actually isn't north and we're just calling it north anyway”.

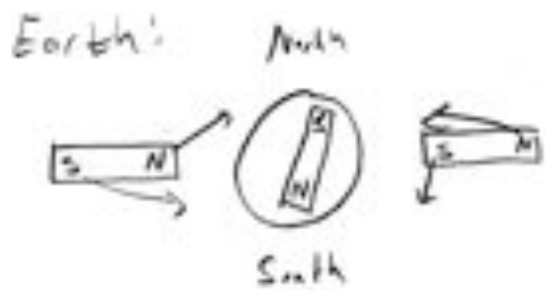


Figure 8.2: Magnetic North is... Magnetic South

Monopoles

There is no such thing as a magnetic monopole. If you separate a north and a south pole, you just end up with 2 of each. This entirely unlike electrostatics. It is possible to construct an electric dipole, but it is made of 2 opposite charges which can exist and act independently. There simply isn't an analog in magnetism, despite extensive searches by very serious people for a very long time and some pretty interesting ideas about why they *should* exist. This is an idea that I imagine won't ever die, and could even be correct, but has no real support at the current time.

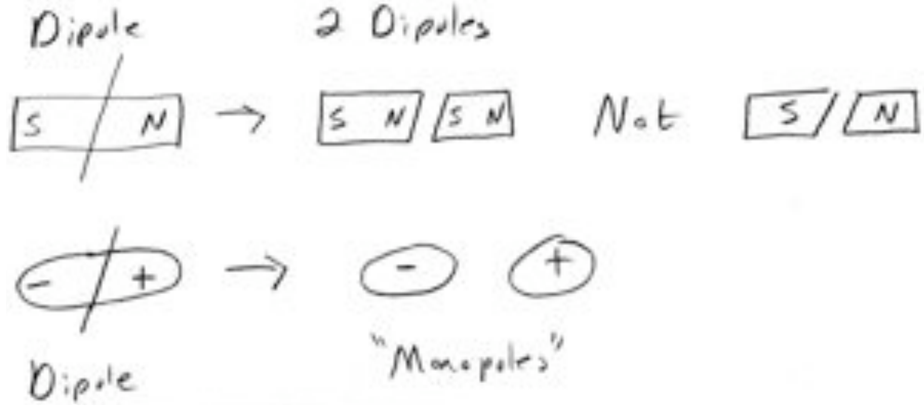


Figure 8.3: No Magnetic Monopoles

Magnetic Fields

We aren't yet ready to define a magnetic field, but even casual experimentation with magnets make a field seem even more real than in the case of electric fields. Holding two reasonably strong magnets and trying to push the like poles together gives a definite sense that *something* is in the way. You can even feel out the shape of the field lines to a certain extent.

Magnetic field lines are similar to electric field lines but also quite different. They represent a vector field, as do electric field lines, but the meaning of their direction is less straightforward and we must wait to define it. They also, unlike electric field lines, have no starting and stopping points. Magnetic field lines form continuous loops that *exit* from a magnet at the north pole and *enter* at the south pole, but continue inside and in this region travel from south to north. This is actually the same as saying that there are no magnetic monopoles. You can split an electric dipole by separating the sources of field lines from the sinks, but with a magnet there are no ends, so sources and no sinks. We denote the magnetic field as $\vec{B}$. I actually have no idea why we use the letter B for the magnetic field, but everyone does.

Magnetic field lines, like electric field lines, cannot cross, and will try and evenly distribute themselves. The field of a bar magnet looks much like the field of an electric dipole for good reasons, even tho they have different

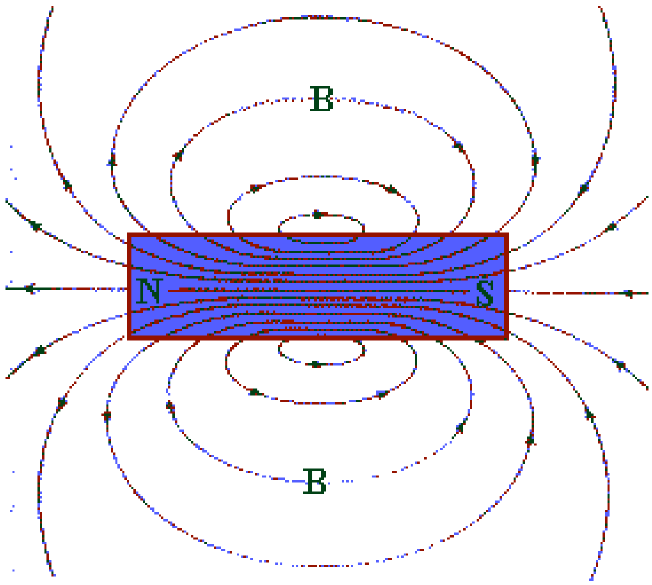


Figure 8.4: Field of a Bar Magnet

behavior, as we will see. Since magnetic field lines emerge from N poles and enter into S poles, it is possible to construct a region of uniform magnetic field in much the same way as we can use two charged plates to construct a uniform electric field. The positively charged plate will emit lines which the negative plate receives: likewise, lines (externally) will flow from N to S.

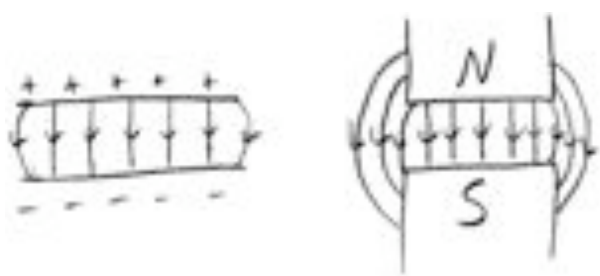


Figure 8.5: Uniform Magnetic Field

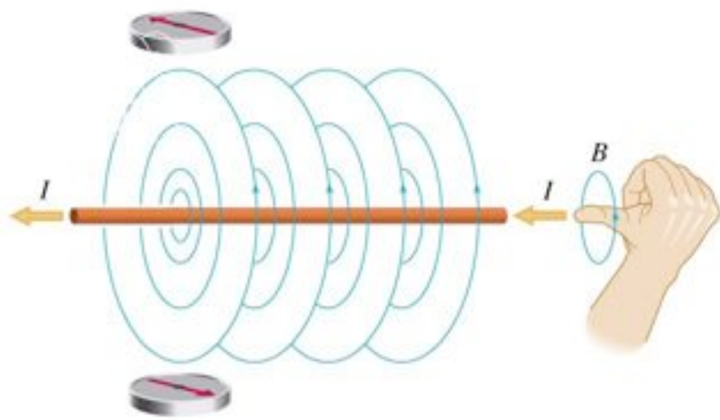


Figure 8.6: Magnetic Field from a Straight Current

8.2.2 Electromagnets

With the invention and relatively easy access to controllable and known electrical currents, it was possible to discover and study the effect by which moving electrical charges (currents) produce magnetic fields. The resulting fields are simple, but unlike what we have seen so far. Rather than in any way emitting from the wire, the field lines will form perfect concentric circles around a straight wire. If you bend the wire (into a loop, for instance), then these circles will distort against one another and change a bit, but close to the wire you always have perfect little circles.

So there are a few strange things about these fields already. First, they aren't produced by a charge of any sort, but the motion of a charge. This is stranger than the difference between gravitation and electrostatics in a sense, because there all we did was change what the "charge" was, and we were in business. The force formula was even the same! Now we have to worry about charges movement. Not only that, however. Notice that the direction of flow of (positive) charge is along the current carrying wire, while the field is going *around* the wire. Not towards, or away, or with, but *around it*. This means that the field is at a right angle to the direction of flow of the current. If it had been radially outward, we could have convinced ourselves that the charge for magnetic fields is just moving charge, and then everything else would be the same. But alas, things are not so straightforward. Hang in there, however, because there is a charge analogy to be made later down the road.

I specified that the field goes around the wire. Around which way? Magnetic fields obey various right-hand-rules, and the first is that if you have a current, you can find the directon of the associated field by aligning your (right) thumb along the current and curling your fingers. The direction of curl will indicate which way the field goes. At this point you should ask yourself where else you've seenthe RHR used, and the answer is: anywhere a vector (cross) product was lurking (such as torque). Sure enough, we are zeroing in on a cross product, but we aren't there yet.

8.3 Forces

8.3.1 Force Exerted by a Magnetic Field on a Length of Current

We saw that magnets exert a force on one another (tho we didn't fully define what it was), and we've seen that electric currents produce magnetic fields. It should not then be a surprise to find that magnetic field exert a force on currents. What may be a surprise, however, is that the force applied is not attractive or repulsive but rather orthgonal. A current bearing wire passing through a magnetic field will feel a force which is perpendicular to both the direction of current flow, and the magnetic field direction. Three mutually orthogonal related vectors is another good sign that the three are connected by a cross product, and in fact once again we use the right hand rule to identify the relationship between the directions of the problem. There is a special memnonic for the right hand rule when used for force, magnetic field, and current based on the acronym FBI. Put your thumb, pointer, and middle finger in a mutually orthogonal position. Now in order your middle finger is the force ($\vec{F}$), pointer is magnetic field ($\vec{B}$) and your thumb is in the direction of current ($\vec{I}$). The acronym is in the order of your fingers, starting in the middle with F. Alternatively (and resulting in an alternative set of assignments, so pick one way and stick with it!) we can use the standard technique based on the order of a cross product once we define the force equation. While these 2 methods will assign a different quantity to different fingers, the resulting relationship is the same, so either one works but you are probably better off sticking with one way to remember.

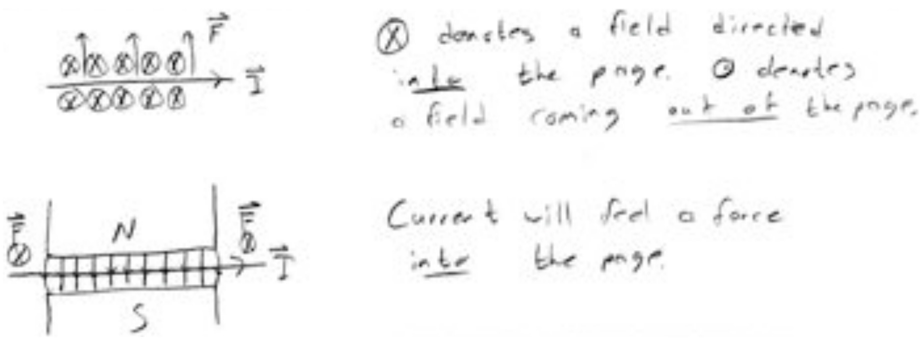


Figure 8.7: Force on current due to magnet

Experimentation allows us to fully identify the form of the force felt by the wire, which isfound to vary with the angle between the current and field as

$\vec{F} = I\vec{\ell} \times \vec{B} = I\ell B \sin\theta_{\ell B} [F] = \frac{C}{s} mT$ where $T = N\frac{s}{mC} = \frac{N}{A \cdot m}$ is a Tesla.

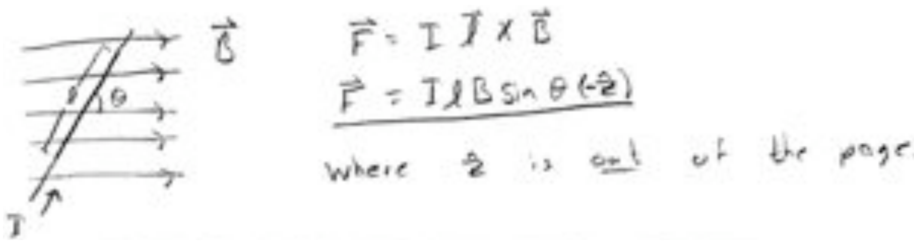


Figure 8.8: Magnetic Force on a Current

where $\vec{\ell}$ is the wire segment within the field. The field was initially defined by this relation, tho we find a more fundamental definition later. The strength of the force is proportional to the length of the wire, and this formula can be applied for curved or non-uniform currents by, as usual, turning things inter differential elements:

$d\vec{F} = I d\vec{\ell} \times \vec{B}$.

This is differential force element due to a small piece of wire. We could also find the force element on a wire by a differential element of field:

$d\vec{F} = I\vec{\ell} \times d\vec{B}$.

This is actually more analogous to what we would use for Coulomb's law but isn't useful until we define a way of calculating the field $\vec{B}$. We won't do this for a little while yet, but there is plenty we can do without such a defnition.

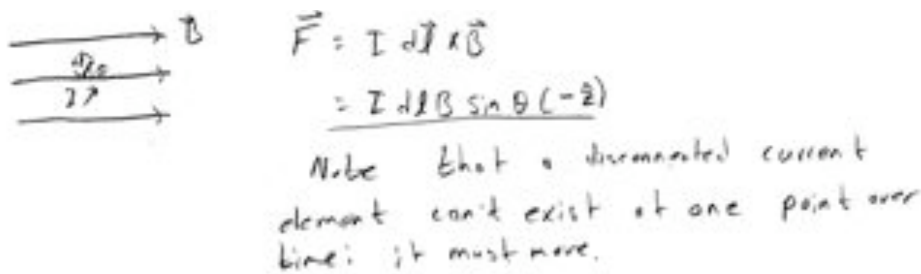


Figure 8.9: Force on current in uniform magnetic field

8.3.2 Force on a Single Moving Charge

We know from our study of circuits that a current is just a collection of moving electric charges. If this leads you to expect that individual moving charges also feel a force due to a magnetic field, you are correct. We can find the force on a single charge simply by dividing the force on a current up amongst all of its constituent charges. Or, equivalently, we can define a single moving charge as a tiny current $\vec{I}_q = q\vec{v}$.

$\vec{F} = q\vec{v} \times \vec{B}$

The force on a wire with current flow is actually the sum of the forces on all of the moving electrons inside the wire. Because the electrons cannot escape the wire, they keep getting pushed against the sides in the direction of the force applied by the magnetic field, and this pushing translates into a force on the wire as a whole. When a particle passes through an electric field on its own, however, there is nothing to push back and keep it going in a straight line. Because of this, a particle passing through a magnetic field will be deflected in a direction determined by the right-hand-rule. However, once the particle is deflected, the force it will experience is in a different direction because of the cross product. This will cause the particle to be deflected further, which results in a new direction of force, and so on and so forth. It is in fact not possible to construct a magnetic field which will only exert a force on a particle in a single direction, as it is with the electric field. The dependence upon $\vec{v}$ means that the force will always depend on how the particle is moving, and since the force will *change* how that particle is moving, charged particles never travel in straight lines through a magnetic field. This is manifestly different from electric or gravitational forces. It is possible to have curved trajectories in either of those fields, but not *necessary*. It is quite possible to fall straight down, and two opposite charges at rest will attract one directly together in a straight line. This simply cannot happen to a free charged particle in a magnetic field. By restricting a charge to a wire, we can force it into a straight line, but even the wire will try to bend and move too accomodate the flowing electrons need to travel in a curve.

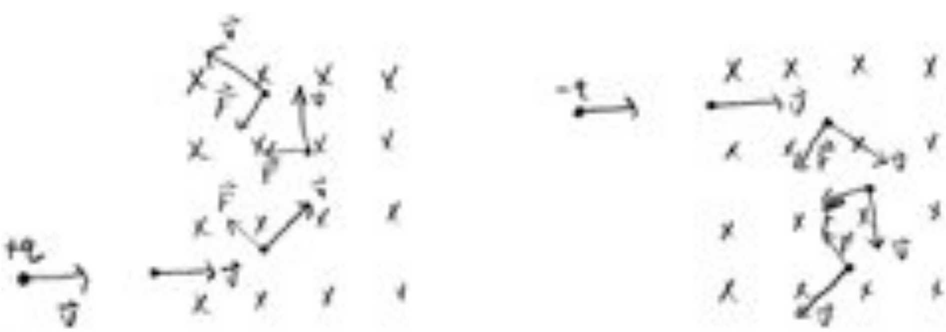


Figure 8.10: Magnetic force on a moving charge

In the special case of a constant magnetic field, a charged particle will travel in a closed circle. This is because it will feel a constant force perpendicular to its direction of motion, exactly as a ball on a wire or any other forced circular motion in mechanics.

8.3.3 Examples

We can calculate forces on currents and moving charges using the equations above. As with electric fields, the principal of superposition still applies: if we have multiple magnetic fields, the resulting total force will just be the sum of the forces due to each field. Also, in the case of currents we need to consider forces on different pieces of a current carrying wire. In our first example, there are 3 sections of wire in a uniform magnetic field. Each segment feels a different force and must be calculated separately and then added together.

Wire loop inserted into a uniform magnetic field

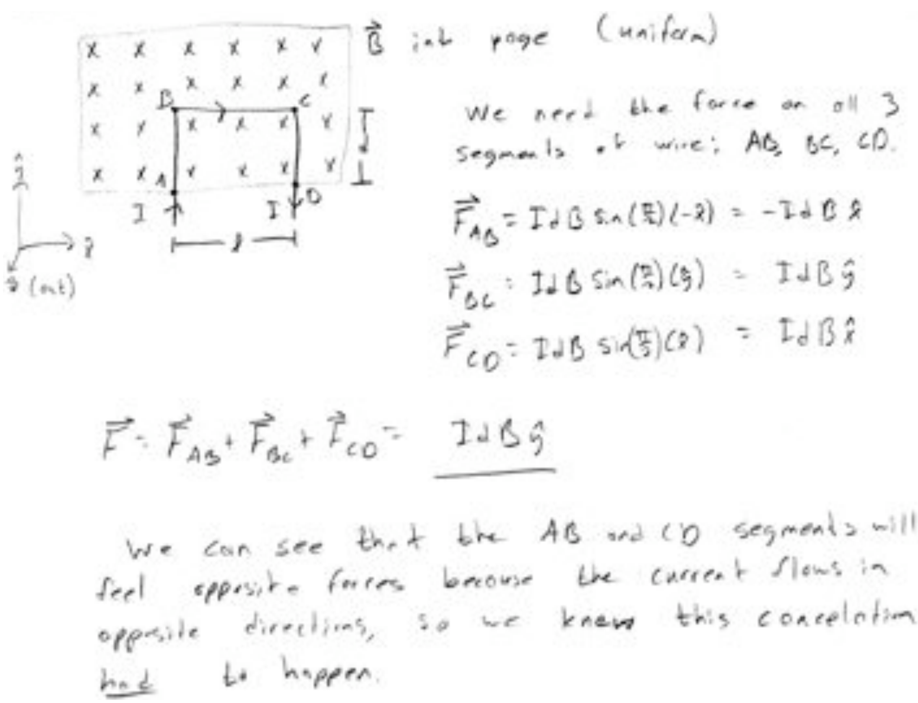


Figure 8.11: Force on curent loop partially inserted into magnetic field

We can measure the strength of a magnetic field by inserting a known current of known length and direction into the field. Note that in this particular example we allow 2 forces which act in different places to cancel. This is correct, the total force does become 0, but if the loop were not already perpendicular to the field, this would create a torque (as there is whenever two forces act in opposition from different points on the same object).

Curved wire in uniform magnetic field

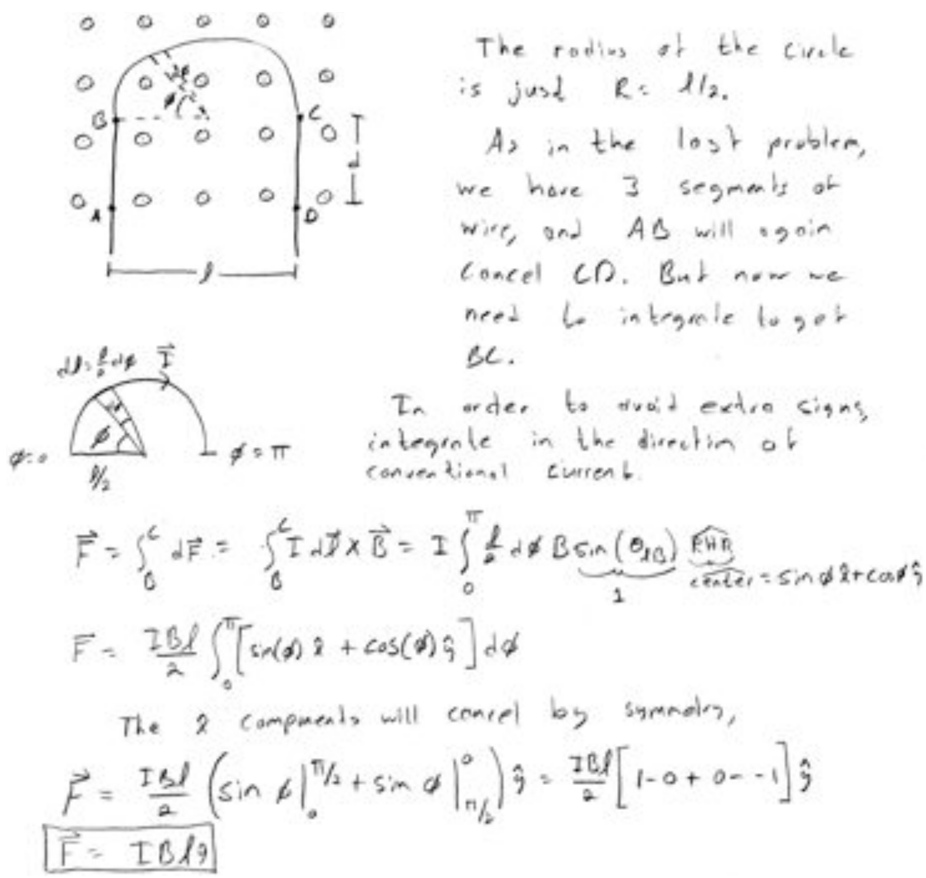
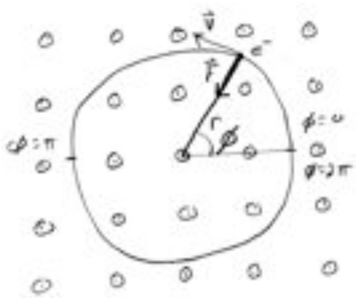


Figure 8.12: Measuring $\vec{B}$ with a curved current loop

This is the same a the last situation, except we are insering a curved section of wire instead of a straight one. This requires us to do an integral over the angle of the semicircle of the current loop. Notice, however, that the force is unchanged by this modification. The force is acting on components of charge from left to right to produce the $\hat{y}$ component of force, and there is just as much of this in one case as the other. The $\hat{x}$ component just cancels out by symmetry.

Single Charge in a uniform magnetic field



Because we are doing this problem w/ an electron, the rotation is opposite the direction of the "current".

We include this sign in RHR

$\vec{F} = -e\,\vec{v} \times \vec{B}$

$\vec{F} = evB\sin(\pi/2)\left[-\cos(\phi)\hat{x} - \sin(\phi)\hat{y}\right]$

$= -evB\left(\cos(\phi)\hat{x} + \sin(\phi)\hat{y}\right)$

$\underline{\vec{F}_B = -evB\,\hat{r}}$

At any given moment, the force on a point charge is simply given by the formula above,

$\vec{F} = q\vec{v} \times \vec{B}$.

There isn't much more to it. However, if we wish to calculate the trajectory a charged particle takes, we would use the equation

$d\vec{F} = q\vec{v} \times d\vec{B} \longrightarrow d\vec{a} = \frac{d\vec{F}}{m}$

and use it in the expression for the arbitrary position of a particle under a force,

$\vec{r}\,(q,T) = \int_0^T \left(\vec{v}_0\,(q) + \int_0^t d\vec{a}\,(q)\right) dt.$

However, there really isn't anything particularly interesting to be gained by wading through the application of this bit of nested calculus. I mention it just to point out that it is doable and to give a sense of the added complexity of dealing with magnetic forces in general. The electrostatic analog would look the same, except that the second integral would just be some number, From mechanics, we know that a particle in circular motion has a centripetal acceleration

$a = \frac{v^2}{r}$, associated central force $\vec{F}_c = -m\frac{v^2}{r}\hat{r}$

Since it is the magnetic force keeping the electron moving in a circle,

$$\vec{F}_B = \vec{F}_c \Rightarrow e\cancel{v}B = m\frac{v^{\cancel{2}}}{r}$$

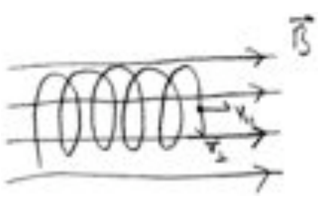
$$\boxed{\frac{v}{r} = B\frac{e}{m}}$$

Depending on what we are given and asked for, we can solve this for whichever is unknown. $\frac{e}{m}$ for an electron is a known physical constant, so we need 2 of v, r, B to find the 3rd.

or a very simple function, rather than an integral.

The only case of single particle motion we want to consider carefully at this point is a charge in a uniform field. In this case, we have already argued that the motion will be circular. Using what we already know, we can get more specific.

Helical motion of a charged particle



Positive Charge with component of velocity $v_\parallel$ to $\vec{B}$ as well as circular $\vec{v}_\perp$ motion.

Figure 8.13: Helical trajectory

If a particle has components of motion both parallel to and perpendicular to an external magnetic field, its motion will be circular in the plane perpendicular to the magnetic field, while traveling unaffected in the parallel direction. The path traced out by such motion is a single helix.

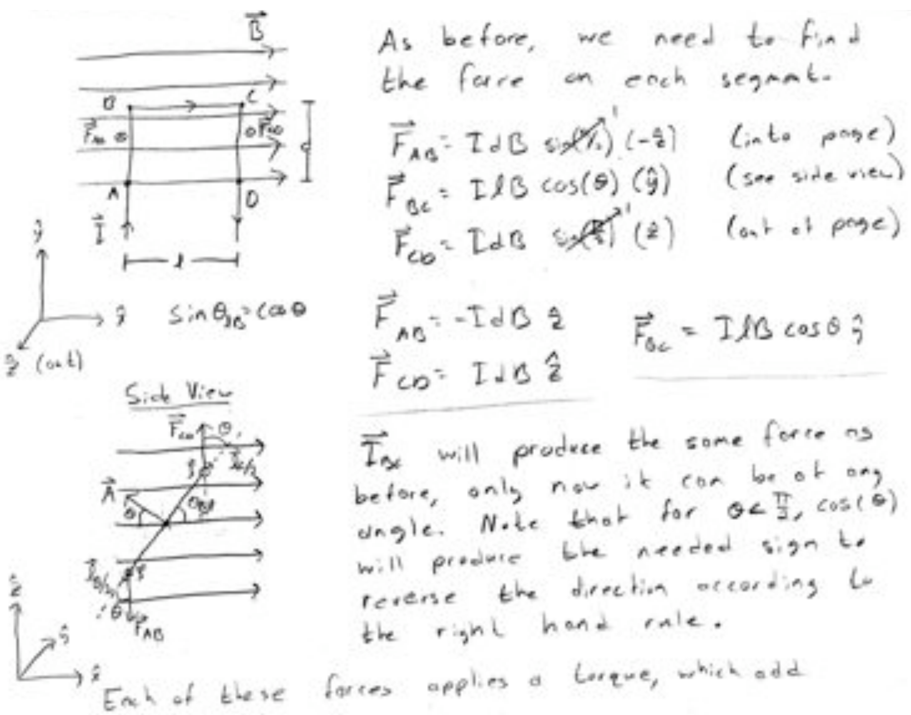
Aurora Borealis

Figure 8.14: Aurora Borealis: Helical motion at work!

The Aurora Borealis (and their southern counterpart the Aurora Australis) are perhaps the most naturally beautiful examples of electric or magnetic phenomena. Charged particles from the sun ("Solar Wind") get caught by the earth's magnetic field and then spiral down to the surface (much like the helical example above) until they reach the atmosphere. Once these charged particles (which have been accelerated by the magnetic field) hit atoms in the atmosphere, they heat it up and cause it to glow in a variety of colors and patterns.

8.3.4 Torque

If we repeat the example of inserting a current loop into a uniform magnetic field, but this time insert the loop rotated 90 such that the end segment $\left(\vec{\ell}\right)$ is parallel to the magnetic field, we find that the forces on the other two lengths exert a torque on the loop. This is analogous to the case of an electric dipole in a constant electric field feeling a torque around its center.



Just like in the case of an electric dipole, a magnetic dipole in a field has an associated potential energy given by the torque and angle through which it must be rotated:

$U = \int \tau d\theta = NIAB \int \sin\theta d\theta = \mu B\left(-\cos\theta\right) + U_0$.

If we chose the 0 of our potential to be at $\theta = \frac{\pi}{2}$, this becomes

$U = -\vec{\mu} \cdot \vec{B}$

which we can readily compare to the potential energy of an electric dipole in a constant electric field $\vec{E}$:

$U_E = -\vec{p} \cdot \vec{E}$.

Each of these forces applies a torque, which add

$$\vec{\tau} = \vec{\tau}_{AB} + \vec{\tau}_{CD}$$

$$\vec{\tau}_{AB} = \frac{\vec{l}_B}{2} \times \vec{F}_{AB} = \frac{I l d B}{2}\left(\hat{l}_B \times \hat{z}\right) = \frac{I l d B}{2}\sin(\theta)\ (-\hat{y})$$

$$\vec{\tau}_{AB} = \frac{I l d B}{2}\sin\theta\,(-\hat{y})$$

$$\vec{\tau}_{CD} = \frac{\vec{l}}{2} \times \vec{F}_{CD} = \frac{I l d B}{2}\left(\hat{l}_c \times \hat{z}\right) = \frac{I l d B}{2}\sin(\theta)\ (-\hat{y})$$

$$\vec{\tau}_{CD} = \frac{I l d B}{2}\sin\theta\,(-\hat{y})$$

$$\vec{\tau} = \vec{\tau}_{AB} + \vec{\tau}_{CD} = I d l B \sin\theta\,\hat{y}$$

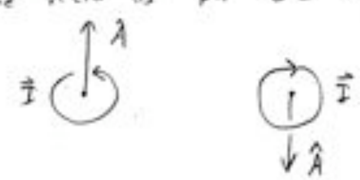
If we make the identification that $dl = A$, we arrive at the more general

$$\boxed{\vec{\tau} = -IAB\sin\theta\,\hat{y}}$$

We can also define an area vector $\vec{A} = A(-\hat{z})$ which points out of the face of the loop. This lets us write:

$$\boxed{\vec{\tau} = I\vec{A} \times \vec{B}}$$

This will only works if we define the direction of $\vec{A}$ such that the current flows around the vector as per the right hand rule. Meaning,



Often, we will loop a wire multiple times. In this case,

$$\boxed{\vec{\tau} = NI\,\vec{A}\times\vec{B}}.$$

Because wire loops & their analogues are so extremely prevelent, we define

$$\boxed{\vec{\mu} = NI\vec{A}}$$ Magnetic Dipole Moment

so then

$$\boxed{\vec{\tau} = \vec{\mu}\times\vec{B}}$$