LECTURE 7

Degree of Polarization

Stokes parameters are additive for waves of uncorrelated phase. That is, if we take the net electric field vector for the sum of waves, no cross terms contribute to the time averaging, that is for $E_i = \sum_n E_{i,n}$ we have

$$\langle E_1 E_2^* \rangle = \sum_n \sum_l \langle E_{1,n} E_{2,l}^* \rangle = \sum_n \langle E_{1,n} E_{2,n}^* \rangle. \tag{207}$$

Thus $I = \sum_n I_n$, $Q = \sum_n Q_n$, $U = \sum_n U_n$, and $V = \sum_n V_n$.

We can then decompose the total Stokes parameters for an arbitrary collection of quasi-monochromatic waves into contributions from the polarized part and the unpolarized part:

$$I, Q, U, V = [I - (Q^2 + U^2 + V^2)^{1/2}, 0, 0, 0] + [(Q^2 + U^2 + V^2)^{1/2}, Q, U, V]. \tag{208}$$

The former is the unpolarized part. The degree of polarization is the ratio of the polarized part of the intensity to the total. (Note that we may be missing constants to make I exactly equal to the total intensity but intensity ratios are independent of this constant).

$$\Pi = (Q^2 + U^2 + V^2)^{1/2}/I = I_{pol}/I. \tag{209}$$

As an example, consider partial linear polarization. Then $V = 0$ and we take the plane of the polarized part to be the x' plane. One can the rotate a polarizing filter which will allow the maximum intensity when the filter is aligned with the electric field. The maximum intensity is the sum of the polarized part + 1/2 the unpolarized part because the intensity for the unpolarized part is shared equally between any two orthogonal directions and the polarized part only contributes when the filter is oriented properly. Thus

$$I_{max} = I_{unpol}/2 + I_{pol}. \tag{210}$$

The minimum intensity is

$$I_{min} = I_{unpol}/2. \tag{211}$$

But $I_{unpol} = I - (Q^2 + U^2)^{1/2}$ and $I_{pol} = (Q^2 + U^2)^{1/2}$. Thus

$$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}}, \tag{212}$$

when $V = 0$, i.e. partial linear polarization.

When does the Macroscopic Transfer theory of "rays" apply?

When size of the area we are interested in, through which radiation passes, approaches the wavelength of the radiation, then the macroscopic classical theory is not applicable. That is, the validity is for $(dp_x dx)(dp_y dy) \sim p^2 dA d\Omega \gtrsim h^2$ or

$$dA d\Omega \gtrsim \lambda^2. \tag{213}$$

In addition we require

$$d\nu dt \gtrsim 1, \tag{214}$$

from the energy uncertainty relation. Thus when λ is greater than the scale of interest, classical transfer theory of rays, as we have studied, fails (e.g. on the scale of atoms).

But now that we have studied waves, we can be more precise. Define rays as the curves whose tangents point along the direction of wave propagation. Thus, rays are well defined only if the amplitude and wave direction are nearly constant over a wavelength. That is the geometric optics limit.

To see the specific quantitative relations for the validity of this limit assume that a wave (electric field vector) is represented by

$$g(\mathbf{r}, t) = a(\mathbf{r}, t) e^{\psi(\mathbf{r}, t)}. \tag{215}$$

For a wave, g satisfies

$$c^{-2} \partial_t^2 g - \nabla^2 g = 0. \tag{216}$$

For constant $\mathbf{a}$, $\mathbf{k} = \nabla\psi$ is direction of propagation and $\omega = -\partial\psi/\partial t$ is frequency. These will be important later because we will show that the approximations which lead to a plane wave approximation using these correspondences to wavevector and frequency.

If we substitute in for g we get

$$\nabla^2 a - \frac{1}{c^2}\partial_t^2 a + ia(\nabla^2 - \frac{1}{c^2}\partial_t^2)\psi + 2i(\nabla a \cdot \nabla\psi - \frac{1}{c^2}\partial_t\psi\partial_t a) - a(\nabla\psi)^2 + \frac{a}{c^2}(\partial_t\psi)^2 = 0. \tag{217}$$

Now in the limit that the following relations are satisfied

$$\frac{1}{a}|\nabla a| << |\nabla\psi|; \ \frac{1}{a}|\partial_t a| << |\partial_t\psi| \tag{218}$$

$$\frac{1}{a}|\nabla^2 a| << |\nabla\psi|^2 \tag{219}$$

$$|\nabla^2\psi| << |\nabla\psi|^2; \ |\partial_t^2\psi| << |\partial_t\psi|^2, \tag{220}$$

(217) reduces to

$$(\nabla\psi)^2 - \frac{1}{c^2}(\partial_t\psi)^2 = 0, \tag{221}$$

which represents the Eikonal equation describing the geometric optics limit.

The above limits represent the case for which the the amplitude is slowly varying and the phase is rapidly varying. For nearly constant a, the direction of propagation which is perpendicular to surfaces of constant phase, is

$$\mathbf{k} = \nabla\psi. \tag{222}$$

and the frequency would be given by

$$\omega = -\partial_t\psi. \tag{223}$$

Using these in (221) then reproduces the usual plane wave relation of $k^2 = w^2/c^2$. Thus Eikonal equation expresses the geometric optics limit, and that limit leads to the relation between wave number and frequency we found earlier for plane waves. This is the relationship that applies in the regime of the classical transfer theory.

Electromagnetic Potentials

E and **B** can be expressed in terms of the electromagnetic 4-vector potential $A_\mu = (\phi, \mathbf{A})$, where ϕ is the scalar potential and $\mathbf{A}$ is the vector potential. We have

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial_t\mathbf{A}. \tag{224}$$

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{225}$$

Then, incorporating the bound and free charges into ρ and $\mathbf{j}$, we have

$$-\nabla \cdot \mathbf{E} = \nabla^2\phi - \frac{1}{c}\partial_t^2\phi + \frac{1}{c}\partial_t(\nabla \cdot \mathbf{A} + \frac{1}{c}\partial_t\phi) = -4\pi\rho \tag{226}$$

and

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c}\partial_t(\nabla\phi + \frac{1}{c}\partial_t\mathbf{A}) = 4\pi\mathbf{j}/c. \tag{227}$$

Gauge Invariance

E and **B** are invariant under gauge transformations, that is under the change

$$\mathbf{A} \to \mathbf{A} + \nabla\psi;\ \phi \to \phi - \frac{1}{c}\partial_t\psi. \tag{228}$$

Only **E** and **B** are measured and so the choice of gauges is merely a choice to make the equations easier to solve. Different gauges are more or less convenient in different contexts.

The Coulomb gauge is such that $\nabla \cdot \mathbf{A} = 0$. The Lorenz gauge employs $\partial_t \phi + \nabla \cdot \mathbf{A} = 0$. For the latter case we obtain for the potentials

$$\nabla^2 \phi - \frac{1}{c^2} \partial_t^2 \phi = -4\pi\rho \tag{229}$$

and

$$\nabla^2 \mathbf{A} - \frac{1}{c^2} \partial_t \mathbf{A} = -4\pi \mathbf{j}. \tag{230}$$

The formal solutions are

$$\phi(\mathbf{r}, t) = \int \frac{[\rho]}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \int \int \frac{\rho(\mathbf{r}', t')\delta(t' - t + |\mathbf{r} - \mathbf{r}'(t')|/c)}{|\mathbf{r} - \mathbf{r}'|} dt' d^3\mathbf{r}' \tag{231}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{1}{c} \int \frac{[\mathbf{j}]}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}' = \int \int \frac{\mathbf{j}(\mathbf{r}', t')\delta(t' - t + |\mathbf{r} - \mathbf{r}'(t')|/c)}{|\mathbf{r} - \mathbf{r}'|} dt' d^3\mathbf{r}', \tag{232}$$

where the brackets indicate evaluation at the retarded time. Thus these are the retarded time potentials. The retarded time means that the quantity is evaluated at a time $t_{ret} = t - |\mathbf{r} - \mathbf{r}'|/c$ from the present time due to finite speed of light travel. Thus e.g. $[\rho] = [\rho(\mathbf{r}', t_{ret})]$.

Formally, given these potentials, we can solve for **E** and **B**.

Potential of Moving Charges

Consider charge q following path $\mathbf{r} = \mathbf{r}_0(t)$, $\mathbf{u} = \dot{\mathbf{r}}_0$. Then the localized charge and current densities are

$$\rho(\mathbf{r}', t') = q\delta(\mathbf{r}' - \mathbf{r}_0(t')) \tag{233}$$

$$\mathbf{j}(\mathbf{r}', t') = q\mathbf{u}(t')\delta(\mathbf{r}' - \mathbf{r}_0(t')). \tag{234}$$

The volume integrals give the total charge q and current $q\mathbf{u}$, for a single particle.

Using the formal solutions (231) and (232), we can calculate retarded potentials for this single particle, these are

$$\phi(\mathbf{r}, t) = q \int \frac{\delta(t' - t + R(t')/c)}{R(t')} dt' \tag{235}$$

and

$$\mathbf{A}(\mathbf{r}, t) = q \int \mathbf{u}(\mathbf{r}') \frac{\delta(t' - t + R(t')/c)}{R(t')} dt', \tag{236}$$

where $R(t') = |\mathbf{R}(t')|$ and

$$\mathbf{R}(t') = \mathbf{r} - \mathbf{r}_0(t'). \tag{237}$$

If we now use $t_{ret} = t - R(t')/c$ or $c(t - t_{ret}) = R(t')$. We then can change variables. Let

$$t'' = t' - t_{ret} \tag{238}$$

so that

$$dt'' = dt' - dt_{ret} = dt' + dt'\frac{\dot{R}(t')}{c} = dt' + dt'\frac{1}{2cR(t')}\frac{dR^2(t')}{dt'} = dt' + dt'\frac{1}{2cR(t')}\frac{d\mathbf{R}^2(t')}{dt'} = dt'(1 - \hat{\mathbf{n}}(t')\cdot\frac{\mathbf{u}(t')}{c}), \tag{239}$$

where $\hat{\mathbf{n}} = \mathbf{R}/R$ and $\mathbf{u}(t') = \dot{\mathbf{r}}_0 = -\dot{\mathbf{R}}$. (Note that we consider t and $\mathbf{r}$ fixed in this differentiation ; we are not integrating or differentiating over t or $\mathbf{r}$, there are just the time and position at which the quantities are being measured.) Thus using (239) in (235) and (236) gives

$$\phi(\mathbf{r}, t) = q \int \frac{1}{R(t')(1 - \hat{\mathbf{n}}(t') \cdot \mathbf{u}(t')/c)} \delta(t'')dt'' = \frac{q}{R(t_{ret})(1 - \hat{\mathbf{n}}(t_{ret}) \cdot \mathbf{u}(t_{ret})/c)} \tag{240}$$

and

$$\mathbf{A}(\mathbf{r}, t) = q \int \mathbf{u}(t')/c \frac{1}{R(t')(1 - \hat{\mathbf{n}}(t') \cdot \mathbf{u}(t')/c)} \delta(t'')dt'' = \frac{qu(t_{ret})}{R(t_{ret})(1 - \hat{\mathbf{n}}(t_{ret}) \cdot \mathbf{u}(t_{ret})/c)}, \tag{241}$$

where we have integrated $\delta(t'')$ by setting $t' = t_{ret}$ using (238). Recall that integrating a function over a δ function works like this: $\int \delta(q - q_0)f(q)dq = f(q_0)$.

Notice the factor appearing on the bottom which "concentrates" the potentials along the direction of motion for strongly relativistic flows. This is related to the concept of relativistic beaming of radiation into narrow cones along the direction of motion that we will derive later. These potentials are also evaluated at the retarded time–again important for very relativistic motion. This retarded time is also the key to getting the radiation part of the electromagnetic fields, that is the part that falls off as $1/r$ rather than $1/r^2$. The potentials above are the Liénard-Wiechart potentials.