9: Sources of Magnetic Fields

July 17, 2008

9.1 Straight Wires

9.1.1 Fields

We learned earlier that current-carrying wires produce a magnetic field, but we didn't get very specific. The right hand rule gives us orientation, but we entirely neglected the strength other than mentioning as an aside that it gets weaker further away, somehow.

Well, this time lets go there.

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\widehat{RHR}$ $\qquad [B] = T = Tesla$, $[\mu_0] = \frac{T \cdot m}{A}$

μ_0 is another constant like ϵ_0 but for magnetism and called the permeability of free space. As opposed to permittivity. Don't worry about it, its just the new constant. This equation defines our unit of magnetic field, the Tesla. A 1 Tesla magnetic field is ridiculously strong. The only case where you might have come in contact with any field at the Tesla level would be if you've had an MRI or related magnetic imaging. Fields of this level are really hard to create and maintain but they can do crazy stuff like align all of the protons in your body so we can take cool pictures of your insides. Physics is useful!

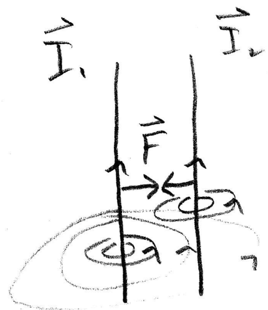
Once again, the principal of superposition applies and fields from multiple wires can be constructed by adding:

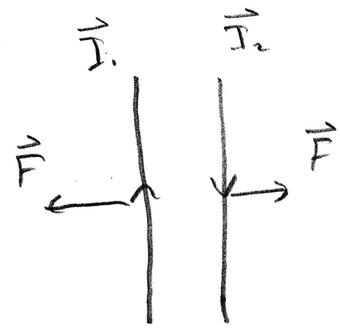
$\vec{B} = \vec{B}_1 + \vec{B}_2 + \cdots$

Note when using this that the field from each wire is going to have a different $\widehat{RHR}$ at each point. At some points these will add simply (if both $\widehat{RHR}$ point in the pure $\hat{x}$ direction at that point, for instance), while elsewhere it won't be so simple. This is no different from having to be careful about how you add electric fields from different charge distributions from different coordinate systems, except that everything dealing with magnets is curved and harder to picture.

9.1.2 Interaction Force

If a current feels a magnetic force, and another wire generates a magnetic field, two wires must do something. We can use the right hand rule and our knowledge of forces on currents to discover the direction. Take 2 parallel wires. Wire 1 creates field $\vec{B}_1$. Aligning our hand appropriately along wire 2, we find that the force must be *towards* the other wire. *Un*like in electrostatics or the poles of permanent magnets, like attracts like. If we flip either current, the right hand rule tells us that the wires repel. Also, we can use the right hand rule on the other wire to find that, as they must, the forces are in opposite directions (that Newton knew his stuff).





But we can do better now. Remember that the force on a wire due to a magnetic field was

$\vec{F} = I\vec{\ell} \times \vec{B}$

and now we have an expression for $\vec{B}$,

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\widehat{RHR}$

Note that the currents appearing in these equations *are different currents.* We could write more explicitly

$\vec{F} = I_{subject}\vec{\ell} \times \vec{B}$

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I_{actor}}{r}\widehat{RHR}$

$I_{subject}$ is the current which is feeling the force, while I_{actor} is the current producing a field. So if I have 2 wires with currents I_1, I_2, I_1 produces a field $\vec{B}_1$ which will be felt by I_2:

$\vec{B}_1 = \frac{\mu_0}{2\pi}\frac{I_1}{r}\widehat{RHR}_1$ $\qquad I_1 = I_{actor}$

$\vec{F}_2 = I_2\vec{\ell}_2 \times \vec{B}_1 = I_2\left(\frac{\mu_0}{2\pi}\frac{I_1}{r}\right)\vec{\ell}_2 \times \widehat{RHR}_1 = \frac{\mu_0}{2\pi}I_1I_2\frac{\ell}{r}\left(-\hat{r}_{12} = \hat{r}_{21}\right)$ $\hat{r}_{12}$ points from 1 to 2

with $\vec{F}_1$ just being in the opposite direction.

9.2 Ampère's Law

The unnamed force law between straight current carrying wires is great, but of limited utility in the real world, which tends to be filled with currents which don't flow in infinite straight wires. Luckily, we can follow a variation on our standard "break it up into little pieces and make an integral" strategy to arrive at Ampère's Law. You know it has to be more useful because it has a whole name associated with it!

9.2.1 Gauss's Law Redux

Motivation

Ampère's Law is based on a similar idea to Gauss's law, but its different. In order to understand what Ampère's Law tells us, lets backtrack for a bit and talk about Gauss's law again. When we discussed Gauss's law, we never really motivated what we were doing. One day I came into class and started talking about "Electric Flux" and we talked about what that was and how to calculate it, but we never told you why we cared. Somehow we defined the flux, we did some calculus, and we ended up with Gauss's law relating some integral we don't actually know how to do most of the time and the charge inside some imaginary surface. And yet somehow we use this law to find the electric field and, if things are working properly, we don't actually ever do an integral or talk about flux. What was the point of those pieces of the problem?

Geometry

Gauss's law is at its heart a very simple statement about geometry. That statement is the following:

1. IF

(a) I have:

  i. some stuff

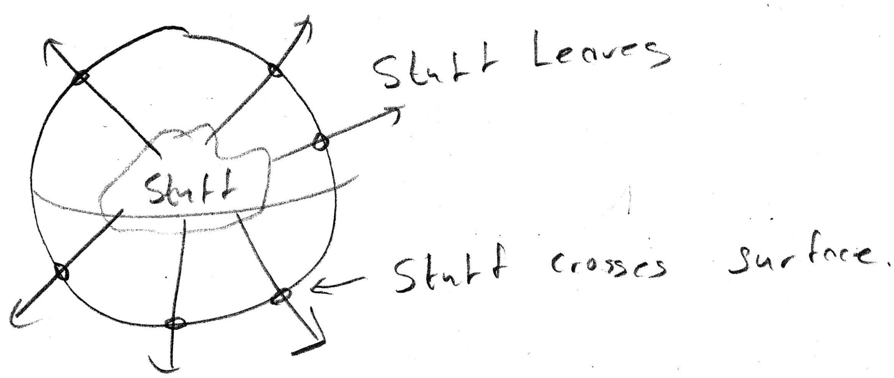
  ii. a box that the stuff is in

(b) I then move the stuff from inside of the box to outside the box

2. THEN

  (a) ALL of the stuff must pass through the box.

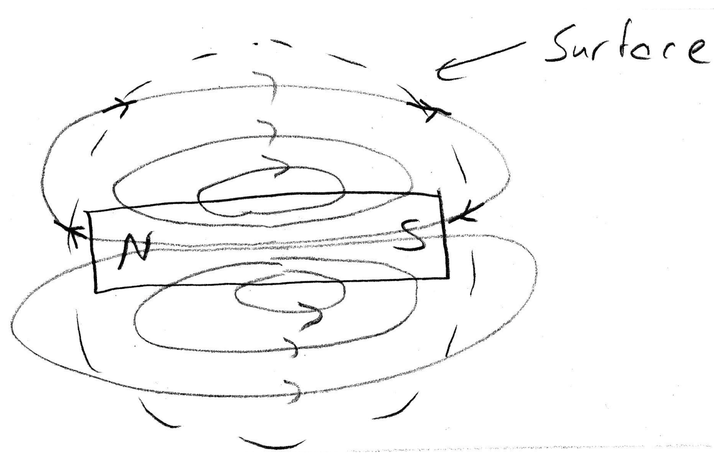
That's it! The stuff is (the ends of) electric field lines, the box is the Gaussian surface. By "moving the stuff" I mean we follow the field lines from their origin at charge out to wherever they end. We place the "box" such that it contains the beginning of all of the field lines, but the amount of "stuff" (ends of field lines) is proportional to the amount of charge (since lines start/end on charges) Q_{enc}.



We define electric flux so that we can express this process mathematically in terms of the surface integral $\int \vec{E} \cdot d\vec{A}$. Our ability to express the geometrical statement above mathematically in terms of electric field is what makes it a useful calculational tool. But the fact remains, *Gauss's Law is an observation about geometry*, and we just cleverly use that observation to calculate stuff about electric fields (which live in space, so they care about geometry).

Gauss's Law on Magnetic Field

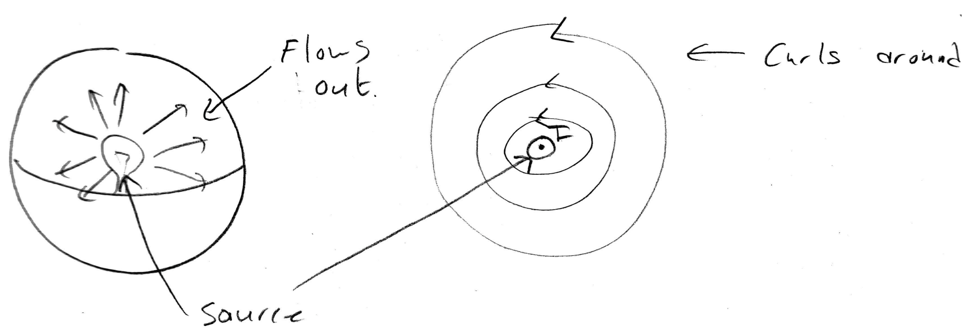
If Gauss's Law is such a great geometrical observation, why not apply it to magnetic fields? Well, magnetic fields don't have a "start" and "end" point, so there is no "stuff" to move from inside a box to outside. If we were to define a flux of magnetic field (which we will for a different purpose, later) and integrate it over a closed surface, we would *always get 0* because there are no monopoles on which the field lines can start and stop. Any line that enters a surface must also exit it, because they form continuous loops. Thus, Gauss's Law for magnetic fields is entirely *valid*, but because "magnetic charge" is *always 0*, the "Q_{enc}" part is always 0, and we haven't learned anything about anything.



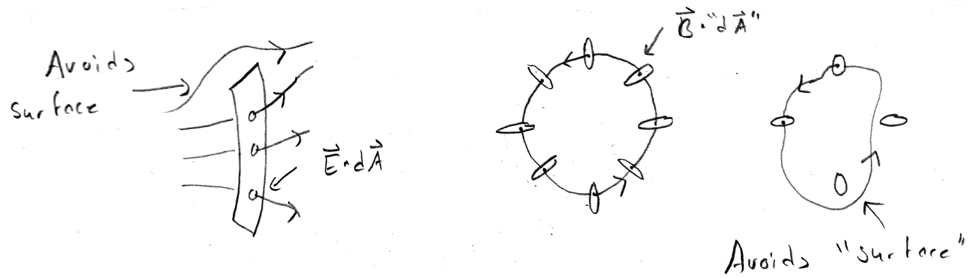
9.2.2 "Twisting" around a current.

Ampère's Law is another clever observation about geometry, but this time we will use it to do useful things with the magnetic field rather than the electric field. Unfortunately, the geometrical basis of Ampère's Law isn't as clear or obvious as it is for Gauss's Law. In fact, the purely geometrical concept underlying Ampère's Law in the way that "stuff goes out" underlies Gauss's Law is fairly obscure and very rarely used. So rare that I am not aware of a single good analogy with something 'everyday'. None of the mediocre analogies available seem to promote understanding, so we'll just carry on with a brand new concept.

In Gauss's Law, charge is the *source* of a flow or *flux* outward. A light bulb creates a flux of light which flows outward through any closed surface you put around it. In Ampère's Law, current is the *source* of a *twisting around* the center. The magnetic field curls around the current, and the more current the more curling is happening. There are plenty of things in nature that *curl* (e.g. hurricanes and whirlpools), but none have a clear source in this same sense that I am aware of. However, if we just accept for the sake of argument (and in trust of mathematicians and such who prove all of these things meticulously for us) that the curling of a magnetic field around the current in a wire is the rotational equivalent to the outflow of field lines from a charge, we can proceed. (Remember that angular momentum is just the rotational equivalent to linear momentum. The analogy to our current situation is incomplete, but the idea of a rotational equivalent isn't entirely new.)



In Gauss's law, we introduce a box around the stuff, or a surface around the charge. This box exists because it contains all of the relevant motion of stuff. In that case the motion of stuff is the outward flow of lines. Here, the relevant motion of stuff is the circular motion of magnetic fields around in a circle. In order to capture all of this flow with Gauss's law, we construct a closed surface. With no gaps or holes, there is nowhere for the flux to escape without being calculated. With Ampère's Law, we construct a closed *loop* oriented around the current. Think of this loop being constructed of lots of infinitely small areas ("$d\vec{A}$") around the ring through which the magnetic field lines must pass. If the spacing between each area segment grows, more field lines could snake in between without being counted, and yet still be curling around.



The limit of our series of small areas which form a sort of "tube" through which field lines pass becomes a continuous ring (of 0 thickness!) around a loop. Its an infinitely thin but continuous tube. So now the sum of magnetic fluxes through "$d\vec{A}$"s becomes an integral of the field (no longer a real flux since we took the area away) around the loop,

"*twisting*" $= \oint \vec{B} \cdot d\vec{\ell}$

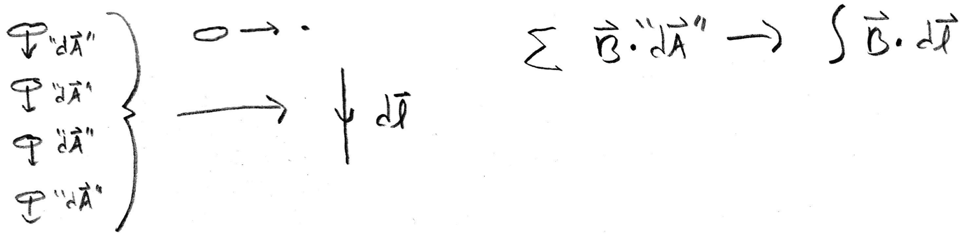
This is getting closer to being Ampère's Law. (We use the integral sign with the circle to remind us that it must be a closed loop and not an open line. This is similar to the use in Gauss's Law, but in that case it refers to a closed surface rather than loop. But the circle means "closed" in both cases.) Right now it is roughly analogous to

$\Phi_E = \oint \vec{E} \cdot d\vec{A}$

except that we haven't defined the analog to flux precisely (and won't). It is ***not*** the magnetic flux. That is a real quantity which is perfectly analogous to Φ_E (just replace E with B everywhere!) rather than a related concept within the context of our particular geometrical argument. The total amount of "twisting" serves the same *function* as flux does in Gauss's law, but it is *not* a flux.

Ampère's Law at last

Gauss's Law wasn't useful until we identified the total flux with Q_{enc}. The same is true here: we need to know what the integral around this loop is in terms of something physical (the current) before it is anything but a



pointless exercise in geometry and calculus. To make this connection, we do the integral explicitely once. This will allow us to find a relationship analogous to that between area, E, and Q_{enc} from Gauss's Law. For a simple loop at constant radius, we find that:

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I.$

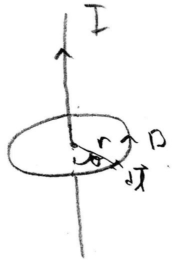
We derived this using a very particular loop which makes the integral easy. While it is far less obvious than the Gauss's Law analog, it is equally true that this relationship will hold for *any* closed loop which *encloses* I. This is a known result of vector calculus and you can see a hint of how it works out if you instead construct the loop of a series of straight elements which aren't necessarily tangent to the $\vec{B}$ field. The dot product will still, in the aggregate, give you the same total for the integral. (With a loop that isn't a perfect circle like this, the loop will have a longer perimeter, but the dog product will at places be less than one. The combination of these two effects is that the integral stays the same.) It doesn't actually matter *why* this result is general: as long as you find it reasonable and believe it, we can use it and write the more general form:

$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc}$ Ampère's Law

Whew! We are now equiped with the closest thing to analogue to Gauss's Law as exists for magnetic fields. It will prove quite useful, but there are a few issues with interpretation left to address.

9.2.3 What does "enclosed" mean for a loop?

A loop doesn't have an "inside" and an "outside" in 3 dimensions, so how do we know what is "enclosed" in it? To answer this, think of another way to define what it means to be enclosed in a surface. If an object is



$$\vec{B} = \frac{\mu_2}{2\pi}\frac{I}{r}\,\widehat{RHR}$$

$$d\vec{l} = r\,d\theta\,\widehat{RHR}$$

$$\oint \vec{B}\cdot d\vec{l} = \int_0^{2\pi} \frac{\mu_0}{2\pi}\frac{I}{\cancel{r}}\cancel{r}\,d\theta \quad \widehat{RHR}\cdot\widehat{RHR}$$

$$= \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\theta$$

$$\boxed{\oint \vec{B}\cdot d\vec{l} = \mu_0 I}$$

enclosed in a surface, it is constrained from going certain places in space without crossing the surface. This is what we intuitively mean by the phrase enclosed: you can't escape without crossing the boundary. A point cannot, by this reasoning, be enclosed in a loop. Neither can a line segment. I can always take any piece of a line and pass it into, out of, and around a closed loop without the 2 crossing. However, a current must be a closed loop, somehow, because of charge conservation. Whenever we draw an infinite current, we are ignore the implied return current that must be traveling backwards out somewhere at infinity. But no matter how big a loop, this current is still a loop. If we nest too loops, they cannot be pulled apart without crossing one another. This is why chains work. If two interlocked loops could become un-interlocked without crossing or breaking, chains, far from holding things securely, would just fall apart.

So: an enclosed current is a current which is, taken as an entire circuit, interlocked with the loop of integration. The loop of integration is *not* a loop of current and is just as imaginary as a Gaussian surface.

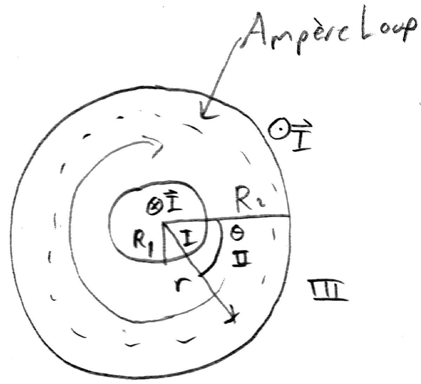
9.2.4 Caveates

Ampère's Law only applies when the currents and fields aren't changing, and there are no magnetic materials (which we define later) in the area. If either of these conditions is violated, the law is modifed. There will still be some related law which holds true, however.

Ampère's Law is only particularly useful if there exists a symmetry which we can exploit to make the integral simplify, as when we pull E out of the integral in G auss's law so we can just use the area of the surface.

9.2.5 Examples

Coaxial Cable



Region I:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$B \int_0^{2\pi} r \, d\theta = \mu_0 I_{enc}$$

$$2\pi \cancel{r} B = \mu_0 I \frac{r^{\cancel{2}}}{R_1^2}$$

$$B = \frac{\mu_0 I}{2\pi} \frac{r}{R_1^2}$$

$$\boxed{\vec{B}_I = \frac{\mu_0 I}{2\pi} \frac{r}{R_1^2} \ \widehat{RHR}}$$

$$I_{enc} = I \frac{A(r)}{A(R_1)}$$

$$= I \frac{\pi r^2}{\pi R_1^2}$$

$$= I \frac{r^2}{R_1^2}$$

Region II:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

$$B \int_0^{2\pi} r\, d\theta = \mu_0 I_{enc}$$

$I_{enc} = I$

$$2\pi r B = \mu_0 I$$

$$B = \frac{\mu_0}{2\pi} \frac{I}{r}$$

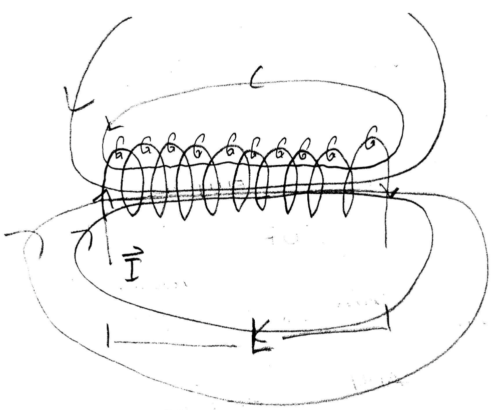
$$\boxed{\vec{B}_{II} = \frac{\mu_0}{2\pi} \frac{I}{r} \widehat{RHR}}$$

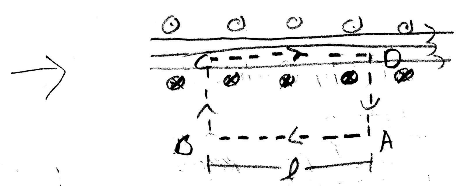
Region III:

Since the current on the outside is the same magnitude as inside but the opposite direction,

$$\boxed{\vec{B}_{III} = 0}$$

Solenoid





$$\oint \vec{B}\cdot d\vec{l} = \underbrace{\int_A^B \cancelto{0}{\vec{B}}\cdot d\vec{l}}_{B=0} + \underbrace{\cancel{\int_B^C}}_{\vec{B}\perp\vec{l}} + \int_C^D + \underbrace{\cancel{\int_D^A}}_{\vec{B}\perp\vec{l}}$$

$$= \int_C^D \vec{B}\cdot d\vec{l}$$

$$= B\int_0^l dl$$

$$= \underline{Bl}$$

$$Bl = \mu_0 I_{enc}$$

$$B\cancel{l} = \mu_0 I N \frac{\cancel{l}}{L}$$

$$B = \mu_0 I \frac{N}{L}$$

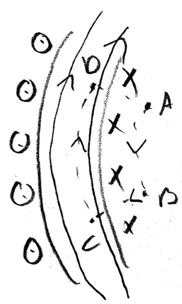
$$\boxed{B = \mu_0 n I}$$

$$\boxed{\vec{B} = \mu_0 n I \,\hat{x}}$$

Toroid

9.3 Biot-Savart

If Ampère's Law is the analogue to Gauss's Law, then Biot-Savart is the analogue to the differential for of Coulomb's Law for the electric field. Am-



$$\oint \vec{B}\cdot d\vec{\ell} = \int_C^D \vec{B}\cdot d\vec{\ell} = \mu_0 I_{enc}$$

$$= B\int_0^{2\pi} R\,d\theta = \mu_0 I N$$

$$= 2\pi R B = \mu_0 I N$$

$$\boxed{B = \frac{\mu_0}{2\pi R} I N}$$

père's Law, like Gauss's, is restricted in usefullness to situations in which an appropriate symmetry exists. We can find Biot-Savart from

$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\widehat{RHR}$.

The conversion into a differential for this is actually fairly involved with a lot of unenlightening trigonometry, but basically you take $\frac{dB}{d\ell}$ and rearrange it to find:

$d\vec{B} = \frac{\mu_0}{4\pi}\frac{Id\vec{\ell}\times\hat{r'}}{r'^2}$

where $\vec{r'}$ is the vector from the current element $d\vec{I} = Id\vec{\ell}$ to the test point. We can turn this into an integral:

$\vec{B} = \frac{\mu_0 I}{4\pi}\int\frac{d\vec{\ell}\times\hat{r}'}{r'^2}$

9.3.1 Examples

Straight Wire

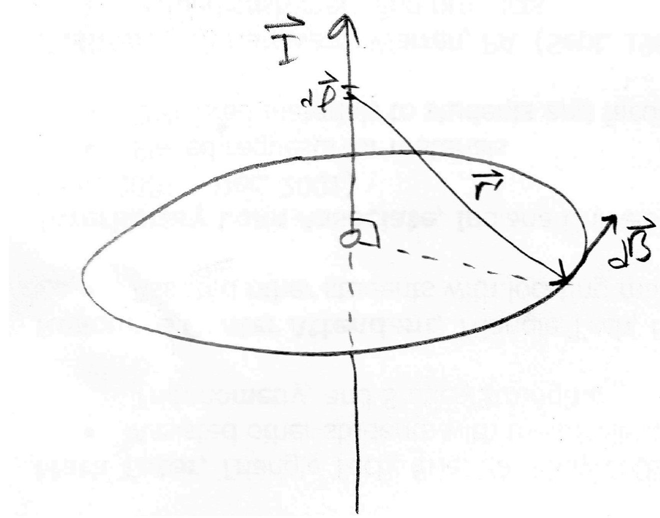
$d\vec{\ell} = dy\hat{y}$

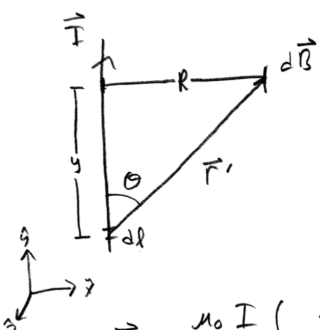
$r^2 = R^2 + y^2$

$B = \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{dy\sin\theta}{r^2}$

$y = -\frac{r}{\tan\theta}$

$$\vec{B} = \frac{\mu_0}{2\pi}\frac{I}{r}\ \widehat{RHR}$$





$$dl = dy\,\hat{y}$$

$$\tan\theta = \frac{R}{-y} \rightarrow y = \frac{-R}{\tan\theta}$$

$$\frac{dy}{d\theta} = \frac{R}{\sin^2\theta} \longrightarrow dy = \frac{R}{\sin^2\theta}\,d\theta$$

$$\sin\theta = \frac{R}{r'}$$

$$dy = \frac{\cancel{R}(r')^2}{R^{\cancel{2}}}\,d\theta$$

$$\hat{y}\times\hat{r}' = \sin\theta\,(-\hat{z})$$

In lecture I put the $\frac{1}{\sin^2\theta}$ back in here, thus causing the confusion.

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l}\times\hat{r}'}{(r')^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_{-\infty}^{\infty}\frac{dy\ \hat{y}\times\hat{r}'}{(r')^2}$$

$$= \frac{\mu_0 I}{4\pi}\int_0^{\pi}\frac{\cancel{(r')^2}}{R\cancel{(r')^2}}\sin\theta\ d\theta\,(-\hat{z})$$

$$= \frac{\mu_0 I}{4\pi R}\int_0^{\pi}\sin\theta\ d\theta\,(-\hat{z})$$

$$= \frac{\mu_0 I}{4\pi R} \cos\theta \Big|_0^{\pi} \hat{z}$$

$$= \frac{\mu_0 I}{4\pi R} (-1-1) \hat{z}$$

$$\boxed{\vec{B} = \frac{\mu_0 I}{2\pi R} (-\hat{z})}$$

Current Loop/Dipole

9.4 Ferromagnetism

I won't be lecturing on it, but you should read the section in your text and understand the basic idea of domains and how they lead to large scale magnetic effects.