LECTURE 10

Basic Special Relativity

We review some concepts in special relativity very quickly.

First note the basic postulates:

1) laws of physics take the same form in frames in relative uniform motion.

2) speed of light is the same in free space for these frames, independent of their relative velocity.

The appropriate transformations of the coordinates x, t that preserve these relations between two frames K and K' such that K' is moving with positive velocity v on the x axis for a fixed observer in K are

$$x' = \gamma(x - vt);\; y = y'; z = z' \tag{304}$$

and

$$t' = \gamma(t - vx/c^2), \tag{305}$$

where $\gamma = 1/(1-v^2/c^2)^{1/2}$ We talk about a 4-dimensional space-time coordinate taking place at $(t, \mathbf{x})$.

Note that a consequence of the constant speed of light is that for a pulse of light emitted at $t = 0$ where we assume the origins coincide is

$$x^2 + y^2 + z^2 - c^2t^2 = x'^2 + y'^2 + z'^2 - c^2t'^2 = 0. \tag{306}$$

Length Contraction

Consider a rigid rod of length $L' = x_2' - x_1'$ carried in rest frame K'. What is the length in the unprimed frame? The length is given by $L = x_2 - x_1$ where x_2 and x_1 are the rod positions as measured at fixed time in L. We have

$$L' = x_2' - x_1' = \gamma(x_2 - x_1) = \gamma L. \tag{307}$$

Thus the length we measure at a fixed time in L is smaller than the proper length measured in L'. The length of the moving rod is contracted.

Time Dilation

Assume a clock at rest in K' measure a time interval $T_0 = t_2' - t_1'$ at fixed position $x' = 0$. This implies that

$$T = t_2 - t_1 = \gamma(t_2' - t_1') = \gamma T'. \tag{308}$$

Thus we see that more time elapses in the unprimed frame per unit time in the moving frame. Thus "moving clocks run slow." This is the time dilation.

The point is that clocks at two different positions in K are not simulatenously synchronized as seen by K'. When $x_2' = x_1'$ in K', $x_2 \neq x_1$. At these two locations in K, clocks in K do not appear to be synchronized as measured by an observer in K' even if they are synchronized as seen by an observer in K.

Transformation of Velocities and Beaming of Radiation

Write the transformations in differential form for boost along x:

$$dx = \gamma(dx' + vdt');\ dy = dy';\ dz = dz' \tag{309}$$

and

$$dt = \gamma(dt' + vx'/c^2), \tag{310}$$

where we have also taken advantage of the symmetry of relative frames and let $v \rightarrow -v$ and $t', \mathbf{x}' \rightarrow t, \mathbf{x}$.

Then we have

$$u_x = dx/dt = \frac{\gamma(dx' + vdt')}{\gamma(dt' + vdx'/c^2)} = \frac{u_x' + v}{1 + u_x' v/c^2} \tag{311}$$

and

$$u_y = \frac{u_y'}{\gamma(1 + u_x' v/c^2)} \tag{312}$$

and

$$u_z = \frac{u_z'}{\gamma(1 + u_x' v/c^2)}. \tag{313}$$

fig 4.2

In general,

$$u_{||} = \frac{u_{||}' + v}{1 + u_{||}' v/c^2} \tag{314}$$

and

$$u_{\perp} = \frac{u_{\perp}'}{\gamma(1 + u_{\perp}' v/c^2)}. \tag{315}$$

Thus we can take the ratio of the perpendicular to the parallel velocities as measured in the lab frame (K). We have

$$tan\theta = u_\perp/u_{||} = \frac{u'_\perp}{\gamma(1+u'_\perp v/c^2)}\frac{1+u'_{||}v/c^2}{u'_{||}+v} = \frac{u'_\perp}{\gamma(u'_{||}+v)} = \frac{u'sin\theta'}{\gamma(u'cos\theta'+v)}. \tag{316}$$

For $u' = c$, corresponding to photons emitted in the moving frame, we have

$$tan\theta = \frac{sin\theta'}{\gamma(cos\theta'+v/c)} \tag{317}$$

and

$$cos\theta = \frac{cos\theta'+v/c}{(1+cos\theta' v/c)};\ sin\theta = \frac{sin\theta'}{\gamma(1+cos\theta' v/c)} \tag{318}$$

At $\theta' = \pi/2$, we have

$$tan\theta = \frac{1}{\gamma v/c};\ cos\theta = v/c, \tag{319}$$

so that $sin\theta = 1/\gamma$.

This means that if a source is emitting isotropically in its rest frame (K' frame) then in the K frame, 50% of the emission is beamed into a cone of angle $1/2\gamma$.

This has many consequences in astrophysics. Both in the microphysics of radiation from moving particles, and also for relativistic bulk outflows such as Active Galactic Nuclei and Gamma-Ray Bursts. It means for example, that a for sufficiently relativistic and radiating bipolar jet, we would only see emission from the flow moving toward our line of sight.

Doppler Effect

It is particularly important in astrophysics to distinguish between local observers (those omnipresent entities that have a rod and clock at each point in space time) and actual observers (those like us fixed at a point). There is no Doppler effect for local observers.

In the rest frame of K consider the moving source to move from point 1 to point 2 and emit one radiation cycle. The period as measured in the lab frame (K) is given by the time dilation effect

$$\Delta t = \gamma\Delta t' = 2\pi\gamma/\omega', \tag{320}$$

but difference in arrival times must be considered because the source is moving relative to the observer. Consider the distance d to be positive if the source moves toward the observer. (fig 4.4)

Then accounting for the difference in arrival times gives

$$\Delta t_{obs} = \Delta t - d/c = \Delta t(1 - (v/c)cos\theta) = \gamma \Delta t'(1 - (v/c)cos\theta), \tag{321}$$

where θ is now the angle between the direction of motion and the line of sight as measured in the lab frame (K). Rearranging we then have

$$\omega_{obs} = 2\pi/\Delta t_{obs} = \frac{\omega'}{\gamma(1 - (v/c)cos\theta)} \tag{322}$$

or as the inverse

$$\omega' = \omega_{obs}\gamma(1 + (v/c)cos\theta'), \tag{323}$$

where θ' is measured in the K' frame. Note that the γ is a purely relativistic effect whereas the $(1 + (v/c)cos\theta')$ is also a non-relativistic effect.

Combined Doppler Effect and Special Relativity

Rearranging (322) and assuming the case for which $\theta \sim 0$, and $v \sim c$ we have

$$\omega' \simeq \omega_{obs}\gamma(1 - (v/c)cos\theta) = \gamma \frac{(1 - v/c)(1 + v/c)}{(1 + v/c)} \sim \omega_{obs}/2\gamma. \tag{324}$$

or

$$\Delta t_{obs} \sim \Delta t'/2\gamma. \tag{325}$$

Compare this to (320): the γ is in the opposite place! Thus there is no time dilation but rather time *contraction* for a source moving at the observer. "A clock moving at the observer runs fast!" This highlights that for astrophysics, where we the observer are at a fixed location, that the standard time dilation and Lorentz contraction effects must be considered carefully. (See Rees 1966, Nature 211 468 for superluminal motion effects!).

Proper time

Note that the quantity $d\tau$, the proper time, is in an invariant:

$$c^2 d\tau^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2) = c^2 d\tau'^2. \tag{326}$$

It is called the proper time because it measures the time interval between events occurring at the same spatial location. Indeed if we divide by dt, and assume the differentials measure changes with respect to a frame moving with velocity $d\mathbf{x}/dt = \mathbf{v}$, we obtain $dt = \gamma d\tau$. This is just the original time dilation effect.

Notes on 4 vectors

The quantity $x^\mu = (ct, x, y, z)$ is a four-vector, in that it obeys the transformation properties previously described. The Greek indices indicate 4 components. Note also that

$$x_\mu = \eta_{\mu\nu} x^\nu \tag{327}$$

where summation is over repeated indices and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$, -1 for $\mu = \nu = 0$ and 1 for $\mu = \nu > 0$. The Lorentz transformations can be written

$$x^{\mu\prime} = \Lambda^\mu_\nu x^\nu \tag{328}$$

where Λ^μ_ν is the transformation matrix, which, for a boost along x satisfies $\Lambda^\mu_\nu = \gamma$ for $\mu = \nu = 0, 1$ 1 for $\mu = \nu = 2, 3$, $-\beta\gamma$ for $\mu = 0, \nu = 1$, and $-\beta\gamma$ for $\mu = 1, \nu = 0$. General Lorentz transformations have more complicated matrices. We restrict ourselves to isochronous ($\Lambda^0_0 \geq 0$), and proper transformations $det\Lambda = +1$.

The norm of x^μ is given by

$$x^\mu x_\mu = \eta_{\mu\nu} x^\mu x^\nu. \tag{329}$$

Since this is a Lorentz invariant, we have

$$\Lambda^\tau_\nu \tilde{\Lambda}^\nu_\sigma = \delta^\tau_\sigma \tag{330}$$

where δ^τ_σ is the Kronecker delta.

All 4-vectors transform in the same way as x^μ under a Lorentz transformation. Some other 4-vectors include $k^\mu = (\omega/c, \mathbf{k})$ for a electromagnetic wave, the 4-velocity $U^\mu = (\gamma c, \gamma \mathbf{v})$, the current density 4-vector $j^\mu = (\rho c, \mathbf{j})$. the momentum 4-vector $(\gamma E/c, \gamma \mathbf{p})$.

The transformation laws also hold for tensor indices. Thus two Λ matrices are required for a two index tensor. That is for example

$$A^{\mu\nu\prime} = \Lambda^\mu_\sigma \Lambda^\nu_\delta A_{\sigma\delta}. \tag{331}$$

It turns out that the electric and magnetic field are not separately 4-vectors but comprise part of of an electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{332}$$

where $A_\mu = (-\phi, A_i)$ is the 4-vector potential with components (Latin indices mean only 1,2,3 components.) $F_{0i} = -E_i$, $F_{i0} = E_i$, $F_{00} = 0$, $F_{12} = B_z$, $F_{21} = -B_z$, $F_{13} = -B_y$, $F_{31} = B_y$, $F_{23} = B_x$, $F_{32} = -B_x$, and the rest zeros.

Maxwell's equations can then be written

$$\partial^\nu F_{\mu\nu} = 4\pi j_\nu / c \tag{333}$$

which includes $\nabla \cdot \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{B} = c^{-1}\partial_t \mathbf{E} + 4\pi \mathbf{j}/c$. and

$$\partial_\sigma F_{\mu\nu} + \partial_\nu F_{\sigma\mu} + \partial_\mu F_{\nu\sigma} = 0, \tag{334}$$

which includes $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + c^{-1}\partial_t \mathbf{B} = 0$.

Lorentz invariants are $F^{\mu\nu}F_{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$ and $Det(F) = (\mathbf{E} \cdot \mathbf{B})^2$.

Lorentz Transformations of the Fields

For a boost with velocity $\vec{\beta}$ we have

$$\mathbf{E}'_{||} = \mathbf{E}_{||}, \tag{335}$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||} \tag{336}$$

and

$$\mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \vec{\beta} \times \mathbf{B}) \tag{337}$$

and

$$\mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \vec{\beta} \times \mathbf{E}). \tag{338}$$

For a physical idea of the transformations, consider a charged capacitor with surface charge density σ' in its rest frame and plate separation d. (fig)

Place the capacitor plates perpendicular to the direction of motion (x). In the lab frame K the capacitor moves with speed v_x. Since $E'_{||} = 4\pi\sigma'$, and the area does not contract with this motion as seen in K, $E_{||} = E'_{||}$. There is no surface current either.

Now rotate the plates 90 degrees so that they are parallel to the direction of motion, parallel to the x-z plane. Now the perpendicular electric field $E_y = E'_y\gamma$ because the surface charge density as seen in K increases from the Lorentz contraction along the direction of motion (length of plate appears smaller). Now since the current and charge density transform as a 4-vector we have

$$\mathbf{j} = \gamma(\mathbf{j}' + v_x \hat{\mathbf{x}} \sigma'/d) \tag{339}$$

and since $\mathbf{j}' = \mathbf{0}$ we have the only component $j_x = \gamma v_x \sigma'/d$. Now the current is also given by $\nabla \times \mathbf{B}$. Since the only direction of variation is perpendicular to the plates, (the y direction in the K frame) we have

$$(\nabla \times \mathbf{B})_x = \partial_y B_z = 4\pi j_x/c, \tag{340}$$

so that

$$B_z \simeq 4\pi j_x d/c = 4\pi\gamma v_x \sigma'. \tag{341}$$

We can see that this result also follows directly from (338) which gives

$$B_z = \gamma(B_z' + \beta_x E_y') = 4\pi\gamma v_x \sigma', \tag{342}$$

since $B_z' = 0$. This exemplifies one restricted case of the transformations above.

Fields of Uniformly Moving Charge

We want to applying the transformation relations for the fields to a charge moving with constant velocity along x. In the primed (rest) frame we have

$$E_x' = qx'/r'^3;\ B_x' = 0 \tag{343}$$

$$E_y' = qy'/r'^3;\ B_y' = 0 \tag{344}$$

$$E_z' = qz'/r'^3;\ B_z' = 0, \tag{345}$$

with $r' = (x'^2 + y'^2 + z'^2)^{1/2}$. We use the inverse transform for the relations (335-338), ($\beta \to -\beta$, and switch primed and unprimed quantities) to find

$$E_x = qx'/r'^3;\ B_x = 0 \tag{346}$$

$$E_y = q\gamma y'/r'^3;\ B_y = -q\gamma\beta z'/r'^3 \tag{347}$$

$$E_z = qz'/r'^3;\ B_z = q\gamma\beta y'/r'^3. \tag{348}$$

But these are in the primed coordinates. We can transform to unprimed coordinates by using the simple coordinate transformations to get

$$E_x = q\gamma(x - vt)/r^3;\ B_x = 0 \tag{349}$$

$$E_y = q\gamma y/r^3;\ B_y = -q\gamma\beta z/r^3 \tag{350}$$

$$E_z = qz/r^3;\ B_z = q\gamma\beta y/r^3, \tag{351}$$

with now $r = (\gamma^2(x - vt)^2 + y^2 + z^2)^{1/2}$.

Now let's evaluate these fields at the retarded time. Let $t_{ret} = t - R/c$, $z = 0$. fig 4.5

Then we have

$$R^2 = y^2 + (x - vt + vR/c)^2 \tag{352}$$

so solving for R gives

$$R = \gamma^2\beta(x - vt) + \gamma(y^2 + \gamma^2(x - vt)^2)^{1/2}. \tag{353}$$

The unit vector to the field point from the retarded time position is then

$$\hat{\mathbf{n}} = \frac{y\hat{\mathbf{y}} + (x - vt + vR/c)\hat{\mathbf{x}})}{R} \tag{354}$$

and

$$\kappa = 1 - \mathbf{n} \cdot \vec{\beta} = (y^2 + \gamma^2(x - vt)^2)^{1/2}/\gamma R. \tag{355}$$

Thus

$$\frac{q}{\gamma^2 R^2 \kappa^3} = \frac{\gamma R q}{(y^2 + \gamma^2(x - vt)^2)^{3/2}}. \tag{356}$$

If we combine the last 3 equations with (349) (350), and (351) we have

$$\mathbf{E} = \left[q(\hat{\mathbf{n}} - \vec{\beta})(1 - \beta^2)/\kappa^3 R^2\right] \tag{357}$$

which is the same result we get for the velocity field of a moving charge from the Liénard-Wiechart potentials.

LECTURE 11

Field from a strongly relativistic charge

fig 4.6

For $\gamma >> 1$ at field point at $x = 0, y = b$ we have

$$E_x = -\frac{qv\gamma t}{(\gamma^2 v^2 t^2 + b^2)^{3/2}};\ B_x = 0 \tag{358}$$

$$E_y = \frac{q\gamma b}{(\gamma^2 v^2 t^2 + b^2)^{3/2}};\ B_y = 0 \tag{359}$$

$$E_z = 0;\ B_z = \beta E_y \sim E_y. \tag{360}$$

Fields are are concentrated mainly in plane transverse to motion ($E_y direction$) in cone of angle $1/\gamma$ (found from ratio of $(E_x/E_y)_{max}$). Book calculates spectrum for this uniformly moving particle. The key point is that the spectrum has peaks at frequency given by $1/\Delta t = \gamma v/b$ which is the time period for which the electric field is significant–very short–thus a very narrow peak in spectrum. $\tilde{E}^2(\omega) \sim E_y^2(\Delta t)^2 \sim (q\gamma/b^2)^2(b/\gamma v)^2 \sim q^2/(b^2v^2)$.

fig 4.7

Relativistic Mechanics

The four-momentum of a particle is given by

$$p^\mu = mu^\mu = mu^\mu = m(c\gamma, \gamma\mathbf{v}) = (E/c, \gamma m\mathbf{v}). \tag{361}$$

Expanding the zeroth component in the non-relativistic regime

$$p^0 c = mc^2(1-\beta^2)^{-1/2} = m(c^2 + v^2/2) + ... \tag{362}$$

which is the rest energy plus the kinetic energy. The norm is

$$p^\mu p_\mu = -m^2c^2 = -E^2/c^2 + \mathbf{p}^2 \tag{363}$$

or

$$E^2 = m^2c^4 + c^2\mathbf{p}^2 = (\gamma mc^2)^2. \tag{364}$$

For a photon we have

$$p^\mu = \frac{h}{2\pi}(\omega/c, \mathbf{k}), \tag{365}$$

and $p^\mu p_\mu = 0$.

If we take the derivative with respect to the proper time (a Lorentz invariant scalar) we have the acceleration 4-vector.

$$a^\mu = du^\mu/d\tau. \tag{366}$$

The generalization of Newton's law is

$$F^\mu = ma^\mu. \tag{367}$$

Note that

$$F^\mu u_\mu = \frac{m}{2}d(u^\mu u_\mu)/d\tau = 0. \tag{368}$$

This implies that 4-surfaces of u_μ and F^μ are orthogonal, so that as you move along F^μ as a function of its parameters, you are moving along different 4-planes in u^μ. This means that the four force is a function of u^μ so 4-force has a dependence on 4-velocity.

For the case of the electromagnetic field, the covariant generalization of the Lorentz force

$$\mathbf{F} = qE + \mathbf{v} \times \mathbf{B}/c \tag{369}$$

can be written in terms of the electromagnetic tensor as

$$F^\mu = \frac{q}{c}F^\mu_\nu u^\nu = ma^\mu. \tag{370}$$

This indeed has a 0th component which reduces to

$$dW/dt = q\mathbf{E}\cdot\mathbf{v} \tag{371}$$

and the ith components

$$dp_i/dt = qE_i + (\mathbf{v}\times\mathbf{B})_i/c. \tag{372}$$

Total Emission from Relativistic Particles

Move into "local Lorentz frame." Even if particle is accelerating, we can move into a frame for which particle moves non-relativistically for short times away from the initial time. Then we use the dipole formula, and then transform back to lab frame.

The radiation has zero momentum for "front-back symmetric" emission in this comoving frame. Using the lorentz transformation for the momentum and positon 4-vectors only the zeroth components then matters in the primed (co-moving) frame and we have and

$$dW = \gamma dW'; \; dt = \gamma dt'. \tag{373}$$

Thus

$$dW'/dt' = dW/dt. \tag{374}$$

The non-relativistic dipole formula is

$$P = P' = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}a_\mu a^\mu \tag{375}$$

where the last equality follows because in the instantaneous rest frame of the emitting particle, $a^{\mu'} = (0, \mathbf{a})$.

We can write this in terms of the 3-vector acceleration by quoting the result that

$$a'_{||} = \gamma^3 a_{||} \tag{376}$$

and

$$a'_{\perp} = \gamma^2 a_{\perp} \tag{377}$$

so

$$P = \frac{2q^2}{3c^3}\mathbf{a}'^2 = \frac{2q^2}{3c^3}\gamma^4(a_\perp^2 + \gamma^2 a_{||}^2). \tag{378}$$