Lecture 16

Compton Scattering

For low photon energies, the scattering of radiation from free charges reduces to the classical case of Thomson scattering. For that case

$$\epsilon_i = \epsilon_f, \tag{479}$$

$$d\sigma_T/d\Omega = \frac{1}{2}r_0^2(1 + cos^2\theta) \tag{480}$$

and

$$\sigma_T = 8\pi r_0^2/3 = 6.6 \times 10^{-25}\text{cm}, \tag{481}$$

where ϵ_i and ϵ_f are the incident and scattered photon energy, $d\sigma_T/d\Omega$ is the differential cross section for scattering into Ω and σ_T is the Thomson cross section, and θ is the angle between the incident and scattered direction. This is “elastic scattering.”

In reality the scattering is not elastic because the charge recoils. fig 7.1

The initial and final 4-momenta of the photon are $P_{i\gamma} = (\epsilon_i/c)(1, \mathbf{n}_i)$ and $P_{f\gamma} = (\epsilon_f/c)(1, \mathbf{n}_f)$ respectively.

For the electron $P_{ie} = (mc, 0)$ and $P_{fe} = (E/c, \mathbf{p})$ respectively. Conservation of energy and momentum can be written in terms of the energy momentum 4-vectors for the electron and photons:

$$P_{ie} + P_{i\gamma} = P_{fe} + P_{f\gamma}, \tag{482}$$

This leads to the two equations

$$mc^2 + \epsilon_i = E + \epsilon_f \tag{483}$$

and

$$\epsilon_i\mathbf{n}_i + 0 = \epsilon_f\mathbf{n}_f + \mathbf{p}c. \tag{484}$$

Rearranging we have

$$(mc^2 + \epsilon_i - \epsilon_f)^2 = E^2 = m^2c^4 + p^2c^2 \tag{485}$$

and

$$(\epsilon_i\mathbf{n}_i - \epsilon_f\mathbf{n}_f)^2 = p^2c^2. \tag{486}$$

where I used $E^2 = m_c^2c^4 + p^2c^2$. Then solving to eliminate p, we obtain

$$\epsilon_f = \frac{\epsilon_i}{1 + \frac{\epsilon_i}{mc^2}(1 - cos\theta)}. \tag{487}$$

If we write $\epsilon_i = h\nu = hc/\lambda$ and similarly for ϵ_f we have,

$$\lambda_f - \lambda = \lambda_c(1 - cos\theta), \tag{488}$$

where $\lambda_c = \frac{h}{mc} = 0.024$ angstroms for electrons is the Compton wavelength. When $\lambda >> \lambda_c$ or $h_\nu << mc^2$, the scattering is elastic in the rest frame of the electron.

When quantum effects are important, ie. $\epsilon_i \gtrsim mc^2$, the cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - sin^2\theta\right). \tag{489}$$

For $\epsilon_f \sim \epsilon_i$ Eq. this reduces to the classical expression. Note that since $\epsilon_f \leq \epsilon_i$, the quantum regime produces a lower overall cross section than the Thomson regime. Think about the role of the uncertainty principle here.

The total cross section in the non-relativistic regime ($h\nu << mc^2$) is

$$\sigma \sim \sigma_T(1 - 2x + ...), \tag{490}$$

where $x = h\nu/mc^2$, and in the relativistic regime $x >> 1$,

$$\sigma \sim \frac{3}{8}\sigma_T x^{-1}(ln2x + \frac{1}{2}). \tag{491}$$

Inverse Compton energy transfer

When the electron has more kinetic energy in the lab frame than the photon energy, there can be energy transfer from the electron to the photon during the scattering. This is inverse Compton scattering. The scattering in the electron rest frame (K') and the lab frame (K), are shown below. fig 7.2

In frame K' (rest frame of the electron) we have

$$\epsilon_f' = \frac{\epsilon_i'}{1 + \frac{\epsilon_i'}{mc^2}(1 - cos\Theta')}. \tag{492}$$

We also have from the Doppler formulas,

$$\epsilon_i' = \epsilon_i \gamma (1 - \beta cos\theta) \tag{493}$$

and

$$\epsilon_f = \epsilon_f' \gamma (1 + \beta cos\theta_f'). \tag{494}$$

Since electron scattering in the rest frame of the electron is front back symmetric, a typical angle for scattering is $\theta \sim \pi/2$ or $\theta' \sim \pi/2$. Thus we have roughly, from the previous 3 equations

$$\epsilon_f' = \epsilon_i' \tag{495}$$

$$\epsilon_i' \sim \epsilon_i \gamma \tag{496}$$

and

$$\epsilon_f \sim \gamma \epsilon_f', \tag{497}$$

so that $\epsilon_f = \gamma^2 \epsilon_i$. Thus there is a gain by a factor of γ^2 in the energy from Compton scattering where γ is the Lorentz factor of the electron. We assumed that $\gamma\epsilon << mc^2$ in the rest frame of the electron. If $\epsilon_i' = \epsilon_i \gamma$ is too large, then we see from (492) that $\epsilon_f' < \epsilon_i'$ so that the processes is less efficient. Also, when $\epsilon_i' = \epsilon_i \gamma$ is too large then the cross section is reduced, which lowers the scattering probability again making the process less efficient.

Note also that γ^2 is the proportional to the energy of the electron squared.

Inverse Compton power for isotropic photon distribution (single scattering)

To get the emitted power from an isotropic distribution of photons scattering off of an isotropic distribution of electrons incurring single scattering per photon, we must average (492-494). To do this let q be density of photons of energy in range range $d\epsilon$. Let $f(p)$ be the phase space distribution function. Then

$$q d\epsilon_i = g(p) d^3p. \tag{498}$$

Now since in the co-moving frame, the particles have no spread in energy (since the energy is quadratic in the velocity) we have that $\gamma d^3p = d^3p'$ as we also derived earlier when discussing the phase space invariants in relativity. Thus d^3p transforms as energy under Lorentz transformations. Since $g(p)$ is an invariant, we have also that $q d\epsilon_i/\epsilon_i$ is an invariant that is

$$q d\epsilon_i/\epsilon_i = q' d\epsilon_i'/\epsilon_i'. \tag{499}$$

The total power scattered in the electron's rest frame is

$$\frac{dE_f'}{dt'} = c\sigma_T \int \epsilon_f'^2 \frac{q' d\epsilon'}{\epsilon_f'}. \tag{500}$$

Assume that the energy change of photon in rest frame is small compared to that in lab frame: $\gamma^2 - 1 >> \epsilon_i/mc^2$. Then $\epsilon'_f = \epsilon'_i$. We also have

$$\frac{dE'_f}{dt'} = \frac{dE_f}{dt} \tag{501}$$

by Lorentz invariance. Thus

$$\frac{dE_f}{dt} = c\sigma_T \int \epsilon_i'^2 \frac{q' d\epsilon'_i}{\epsilon'_i} = c\sigma_T \int \epsilon_i'^2 \frac{q d\epsilon_i}{\epsilon_i}, \tag{502}$$

using the $\epsilon'_f = \epsilon'_i$ assumption. Using the Doppler formula

$$\epsilon'_i = \epsilon_i \gamma (1 - \beta cos\theta) \tag{503}$$

we have

$$\frac{dE_f}{dt} = c\sigma_T \gamma^2 \int (1 - \beta cos\theta)^2 q \epsilon_i d\epsilon_i. \tag{504}$$

Thus all quantities are now written in the K frame as desired. For isotropic distribution of photons, using $\langle \cos^2\theta \rangle = 1/3$ we have

$$\langle (1 - \beta cos\theta)^2 \rangle = 1 + \frac{1}{3}\beta^2. \tag{505}$$

Thus

$$\frac{dE_f}{dt} = c\sigma_T \gamma^2 U_\gamma (1 + \beta^2/3), \tag{506}$$

where $U_\gamma = \int \epsilon_i q(\epsilon_i) d\epsilon_i$ which is the energy density of incident photons.

Now the norm of the rate of decrease of the initial photon energy is

$$\left|\frac{dE_i}{dt}\right| = |c\sigma_T \int \epsilon_i q d\epsilon_i| = \sigma_T c^2 U_\gamma, \tag{507}$$

and so the energy actually radiated by the electron and converted into radiation is the difference between the energy scattered and the energy lost by the incoming photons. That is, it is the energy out minus the energy in. We have

$$\frac{dE_{rad}}{dt} = \frac{dE_f}{dt} - \left|\frac{dE_i}{dt}\right| = c\sigma_T U_\gamma \left[\gamma^2 (1 + \frac{1}{3}\beta^2) - 1\right]. \tag{508}$$

Using $\gamma^2 - 1 = \gamma^2 \beta^2$ we have

$$P_{compt} = \frac{dE_{rad}}{dt} = \frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_\gamma. \tag{509}$$

(When the energy transfer in the electron rest frame is not ignored, there is a correction term. (Blumenthal & Gould (1970)).)

Note that the synchrotron power we calculated for a single electron was

$$\frac{4}{3}\sigma_T c \gamma^2 \beta^2 U_B. \tag{510}$$

Thus the ratio of Compton to Synchrotron power is

$$P_c/P_{syn} = U_\gamma/U_B. \tag{511}$$

The result also is true for arbitrarily small γ (i.e. the non-relativistic limit, as long as $\gamma\epsilon << mc^2$. This gives an important rough and ready tool to determine which of the two emission processes are more important.