LECTURE 9

Thomson Scattering

We want to apply the dipole formula to the case for a charge radiating in response to an impinging wave. Ignore $\vec{\beta} \times \mathbf{B}$ force for non-relativistic velocities. The electric force is the main contributor and we have the equation of motion for forcing from a linear polarized wave

$$m\ddot{\mathbf{r}} = \mathbf{F} = e\vec{\epsilon}E_0 sin\omega_0 t, \tag{267}$$

or

$$\ddot{\mathbf{d}} = (e^2/m)\vec{\epsilon}E_0 sin\omega_0 t, \tag{268}$$

where e is the charge and $\vec{\epsilon}$ is the direction of E_0. The solution is an oscillating dipole

$$\mathbf{d} = e\mathbf{r} = -(e^2 E_0/m\omega_0^2) sin\omega_0 t. \tag{269}$$

fig (3.6)

Time averaged power is then

$$dP/d\Omega = \frac{\ddot{\mathbf{d}}^2}{4\pi c^3} sin^2\Theta = \frac{e^4 E_0^2}{8\pi m^2 c^3} sin^2\Theta, \tag{270}$$

where we used $\mathrm{Lim}_{T\to\infty}(1/T)\int_0^T sin^2(\omega t)dt = 1/2$. We also have upon integration over solid angle $P = e^4 E_0^2/3m^2c^3$.

Define the differential cross section $d\sigma$ for scattering an initial electromagnetic wave into solid angle $d\Omega$ from interaction with the particle. Then

$$\frac{dP}{d\Omega} = \langle S \rangle \frac{d\sigma}{d\Omega} = \frac{cE_0^2}{8\pi}\frac{d\sigma}{d\Omega}, \tag{271}$$

therefore from (275)

$$\frac{d\sigma}{d\Omega_{polarized}} = r_o^2 sin^2\Theta, \tag{272}$$

where

$$r_0^2 = \frac{e^4}{m^2 c^4}. \tag{273}$$

Note here that $\langle S \rangle$ is the incident flux.

For an electron, $r = 2.8 \times 10^{-13}$cm, and is called the classical electron radius. Integrating over solid angle gives

$$\sigma = 8\pi r_0^2/3 \tag{274}$$

which is $\sigma_T = 6.6 \times 10^{-25}$cm^2, called the Thomson cross section for an electron. This remains valid when $h\nu < 0.5MeV$, the electron rest mass. Above this we are in the quantum regime, and the "Klein Nishina" cross section must be used (discussed later). Dipole approximation is also invalid for relativistic motions.

Polarization of Electron scattered radiation

The scattered radiation is polarized in the plane spanned by initial polarization vector $\vec{\epsilon}$ and $\hat{\mathbf{n}}$.

Differential cross section (as a function of angle into which radiation is scattered) for scattering of initially unpolarized radiation is found by writing the incident radiation as the sum of two orthogonal linearly polarized beams.

Take $\vec{\epsilon}_1$ to be in plane of incident and scattered beam.

Take $\vec{\epsilon}_2$ to be $\perp$ to this plane.

Take Θ as the angle between $\vec{\epsilon}_1$ and $\hat{\mathbf{n}}$. $\vec{\epsilon}_2 \perp \hat{\mathbf{n}}$, and $\mathbf{n}$ is the direction of the scattered wave.

(fig3.7)

Then using (272) the differential cross section for scattering into $d\Omega$ from an initially unpolarized beam is

$$d\sigma/d\Omega|_{unpo} = \frac{1}{2}[d\sigma(\Theta)/d\Omega + d\sigma(\pi/2)/d\Omega] = \frac{1}{2}(1 + sin^2\Theta) = \frac{1}{2}(1 + cos^2\theta), \tag{275}$$

where $\theta = \pi/2 - \Theta$. This depends only on the angle between the direction of the incident and scattered radiation. Note that integrated cross section for polarized radiation is the same as for the unpolarized case (because electron has no intrinsic direction and energy is conserved).

Recall that **n**, the outgoing wave direction, is perpendicular to the polarization vector of the scattered radiation. The two differential cross sections on the right of (275) refer to the intensities of these two planes of polarization.

The ratio of the differential cross section for scattering into the plane and perpendicular to the plane of scattering is $cos^2\theta$, so we have, from (212)

$$\Pi = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{1 - cos^2\theta}{1 + cos^2\theta}. \tag{276}$$

This means that initially unpolarized radiation can be polarized upon scattering. This makes sense. If we look along incident direction, then we get no polarization since there is azimuthal symmetry around the initial wave vector axis and all directions are the same. If we look $\perp$ to the initial wave vector axis $\theta = \pi/2$, then we see 100% polarization. This is because the particle moves in the plane $\perp$ to the initial direction.

Radiation Damping Force

The radiation damping force can be assumed to be perturbation on the particles motion when $T_{rad} >> t_p$, i.e. the time over which the particles energy is changed significantly by radiation is much longer than the time for the particle to change its position (e.g. orbit time).

$$T_{rad} \sim mv^2/P_{rad} = \frac{3mc^3}{2e^2}\left(\frac{u}{\dot{u}}\right)^2. \tag{277}$$

This then requires that $t_p >> \tau = r_0/c = 10^{-23}$s, the light crossing time of a classical electron radius.

For scattering problems, this holds when the distance electrons wander in an atom (typically of order 1 Angstrom) is << than the wavelength of the radiation—that is the wavelength on which the electric field varies (e.g. >> 1 Angstrom.)

The radiative loss of energy, time averaged is

$$\langle dW/dt\rangle = P = (2e^2/3c^3)\langle|\ddot{\mathbf{x}}|^2\rangle, \tag{278}$$

where

$$\langle|\ddot{\mathbf{x}}|^2\rangle = \frac{1}{t_p}\int_{t-t_p/2}^{t+t_p/2} \ddot{\mathbf{x}}^2 dt, \tag{279}$$

where $t_p = 2\pi/\omega$. Since we are considering $\dot{\mathbf{x}} = x_0 e^{i\omega t}$, upon integrating by parts we note that products of odd and even numbers of derivatives produce a real part that cancels out, the only term that survives is

$$\langle |\ddot{\mathbf{x}}|^2 \rangle = \frac{1}{t_p} \int_{t-t_p/2}^{t+t_p/2} \dddot{\mathbf{x}} \cdot \dot{\mathbf{x}} dt, \tag{280}$$

We can then write

$$\langle dW/dt \rangle = \langle F_{rad} \cdot \dot{\mathbf{x}} \rangle, \tag{281}$$

so

$$F_{rad} = \frac{2e^2}{3c^3} \dddot{\mathbf{x}}. \tag{282}$$

Radiation from Harmonically Bound Particles, Classical Line Profile

Assume that particle is bound to a center of force. Then its motion represents oscillation and its equation of motion

$$\mathbf{F} = -k\ddot{\mathbf{r}} = -m\omega_0^2 \mathbf{r}, \tag{283}$$

such that the oscillation is with frequency ω_0. A classical oscillator (Thomson atom) is not ultimately applicable, though quantum results are quoted in relation to the classical results.

The radiation reaction force for a radiating particle damps the oscillator. We assume that the damping is a perturbative effect. From (277), the validity of this classical non-relativistic regime is when $\tau/t_p << 1$ so that, τ, the light crossing time over a classical electron radius is short compared to any particle orbit period. This means the time scale to lose significant energy to radiation is long compared to particle orbit time. For oscillation along the $x - axis$:

$$-\tau \dddot{x} + \ddot{x} + \omega_0^2 x = 0. \tag{284}$$

If the 3rd derivative term is small we can assume the motion is harmonic to lowest order. Then we assume $x = cos(\omega_0 t + \phi_0)$ to this order. We then have $\tau \dddot{x} = -\omega_0^2 \dot{x}$ so

$$\ddot{x} + \omega_0^2 \tau \dot{x} + \omega_0^2 x = 0. \tag{285}$$

To solve assume $x(t) = e^{\alpha t}$ so that

$$\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0. \tag{286}$$

The solution is

$$\alpha = \pm i\omega_0 - \frac{1}{2}\omega_0^2 \tau + O(\tau^2 \omega_0^3). \tag{287}$$

At $t = 0$ take $x(0) = x_0$ and $\dot{x}(0) = 0$. We thus have

$$x(t) = x_0 e^{-\Gamma t/2} cos\omega_0 t = \frac{1}{2} x_0 (e^{-\Gamma t/2 + i\omega_0 t} + e^{-\Gamma t/2 - i\omega_0 t}). \tag{288}$$

where $\Gamma = \omega_0^2 \tau$.

If we Fourier transform we can find the power spectrum of the emitted radiation in the dipole approximation. We have

$$\tilde{x}(\omega) = \frac{1}{2\pi} \int x(t')e^{iwt'}dt' = \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega_0+\omega)} + \frac{1}{\Gamma/2 - i(\omega_0-\omega)}\right] \sim \frac{x_0}{4\pi}\left[\frac{1}{\Gamma/2 - i(\omega_0-\omega)}\right], \tag{289}$$

where we ignore the first term on the right as we are working for the regime of ω near ω_0. The energy radiated per unit frequency is then

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3}|\tilde{d}(\omega)|^2 = \left(\frac{8\pi\omega^4 e^2}{3c^3 16\pi^2}\right) x_0^2 \left[\frac{1}{(\Gamma/2)^2 + (\omega_0-\omega)^2}\right]. \tag{290}$$

The power spectrum is then a Lorentz profile. (fig 3.8)

The power spectrum for a particle damped by radiation–classical line. There is a sharp max near $\omega = \omega_0$, and the width is $\delta\omega \sim \Gamma = \omega_0^2\tau << 1$. The quantity in curved parentheses can be thought of as a spring constant, characterizing the potential energy of the particle. Integrating (290) over frequency gives the total energy of $\frac{1}{2}kx_0^2$. This is the energy that is being radiated away.

Driven Harmonically Bound Particles

In this case we have a forcing from beam of radiation, so the equation of motion is

$$\ddot{x} = -\omega_0^2 x + \tau\dddot{x} + (eE_0/m)cos\omega t. \tag{291}$$

Now take x to be complex so we have

$$\ddot{x} + \omega_0^2 x - \tau\dddot{x} = (eE_0/m)e^{i\omega t}. \tag{292}$$

The solution is

$$x = x_0 e^{i\omega t} \equiv |x_0|e^{i(\omega t+\delta)}, \tag{293}$$

where

$$x_0 = \frac{eE_0/m}{w^2 - \omega_0^2 - i\omega_0^3\tau} \tag{294}$$

and

$$\delta = tan^{-1}(\omega^3\tau/w^2 - \omega_0^2). \tag{295}$$

The damping term from the radiation reaction provides a phase shift. For $w > w_0$, $\delta > 0$ and for $\omega < \omega_0$, $\delta < 0$. This represents an oscillating dipole with frequency ω and amplitude $|x_0|$.

The time averaged power radiated is then

$$P(\omega) = e^2|x_0|^2\omega^4/3c^3 = \frac{e^4E_0^2}{3m^2c^2}\frac{\omega^4}{(\omega_2^2 - \omega_0^2) + (\omega_0^3\tau)^2}. \tag{296}$$

Dividing by the time averaged Poynting flux $\langle S\rangle = cE_0^2/8\pi$ we "project out" the cross section for scattering

$$\sigma = \sigma_T\frac{\omega^4}{(\omega^2 - \omega_0^2) + (\omega^3\tau)^2}. \tag{297}$$

For $\omega >> \omega_0$ $\sigma(\omega) \rightarrow \sigma_T$. Basically this is because at high incident energies, the frequency of the particle motion is not seen by the radiation, so it appears free.

For $\omega << \omega_0$, we have $\sigma \propto \omega^4$. This is the Rayleigh-Scattering regime—bluer frequencies scattered more (thus looking at horizon means we the Sun to be redder). Inertial term in equation of motion is small, so the dipole moment of the particle is directly proportional to the incident radiation. $kx = eE$.

For $\omega \sim \omega_0$ we have $\omega^2 - \omega_0^2 \sim 2\omega(\omega - \omega_0)$ so

$$\sigma(\omega) = \frac{\sigma_T}{\tau}\frac{\Gamma}{4(\omega - \omega_0)^2 + \Gamma^2} = \frac{2\pi^2e^2}{mc}\frac{\Gamma}{4(\omega - \omega_0)^2 + \Gamma^2}, \tag{298}$$

recalling that $\Gamma = \omega_o^2\tau$. This is the Lorentz profile. This result arises because free oscillations of the unforced oscillator can be excited by a pulse of radiation $E(t) = \delta(t)$. The spectrum for this pulse is independent of frequency. (White spectrum) and so the emission is just scattering of a white spectrum. The emission is then proportional to the cross section for this scattering. $P = \langle S\rangle\sigma$

If we integrate $\sigma(\omega)$ over ω we find that

$$\int_0^\infty \sigma(\omega)d\omega = \frac{2\pi^2e^2}{mc} \sim \sigma_T/\tau. \tag{299}$$

But this is not valid because at large ω the classical approximation breaks down (classical requires $\omega\tau << 1$, where τ is light crossing time across classical electron radius). Actually we have

$$\int_0^{\omega_{max}} \sigma(\omega)d\omega = \sigma_T\omega_{max}, \tag{300}$$

because the range of validity of the classical radiation reaction is, $\omega_{max} << 1/\tau = c/r_0$.

For quantum regime however, we write

$$\int_0^\infty \sigma(\omega)d\omega = (\sigma_T/\tau)f_{mm'} \sim \sigma_T/\tau, \tag{301}$$

where $f_{mm'}$ is the oscillator strength for $m \to m'$ transitions. (fig 3.9)

Correspondence between emission and Einstein Coeff.

Note the power radiated in the Larmor formula time averaged over 1 cycle for a harmonic oscillator is

$$P = \frac{e^2\omega_0^4}{3c^3}|x_0|^2. \tag{302}$$

But this should equal $A_{21}h\nu_{21}$ using the form of the Einstein coefficients, where ν_{21} is the resonance line frequency. We can then say that $|x_0|/2 = \langle 2|\mathbf{x}|1\rangle$ is 1/2 the displacement of the oscillation representing the expectation value between the transition from state 2 to state 1. Then we would have

$$A_{21} = \frac{8\pi e^2\omega_{21}^3}{3c^3h}|\langle 2|\mathbf{x}|1\rangle|^2, \tag{303}$$

which turns out to be not a bad approximation for specific cases in in the quantum regime (e.g. when the statistical weights of the two states are non-degenerate).